Final, Math 535, professor Agol, due 5/10, 5 pm, Spring 2007

You may use your book or notes, but may not consult anyone.
p. 184, #5; p. 186, #3; p. 227, # 1

1. Classify twice punctured simply-connected regions in $\mathbb{C}$. That is, suppose $\Omega_i = \Omega_i' - \{w_i, z_i\}$, $i = 1, 2$, $w_i, z_i \in \Omega_i'$, $w_i \neq z_i$, where Ω_i' is simply-connected. Then Ω_1 and Ω_2 are equivalent if there exists a bijective analytic map $f : \Omega_1' \to \Omega_2'$, such that $f(w_1) = w_2, f(z_1) = z_2$. Find a canonical representative for each equivalence class, and show that the equivalence classes are distinct.

2. Compute
$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 e^{itx} dx$$
for real t.

3. (a) Show that
$$f(z) = \frac{1}{e^z - 1} = \sum_{k=-1}^{\infty} B_{k+1} z^k/(k+1)!,$$
where $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_{2k+1} = 0$ for $k \in \mathbb{N}$. Prove that $\overline{\lim_{k\to\infty}} |B_{k+1}/(k+1)!|^{1/k} = 1/2\pi$, and compute B_2, B_4, B_6.

   (b) Compute a Taylor series about $z = 0$ for
$$\frac{e^{mz} - 1}{e^z - 1}$$
two different ways, where $m \in \mathbb{N}$. Compare coefficients, to show that
$$\sum_{k=0}^{m-1} k^n = \frac{1}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} m^j B_{n+1-j}.$$
Find the polynomial expression for $\sum_{k=0}^{m-1} k^6$.

4. Show that if a, b, c lie on the unit circle, and $a + b + c = 0$, then they form the vertices of an equilateral triangle.