

Midterm, Math 535, professor Agol, 3/14, Spring 2007

Due 3/19/07, in class

1. Suppose α is a complex number, $|\alpha| \neq 1$, and compute

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha \cos \theta + \alpha^2}$$

by integrating $(z - \alpha)^{-1}(z - \alpha^{-1})^{-1}$ over the unit circle.

Solution: Parameterize the unit circle C by $z(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$. Then $z'(\theta) = ie^{i\theta} = iz(\theta)$, and we have $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$. We also have $d\theta = -idz/z$, so we have

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{1 - 2\alpha \cos \theta + \alpha^2} &= \int_C \frac{-idz/z}{1 - \alpha(z + z^{-1}) + \alpha^2} = \int_C \frac{idz\alpha^{-1}}{z^2 - (\alpha + \alpha^{-1})z + 1} \\ &= \frac{i}{\alpha} \int_C \frac{dz}{(z - \alpha)(z - \alpha^{-1})}. \end{aligned}$$

If $\alpha = 0$, then

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha \cos \theta + \alpha^2} = \int_0^{2\pi} d\theta = 2\pi = \frac{2\pi}{1 - \alpha^2}.$$

If $0 < |\alpha| < 1$, then $\frac{1}{z - \alpha^{-1}}$ is analytic inside C since $|\alpha^{-1}| > 1$, so by Cauchy's integral formula, we have

$$i\alpha^{-1} \int_C \frac{(z - \alpha^{-1})^{-1} dz}{z - \alpha} = i\alpha^{-1} \frac{2\pi i}{\alpha - \alpha^{-1}} = \frac{2\pi}{1 - \alpha^2}.$$

If $|\alpha| > 1$, then exchanging the roles of α and α^{-1} , we have

$$i\alpha^{-1} \int_C \frac{(z - \alpha)^{-1} dz}{z - \alpha^{-1}} = i\alpha^{-1} \frac{2\pi i}{\alpha^{-1} - \alpha} = \frac{2\pi}{\alpha^2 - 1}.$$

So we have for $|\alpha| \neq 1$:

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha \cos \theta + \alpha^2} = \frac{2\pi}{\alpha^2 - 1} \frac{|\alpha| - 1}{||\alpha| - 1|}.$$

2. Compute

$$\int_{|z-1|=\frac{3}{2}} \frac{3z^2 - 1}{z^3 - z} dz.$$

Solution: Let C be the closed curve parameterizing $|z - 1| = \frac{3}{2}$ using the counterclockwise orientation. Then

$$= \int_C \frac{1}{z-1} + \frac{1}{z} + \frac{1}{z+1} dz = 2\pi i(n(C, 1) + n(C, 0) + n(C, -1)) = 2\pi i(1 + 1 + 0) = 4\pi i.$$

3. Prove that if $f : B_R(a) \rightarrow \mathbb{C}$ satisfies

$$f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k,$$

where R is the radius of convergence of the series, then there is an anti-derivative $F(z)$ defined on the same region. Apply this to the function $f(z) = \frac{1}{z}$ defined on $B_1(1)$ to derive an anti-derivative $F(z)$ on this disk. Prove that one may choose $F(z)$ such that $e^{F(z)} = z$, and give the power series for this $F(z)$.

Solution: Define $F(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(z-a)^{k+1}$. By Theorem 2, p. 38, we have $F'(z) = \sum_{k=0}^{\infty} a_k(z-a)^k$, and the power series for $F(z)$ will have the same radius of convergence as the power series for $f(z)$.

Let $f(z) = 1/z = 1/(1 - (1-z)) = \sum_{k=0}^{\infty} (1-z)^k = \sum_{k=0}^{\infty} (-1)^k (z-1)^k$, which is a series for $f(z)$ converging on $B_1(1)$ by the formula for a geometric series. Then

$$F(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (z-1)^{k+1}.$$

Then $g(z) = e^{F(z)}$ is an analytic function on $B_1(1)$, with $F(1) = 0$. We have $\frac{d}{dz}(z/g(z)) = -ze^{-F(z)}F'(z) + e^{-F(z)} = (-z/z+1)e^{-F(z)} = 0$. Then $z/g(z) = 1$, since its derivative is zero, so it is a constant which is determined by plugging in $z = 1$. So we see that $e^{F(z)} = z$. In this case, we may identify $F(z) = \log(z)$, where we have chosen the principle branch of log on the disk $B_1(1)$.

4. Let $p(k)$ be a degree n polynomial defined by

$$p(k) = p_0 + \binom{k}{1}p_1 + \binom{k}{2}p_2 + \cdots + \binom{k}{n}p_n.$$

Define

$$f(z) = \sum_{k=0}^{\infty} p(k)z^k.$$

Find the radius of convergence of the power series, prove that $f(z)$ is a rational function restricted to the disk of convergence, and give a formula for $f(z)$.

Solution: We have $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ for $|z| < 1$. Then

$$\frac{d^n}{dz^n} \frac{1}{1-z} = \frac{n!}{(1-z)^{n+1}}.$$

We also have

$$\frac{d^n}{dz^n} \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} k(k-1)\cdots(k-n+1)z^{k-n} = \frac{n!}{(1-z)^{n+1}},$$

with the same radius of convergence $R = 1$ by Theorem 2, p. 38 (unless all the $p_i = 0$, in which case $R = \infty$). We may rewrite this as

$$z^n(1-z)^{-(n+1)} = \sum_{k=0}^{\infty} \binom{k}{n} z^k.$$

Thus,

$$f(z) = \sum_{k=0}^{\infty} \sum_{j=0}^n p_j \binom{k}{n} z^k = \sum_{j=0}^n p_j \sum_{k=0}^{\infty} \binom{k}{n} z^k = \sum_{j=0}^n p_j z^j (1-z)^{-j-1},$$

which is a sum of rational functions, and is therefore rational.

5. Let $f(z) = z^3 - 3z$. Find a region in the plane that one may invert the equation $w = f(z)$ by an analytic function $z = F(w)$, such that

$$F(w) = \left(w/2 + ((w/2)^2 - 1)^{\frac{1}{2}} \right)^{\frac{1}{3}} + \left(w/2 - ((w/2)^2 - 1)^{\frac{1}{2}} \right)^{\frac{1}{3}}.$$

Solution: Let $\Omega = \mathbb{C} - \{z \in \mathbb{R} \mid |z| \geq 2\}$. We claim that $F(w)$ may be well-defined on Ω . For $w \in \Omega$, we have $((w/2)^2 - 1)^{\frac{1}{2}} = i(1 - w/2)^{\frac{1}{2}}(w/2 + 1)^{\frac{1}{2}}$. Since for $w \in \Omega$, $1 - w/2 \notin \{z \in \mathbb{R} \mid z \leq 0\}$, and $w/2 + 1 \notin \{z \in \mathbb{R} \mid z \leq 0\}$, we may use the principal branch of $z^{\frac{1}{2}}$ to define $(1 - w/2)^{\frac{1}{2}}$ and $(w/2 + 1)^{\frac{1}{2}}$ analytically in Ω . Thus, the functions $w/2 \pm ((w/2)^2 - 1)^{\frac{1}{2}}$ are well-defined in Ω using the above formula. We also have a principal branch of $z^{\frac{1}{3}}$ defined on $\mathbb{C} - \{z \in \mathbb{R} \mid z \leq 0\}$, using the principal branch of $\log z$, by setting $z^{\frac{1}{3}} = e^{\frac{1}{3} \log z}$. We may use the principal branch of $z^{\frac{1}{3}}$ to define $F(w)$ as long as $w/2 \pm ((w/2)^2 - 1)^{\frac{1}{2}} \notin \{z \in \mathbb{R} \mid z \leq 0\}$. Suppose that $w/2 + ((w/2)^2 - 1)^{\frac{1}{2}} \leq 0$. Since $(w/2 + ((w/2)^2 - 1)^{\frac{1}{2}})(w/2 - ((w/2)^2 - 1)^{\frac{1}{2}}) = 1$, we must have $w/2 + ((w/2)^2 - 1)^{\frac{1}{2}} \neq 0$, and thus also $w/2 - ((w/2)^2 - 1)^{\frac{1}{2}} < 0$. Then we see that $w < 0$ by adding the two formulae, and thus $(w/2)^2 - 1 > 0$ since its square root must be real, so $|w| > 2$, contradicting our assumption that $w \in \Omega$. Thus, we may use the principal branch of $z^{\frac{1}{3}}$ to define both terms in the expression for $F(w)$. We have $\log(1/z) = -\log z$ for the principal branch of \log , since $1/z$ preserves $\{z \in \mathbb{R} \mid z \leq 0\}$, and $e^{\log(1/z) + \log z} = 1$, so $\log(1/z) + \log z = 2\pi in$, for some fixed $n \in \mathbb{Z}$. Since $\log(1/1) = \log 1 = 0$, we see that $n = 0$, and thus $\log(1/z) = \log z$. Thus, we have $(1/z)^{\frac{1}{3}} = e^{\frac{1}{3} \log(1/z)} = e^{-\frac{1}{3} \log z} = 1/z^{1/3}$.

Write $x = (w/2 + ((w/2)^2 - 1)^{\frac{1}{2}})^{\frac{1}{3}}$, then $F(w) = x + 1/x$. We see that

$$f(F(w)) = f(x + 1/x) = (x + 1/x)^3 - 3(x + 1/x) = x^3 + 3x + 3/x + 1/x^3 - 3(x + 1/x) = x^3 + 1/x^3 = w.$$

Thus, $z = F(w)$ inverts $w = f(z)$ on the region Ω .