Fixed-Point Theorems for Frêchet Spaces

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In considering nonlinear partial differential equations defined on functions whose domain is an unbounded subset of Euclidean space, one is led to search for solutions u of operator equations of the form

$$u = Au. \tag{1}$$

Typically, A will be an integral operator. It often happens that while A is nonlinear, it nevertheless presents the zero function as a solution, and what is required is a second, nontrivial solution.

Many of the powerful methods of attacking equations like (1) appear not to be applicable in these circumstances. The contraction mapping principle is difficult to apply in the presence of the trivial solution, unless one has knowledge of an approximation to the nontrivial solution. Further, in the usual Banach spaces of functions used in analyzing (1), A is not a compact operator, owing to the unboundedness of the underlying Euclidean domain. In particular the results of Krasnosel'skii [K] on mappings of Banach spaces ordered by a cone are not applicable.

It very often happens, however, that the operators A <u>are</u> compact in wider function spaces, which are generally not normed spaces, but more general metric linear spaces. It is therefore of interest to extend the 'cone theorems' to a broader class of spaces, and this note contributes a result in this direction.

The proofs of the cone theorems in a Banach space setting are accomplished by Krasnosel'skii [K, ch. 4] by intricate arguments which appear difficult to extend. However, a recent and simplified proof of

Krasnosel'skii's result has been obtained by Benjamin [B, appendix 1], and the method proposed there can be generalized by using the degree theory in locally convex topological linear spaces developed by Nagumo [N].

We let E denote a Fréchet space (complete, separable metric linear space, over $\mathbb{R}$) and K a cone in E ($K \subset E$ is not the zero element alone, is convex, closed, $x \in K \Rightarrow \alpha x \in K$ for $\alpha > 0$, and $x, -x \in K \Rightarrow x = 0$). Let d denote the metric in E. Without loss of generality, we can suppose d has the form

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} p_j(x-y)/(1+p_j(x-y)), \tag{2}$$

where $p_1, p_2, \ldots$ is an increasing sequence of pseudo-norms [T, ch. 8]. Further, we may suppose that for some $w \in K$, $p_1(w) > 0$. Let $A: K \to K$ be a continuous mapping which is 'compact' in the sense that $A(S_r(0) \cap K)$ is a precompact set, for $r<1$, where $S_r(0) = \{x \in E: d(x,0) < r\}$.

It follows from Dugundgi's theorem [H, theorem 14.1, p. 57] that A can be extended to a continuous mapping $\tilde{A}$ of E into K such that $\tilde{A}(S_r(0))$ is precompact, for $r<1$ (apply the extension theorem sequentially to the closed sets $\overline{S_{r_n}(0)}$ where $r_n \uparrow 1$ and use the fact that the convex hull of a precompact set is precompact). Define $\Phi = I - \tilde{A}$. Then solutions of (1) for A correspond to zeros of Φ.

Suppose G is an open subset of E which $\tilde{A}$ maps to a precompact set and for which $\Phi \neq 0$ on ∂G $(= \overline{G} \setminus G)$. Then the rotation $\mathrm{rot}[\Phi, G]$ of Φ on G is an integer defined, in a locally convex linear space setting, by Nagumo [N]. In Nagumo's notation, $\mathrm{rot}[\Phi, G] = A[0, G, \Phi]$. This integer obeys all the usual rules of the usual rotation in n-dimensional Euclidean space [N, §4]. Here is our result.

THEOREM. Let K be a cone in the Fréchet space E as above and let $A: K \to K$ be a continuous and compact mapping. Suppose that for $0<r<R<1$,

(i) $u - Au \notin K$ for $u \in K$ and $d(u,0) = r$,

(ii) $Au - u \notin K$ for $u \in K$ and $d(u,0) = R$.

then A has a solution ϕ to (1) such that $r<d(\phi,0)<R$.

REMARK. The same conclusion may be drawn if instead of (i) and (ii), we suppose

(iii) $Au - u \notin K$ for $u \in K$ and $d(u,0) = r$,

(iv) $u - Au \notin K$ for $u \in K$ and $d(u,0) = R$.

The proof is a trivial modification of the proof given below assuming (i) and (ii).

Proof. We let $\tilde{A}$ and Φ be as defined above. First, if $d(u,0) = r$, $u - \tilde{A}u \notin K$. If $u \in K$, this is provided by (i) since then $\tilde{A}u = Au$. If $u \notin K$, and $u - \tilde{A}u = v \in K$, then $u = \tilde{A}u + v \in K$, a contradiction.

Next, there is a $v \in K$ such that $v \notin \Phi(\overline{S_r(0)})$. In fact, a large positive multiple of the element $w \in K$ such that $p_1(w)>0$ works. For if $\Phi(\overline{S_r(0)})$ contains the entire half ray $\{\alpha w: \alpha \geq 0\}$, then since $d(\alpha w,0) \to 1$ as $\alpha \to +\infty$ (from (2) and the condition $p_1(w)>0$), there would be a sequence $u_n - \tilde{A}u_n$ such that $d(u_n-\tilde{A}u_n,0) \to 1$. From (2), this can happen only if $p_1(u_n-\tilde{A}u_n) \to +\infty$. We can assume that $\tilde{A}u_n \to y$ in the metric d (by compactness of $\tilde{A}$) and hence that $\tilde{A}u_n \to y$ for the pseudo-norm p_1. It then follows that $p_1(u_n) \to +\infty$, which implies $d(u_n,0) \to 1$ from (2), a contradiction because $u_n \in \overline{S_r(0)}$.

Consider the homotopy $M: \overline{S_r(0)} \times [0,1] \to E$ given by $M(u,t) = \tilde{A}u - tv$. For $u \in \partial S_r(0)$, $u - M(u,t) \neq 0$, $0 \leq t \leq 1$. For $u - M(u,t) = 0 \Rightarrow \Phi(u) = tv \in K$, which we have shown cannot hold. For fixed t, M is a compact mapping of $\overline{S_r(0)}$, since $\tilde{A}$ is, and for fixed u, M is uniformly continuous in t. The conditions of [N, theorem 7, p. 504] are met, and we may conclude that the rotation of $I - M(u,0)$ and $I - M(u,1)$ are the same on $S_r(0)$. Thus,

$$\mathrm{rot}[\Phi,S_r(0)] = \mathrm{rot}[\Phi-v,S_r(0)]. \tag{3}$$

The right side of (3) is zero, for if not, then by [N, theorem 5, p. 503], $\Phi - v$ would have a zero in $S_r(0)$, a contradiction to our choice of v.

Finally, consider the homotopy $F: \overline{S_R(0)} \times [0,1] \to E$ defined by $F(u,t) = t\tilde{A}u$. If $u - F(u,t) = 0$ for $u \in \partial S_R(0)$, then $u = t\tilde{A}u \in K$ (and hence $t \neq 0$). Thus $\tilde{A}u = Au$ and so $t(Au-u) = (1-t)u \in K \Rightarrow Au - u \in K$ which contradicts (ii). Again, for fixed t, F is compact on $\overline{S_R(0)}$, and for fixed u, F is uniformly continuous in t. Thus we derive that

$$\mathrm{rot}[\Phi, S_R(0)] = \mathrm{rot}[I, S_R(0)]$$

and the latter rotation is 1 [N, p. 503].

Now, let $T = \{x \in E: r<d(x,0)<R\}$. T is an open set, and $T \cap S_r(0) = \emptyset$, $\overline{T} \cup \overline{S_r(0)} = \overline{S_R(0)}$, $T \cup S_r(0) \subset S_R(0)$ and $\Phi \neq 0$ on $\partial S_r(0)$ or ∂T by (i) and (ii). It follows from [N, theorem 6, p. 503] that

$$\mathrm{rot}[\Phi, S_R(0)] = \mathrm{rot}[\Phi, T] + \mathrm{rot}[\Phi, S_r(0)],$$

from which we conclude $\mathrm{rot}[\Phi, T] = 1$. Hence Φ has a zero in T, [N, theorem 5], and this is the required result.

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