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A model for the two-way propagation of water waves in a channel

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Global existence, uniqueness and regularity of solutions and continuous dependence of solutions on varied initial data are established for the initial-value problem for the coupled system of equations

$$\eta_t + u_x + (u\eta)_x - \tfrac{1}{3}\eta_{xxt} = 0,$$

$$u_t + \eta_x + uu_x - \tfrac{1}{3}(u_t + \eta_x)_{xx} = 0.$$

This system has the same formal justification as a model for the two-way propagation of (one-dimensional) long waves of small but finite amplitude in an open channel of water of constant depth as other versions of the Boussinesq equations. A feature of the analysis is that bounds on the wave amplitude η are obtained which are valid for all time.

1. *Introduction.* In the scientific literature on water waves there are numerous model equations which approximately describe the two-way propagation of long surface waves in constant depth open channel flow in regimes where the competing effects of nonlinearity and dispersion are of the same small order (Boussinesq (**4**), Byatt-Smith (**5**), Long (**7**), Madsen and Mei (**8**), to cite but a few). One reason for this variety of 'Boussinesq' equations is that the zeroth order approximation (the one-dimensional wave equation) made without regard for non-linearity or dispersion can be used to change the form of the first-order terms, added to account for nonlinear and dispersive effects, without changing the overall formal level of approximation (cf. Benjamin, Bona and Mahony (**2**) for similar remarks regarding one-dimensional equations for uni-directional propagation of water waves and Peregrine (**10**) for remarks concerning 'Boussinesq'-type equations). For example, in dimensionless coordinates scaled so that the size of individual terms is shown explicitly, the model derived by Long (**7**) is

$$\eta_t + U_x + \epsilon[U\eta_x - \eta\eta_t + \tfrac{1}{6}\eta_{ttt}] = O(\epsilon^2),$$

$$U_t + \eta_x + \epsilon[-U\eta_t + \tfrac{1}{2}\eta_{xtt}] = O(\epsilon^2),$$

as $\epsilon \downarrow 0$, where η is the wave height above the undisturbed depth (taken to be 1), U the horizontal velocity at the bottom of the channel and ϵ typifies the ratio of the dimensional wave height to the water depth. For this model, the approximate identities

$$\eta_t + U_x = O(\epsilon) \quad \text{and} \quad U_t + \eta_x = O(\epsilon)$$

as $\epsilon \downarrow 0$ are formally valid. These may be used systematically to alter the nonlinear and dispersive terms without changing the basic $O(\epsilon^2)$ formal approximation.

Another reason for the variety of model equations lies in the large choice of relevant dependent variables available to describe the basic physical situation (cf. Peregrine (10)). In Long's model above for example, the dependent variable U can be replaced by some other velocity, such as the velocity u at a height y above the channel bed,

$$u = U - \frac{\epsilon}{2} y^2 U_{xx} + O(\epsilon^2)$$

as $\epsilon \downarrow 0$ where $0 \leqslant y \leqslant 1 + \epsilon\eta$, the velocity at the free surface

$$u = U - \frac{\epsilon}{2}(1+\epsilon\eta)^2 U_{xx} + O(\epsilon^2)$$

or a velocity based on the flow of kinetic energy

$$\left(\int_0^{1+\epsilon\eta} u^3\,dy\right)^{\frac{1}{3}} = U + \epsilon(\tfrac{1}{3}U\eta - \tfrac{1}{6}U_{xx}) + O(\epsilon^2)$$

as $\epsilon \downarrow 0$. These do not exhaust the possibilities. Thus from Long's model many different models may be devised, all with the same formal error $O(\epsilon^2)$ as $\epsilon \downarrow 0$. The particular choice $y = 1/\sqrt{3}$ above leads to Boussinesq's original formulation

$$\eta_t + u_x + \epsilon(u\eta)_x = O(\epsilon^2),$$
$$u_t + \eta_x + \epsilon u u_x - \tfrac{1}{3}\epsilon u_{xxt} = O(\epsilon^2)$$

in which the dispersion term $-\frac{1}{3}\epsilon u_{xxt}$ is appended to the momentum equation in the usual shallow water equations.

This degree of freedom in the choice of the model system may be used to ensure the system has the correct sort of qualitative mathematical properties for the modelling job at hand. As pointed out by Benjamin *et al.* (2) and by Benjamin (1) for the case of uni-directional long-wave models, this is not just an academic exercise. For while it is true that a physical long flat initial wave profile is not expected to come upon subsequent singularities, such as breaking or the channel bed running dry, the corresponding property for the model system should be proved rather than assumed. In fact, such qualitative information about a model system is one of the important ways available for judging the model.

In designing a model equation for long waves it appears desirable that the model should not respond too strongly to shorter wave components. For instance, if an approximation to the solution of a model system is attempted by an explicit finite-difference scheme, the scheme may present stability problems if the model responds violently to waves as short as the grid size being used. (For references to work on numerical solutions of Boussinesq-type equations together with a wide-ranging discussion of model equations for water waves, see Peregrine (9).) A heuristic way of assuring a suitably weak response to short waves is to build the model system in such a way that the dispersion relation determined by the linear terms of the system is particularly well-behaved for short-wave components. The dispersion relation for Boussinesq's model is

$$\omega^2 = \frac{k^2}{1+\frac{1}{3}\epsilon k^2} \quad \text{with} \quad \frac{d\omega}{dk} = \pm\frac{1}{(1+\frac{1}{3}\epsilon k^2)^{\frac{3}{2}}},$$

and it follows that large wave-number components simply lead to standing oscillations of finite frequency. It happens that the dispersion relations for the model proposed by Byatt-Smith (5) and for the model subsequently studied in this paper are identical with the dispersion relation for Boussinesq's model. The dispersion relation for the model used by Madsen and Mei (8) is

$$\omega^2 = \frac{k^2(1+\frac{1}{6}\epsilon k^2)}{1+\frac{1}{2}\epsilon k^2} \quad \text{with} \quad \frac{d\omega}{dk} = \pm\frac{1+\frac{1}{3}\epsilon k^2+\frac{1}{12}\epsilon^2 k^4}{(1+\frac{1}{6}\epsilon k^2)^{\frac{1}{2}}(1+\frac{1}{2}\epsilon k^2)^{\frac{3}{2}}}$$

and large wave-number components propagate without distortion with group and phase velocity $3^{-\frac{1}{2}}$, which suggests that the long term effect of order ϵ non-linearities might be to cause discontinuities on a time scale of order ϵ^{-1}. The dispersion relation for Long's model has two branches:

$$\omega^2 = \frac{3}{\epsilon}[1+\tfrac{1}{2}\epsilon k^2 \pm \{(1+\tfrac{1}{2}\epsilon k^2)^2 - \tfrac{2}{3}\epsilon k^2\}^{\frac{1}{2}}]$$

and on the upper branch large wave-number components have large phase and group velocities making it conceivable that short waves in far-separated places could come together on a time scale of order 1 causing a discontinuity (cf. the critique of the linearized Korteweg de Vries equation in the paper of Benjamin *et al.* (2)).

Another useful property that a model system might have is an 'energy' conservation law, or more precisely, a functional which, when applied to solutions of the system, is positive and is conserved as time evolves. Such a conservation law has many useful implications for both formal and rigorous investigations of the model. For example, while it is relatively easy to conclude local existence (existence over a sufficiently small time interval) for many of the choices of equations for the two-way propagation of long waves, the more interesting questions concern global existence results, together with uniqueness and continuous dependence on the prescribed data. In order to establish existence and other qualitative aspects of global solutions, some sort of *a priori* knowledge is generally required, an energy invariant being one of the simplest examples of such information.

The two constraints discussed do not uniquely determine the model, but the model system

$$\left.\begin{aligned} \eta_t + u_x + (u\eta)_x - \tfrac{1}{3}\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x - \tfrac{1}{3}(u_t+\eta_x)_{xx} &= 0, \\ \eta(x,0) = f(x), \quad u(x,0) &= g(x), \end{aligned}\right\} \qquad (1)$$

has the added features of retaining the non-linear terms of shallow water theory and of having a velocity u with a simple physical interpretation (namely velocity at the free surface level). Here the variables are dimensionless, but unscaled since in the rigorous theory the small parameter plays no explicit rôle.

The main accomplishment of the following sections is to prove that the model (1) has global solutions corresponding to appropriately restricted initial data. It is further shown, for such data, that the wave height is bounded, solely in terms of the data, for *all* time. Section 2 is devoted to definitions and a few elementary lemmas. In section 3

Table of function spaces

Space	Defining properties of the space	Norm
L_2	$g\colon \mathbb{R}\to\mathbb{R}$ that are square integrable	$\|g\|^2 = \int_{-\infty}^{\infty} g(x)^2\,dx$
H^s	L_2-functions whose derivatives up to order s also lie in L_2	$\|g\|_s^2 = \sum_{j=0}^{s} \|g^{(j)}\|^2$
C_b^s	$g\colon \mathbb{R}\to\mathbb{R}$ that have s continuous and bounded derivatives	$\|g\|_{C_b^s} = \sum_{j=0}^{s} \sup_{x\in\mathbb{R}} \|g^{(j)}(x)\|$
$\mathscr{L}_T$	$u\colon R\times[0,T]\to\mathbb{R}$ such that $u(\cdot,t)\in L_2$ for each t in $[0,T]$ and such that $t\mapsto u(\cdot,t)$ is continuous from $[0,T]$ to L_2	$\|u\|_{\mathscr{L}_T} = \sup_{0\leqslant t\leqslant T} \|u(\cdot,t)\|$
$\mathscr{H}_T$	$\eta\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ such that $\eta(\cdot,t)\in H^1$ for each t in $[0,T]$ and such that $t\mapsto\eta(\cdot,t)$ is continuous from $[0,T]$ to H^1	$\|\eta\|_{\mathscr{H}_T} = \sup_{0\leqslant t\leqslant T} \|\eta(\cdot,t)\|_1$
$\mathscr{H}_T^s$	$\eta\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ as above with H^s replacing H^1. (Thus $\mathscr{H}_T^1=\mathscr{H}_T$ and $\mathscr{H}_T^0=\mathscr{L}_T$)	$\|\eta\|_{\mathscr{H}_T^s} = \sup_{0\leqslant t\leqslant T} \|\eta(\cdot,t)\|_s$
$\mathscr{H}_T^{s,r}$	$\eta\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ such that $\partial_t^j\eta$ is in $\mathscr{H}_T^s$ for $0\leqslant j\leqslant r$	$\|\eta\|_{\mathscr{H}_T^{s,r}} = \sum_{j=0}^{r} \|\partial_t^j\eta\|_{\mathscr{H}_T^s}$
$\mathscr{C}_T^s$	$u\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ such that $u(\cdot,t)\in C_b^s$ for each t in $[0,T]$ and such that $t\mapsto u(\cdot,t)$ is continuous from $[0,T]$ to C_b^s	$\|u\|_{\mathscr{C}_T^s} = \sup_{0\leqslant t\leqslant T} \|u(\cdot,t)\|_{C_b^s}$
$\mathscr{C}_T^{s,r}$	$u\colon \mathbb{R}\times[0,T]\to\mathbb{R}$ such that $\partial_t^j u\in\mathscr{C}_T^s$ for $0\leqslant j\leqslant r$. (Thus $\mathscr{C}_T^{s,0}=\mathscr{C}_T^s$)	$\|u\|_{\mathscr{C}_T^{s,r}} = \sum_{j=0}^{r} \|\partial_t^j u(\cdot,t)\|_{\mathscr{C}_T^s}$

the question of uniqueness is settled. Section 4 turns to existence theory of weak solutions over small time intervals while in section 5 the smoothness properties of weak solutions are investigated. Global existence is covered in section 6 and the question of continuous dependence of solutions on the initial data is addressed in section 7. Section 8 touches briefly on further 'Boussinesq' equations to which the theory presented in the earlier sections is applicable and closes with a few remarks concerning other model equations in the literature.

2. *Definitions and elementary results.* Let $\mathbb{R}$ denote the real line. To describe the initial conditions imposed upon the model (1) use will be made of the Hilbert space $L_2 = L_2(\mathbb{R})$, the Sobolev spaces $H^s = H^s(\mathbb{R})$ and the Banach spaces $C_b^s = C_b^s(\mathbb{R})$. To describe the evolution in time of the waves three analogous families of Banach spaces will be used, namely $\mathscr{L}_T$, $\mathscr{H}_T^s$ and $\mathscr{C}_T^{s,r}$. For future reference, these spaces are defined above.

The notation $\|u\|$, $\|u\|_1$, etc., will be used subsequently instead of $\|u(\cdot,t)\|$, $\|u(\cdot,t)\|_1$, etc. There should be no confusion associated with this shorthand since norms are always applied either in the spatial variable or in both the spatial and temporal variables.

An inequality that will be used several times in the sequel applies to functions f in H^1.

Such functions are necessarily in C_b and moreover $\|f\|_{C_b} \leq 2^{-\frac{1}{2}}\|f\|_1$ (a variant of this result is proved as the first step in Theorem 3). From this there follows instantly that if $\eta \in \mathscr{H}_T$, then $\eta \in \mathscr{C}_T$ and

$$\|\eta\|_{\mathscr{C}_T} \leq 2^{-\frac{1}{2}}\|\eta\|_{\mathscr{H}_T}.$$

The first step in the analysis of the initial-value problem (1) is to establish uniqueness and existence over a small time interval. This is most easily accomplished by recasting (1) into a coupled system of integral equations. Write the system (1) in the form

$$(1-\tfrac{1}{3}\partial_x^2)\,\eta_t = -\,[u(1+\eta)]_x,$$
$$(1-\tfrac{1}{3}\partial_x^2)\,u_t = -\,\eta_x - uu_x + \tfrac{1}{3}\eta_{xxx},$$

and invert the operator $(1-\frac{1}{3}\partial_x^2)$ subject to zero boundary conditions at $\pm\infty$. If a formal integration by parts is performed and the resulting system integrated over $[0,t]$ there appears

$$\left.\begin{aligned} \eta(x,t) &= f(x) + \int_0^t K*\{u(1+\eta)\}\,d\tau \\ u(x,t) &= g(x) - \int_0^t \eta_x\,d\tau + \frac{1}{2}\int_0^t \{K*u^2\}\,d\tau \end{aligned}\right\} \tag{2}$$

where $*$ denotes convolution over $\mathbb{R}$, that is,

$$K*w = \int_{-\infty}^{\infty} K(x-y)\,w(y,\tau)\,dy,$$

and

$$K(z) = \tfrac{3}{2}\,\mathrm{sgn}\,(z)\exp\,(-\sqrt{3}\,|z|). \tag{3}$$

It is essential to understand the way in which convolution with K maps various function classes.

Proposition 1. *Convolution with K has the following properties:*

(i) *if $v \in L_2$ then $K*v \in H^1$ and $\|K*v\|_1 \leq 3\|v\|$,*

(ii) *if $v, w \in L_2$ then $K*(vw) \in L_2$ and $\|K*(vw)\| \leq 2\|v\|\,\|w\|$,*

(iii) *if $v \in C_b^s$ then $K*v \in C_b^{s+1}$ and $\|K*v\|_{C_b^{s+1}} \leq M_s\|v\|_{C_b^s}$ where the constant M_s depends only on s,*

(iv) *if $v \in C_b$ and $v \to 0$ as $x \to \pm\infty$, then $K*v \to 0$ as $x \to \pm\infty$.*

Proof. For (i) utilize the fact that convolution reduces to multiplication in the Fourier transformed variables and that

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(k)|^2\,dk, \quad \|f\|_1^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} (1+k^2)\,|\hat{f}(k)|^2\,dk,$$

where

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

denotes the Fourier transform of f. Thus

$$\begin{aligned} \|K*v\|_1^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (1+k^2)\left|\frac{3ik}{3+k^2}\hat{v}(k)\right|^2 dk \\ &\leq \sup_{k\in\mathbb{R}} \frac{9k^2(1+k^2)}{(3+k^2)^2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{v}(k)|^2\,dk = 9\|v\|^2. \end{aligned}$$

(ii) follows from the elementary convolution inequality

$$\|f*g\| \leqslant \|f\|\,\|g\|_{L_1},$$

the Cauchy–Schwarz inequality and the fact that $\|K\| = 3^{\frac{3}{4}}/2 < 2$. The bound (iii) on $\phi = K*v$ follows from the inequality

$$\|f*g\|_{C_b} \leqslant \|f\|_{C_b}\|g\|_{L_1}$$

used in conjunction with the formulae

$$\phi' = K'*v+3v \quad \text{and} \quad \phi^{(k+2)} = 3\phi^{(k)}+3v^{(k+1)},$$

for $k = 0, 1, \ldots$. Finally (iv) may be established by using the inequality

$$|(K*v)(x)| \leqslant e^{-\sqrt{3}|x|}\int_{-|\xi|}^{|\xi|} \tfrac{3}{2}e^{\sqrt{3}|z|}\,|v(z)|\,dz+\sqrt{3}\sup_{|z|\geqslant|\xi|}|v(z)|$$

valid for $|x| \geqslant |\xi|$. The second term on the right may be made as small as desired by choosing $|\xi|$ large. For fixed ξ, the first term on the right is then asymptotically null as $|x| \to +\infty$, and the result follows.

The following result is a generalization of the lemma in appendix 2 of Benjamin *et al.*(2).

PROPOSITION 2. *Let s be a non-negative integer. Then*

(i) *if $v \in C_b^{s+1}$ and $v(x) \to 0$ as $|x| \to +\infty$ (e.g. if $v \in L_p$ for some p in $(0,\infty)$) then $\partial^j v(x) \to 0$ as $|x| \to +\infty$ for $0 \leqslant j \leqslant s$,*

(ii) *if $v \in H^{s+1}$ then $\partial^j v(x) \to 0$ as $|x| \to +\infty$ for $0 \leqslant j \leqslant s$,*

(iii) *if $u \in \mathscr{C}_T^{s+1}$ and $u(x,t) \to 0$ as $|x| \to +\infty$ (e.g. if $u \in \mathscr{L}_T$) then $\partial_x^j u(x,t) \to 0$ as $|x| \to +\infty$ for $0 \leqslant j \leqslant s$ uniformly for t in $[0,T]$,*

(iv) *if $u \in \mathscr{H}_T^{s+1}$ then $\partial_x^j u(x,t) \to 0$ as $|x| \to +\infty$ for $0 \leqslant j \leqslant s$ uniformly for t in $[0,T]$.*

The proof of (i) is a straightforward adaptation of the last quoted reference. (ii) follows from the Riemann–Lebesgue lemma. The rest is a simple extension of (i) and (ii).

3. *Uniqueness.*

THEOREM 1. *There is at most one solution pair η, u on $\mathbb{R}\times[0,T]$ to the integral equations (2) such that $\eta \in \mathscr{H}_T^1$ and $u \in \mathscr{L}_T$.*

Remark. In order that such an η, u exist, it must be the case that $f \in H^1$ and $g \in L_2$. This latter fact follows from the integral equation by taking the limit $t\downarrow 0$, but is without crucial significance in the proof of uniqueness.

Proof. Suppose η_1, u_1 and η_2, u_2 are two solutions of (2). Let $\gamma = \eta_1 - \eta_2$ and $w = u_1 - u_2$. Then γ, w satisfy the following coupled system of integral equations:

$$\gamma = \int_0^t K*(w(1+\eta_1)+u_2\gamma)\,d\tau,$$

$$w = -\int_0^t \gamma_x \, d\tau + \frac{1}{2}\int_0^t K*((u_1+u_2)w)\,d\tau.$$

The properties (i) and (ii) of convolution with K expounded in Proposition 1 can be used, along with the estimates

$$\left.\begin{aligned} \|w\eta_1\| &\leqslant \|w\| \sup_{x\in\mathbb{R}} |\eta_1(x)| \leqslant \|w\|\,\|\eta_1\|_1, \\ \|u_2\gamma\| &\leqslant \|u_2\|\,\|\gamma\|_1, \end{aligned}\right\} \tag{4}$$

to derive the integral inequalities

$$\left.\begin{aligned} \|\gamma\|_1 &\leqslant 3\int_0^t (\|w\| + \|w\|\,\|\eta_1\|_1 + \|u_2\|\,\|\gamma\|_1)\,d\tau, \\ \|w\| &\leqslant \int_0^t (\|\gamma\|_1 + (\|u_1\| + \|u_2\|)\,\|w\|)\,d\tau. \end{aligned}\right\} \tag{5}$$

Let
$$\begin{aligned} C &= \sup_{0\leqslant t\leqslant T} 3(1+\|\eta_1\|_1 + \|u_1\| + \|u_2\|) \\ &\leqslant 3(1+\|\eta_1\|_{\mathscr{H}_T} + \|u_1\|_{\mathscr{L}_T} + \|u_2\|_{\mathscr{L}_T}). \end{aligned}$$

Because of the regularity assumptions made on η, u_1 and u_2, C is finite. Upon summing the two inequalities in (5) and making the obvious estimates, there appears

$$\|w\| + \|\gamma\|_1 \leqslant C\int_0^t (\|w\| + \|\gamma\|_1)\,d\tau, \tag{6}$$

valid for t in $[0,T]$. Gronwall's lemma implies immediately that $\|w\| + \|\gamma\|_1 \equiv 0$ for t in $[0,T]$ and uniqueness is established.

An analogous uniqueness proof can be given for solutions of (2) in $\mathscr{C}_T^1 \times \mathscr{C}_T$. The estimates are the same except that (iii) of Proposition 1 and the inequalities

$$\|uv\|_{\mathscr{C}_T^s} \leqslant K_s \|u\|_{\mathscr{C}_T^s} \|v\|_{\mathscr{C}_T^s}, \tag{7}$$

are used instead of (i)–(ii) of Proposition 1 and the inequalities (4). The inequality (7) is valid for $u, v \in \mathscr{C}_T^s$ where K_s is a constant depending only on s.

4. *Existence for small time intervals.* It is now established that for given initial data there exists a time interval $[0,T]$ and a pair η, u defined at least on $[0,T]$ which is a solution of the integral equations (2) satisfying the hypotheses of Theorem 1.

PROPOSITION 3. *Let $f\in H^1$ and $g\in L_2$ and let $b = \|f\|_1 + \|g\|$. Then there exists $T = T(b) > 0$ such that the integral equations* (2) *have a solution (η, u) with $\eta\in\mathscr{H}_T$ and $u\in\mathscr{L}_T$.*

Proof. Write the pair of integral equations (2) symbolically as $(\eta,u) = \mathbf{A}(\eta,u)$. Looked at in this way, $\mathbf{A}$ is seen to be a mapping of the product Banach space $E = \mathscr{H}_T \times \mathscr{L}_T$ (with the product norm $\|(\eta,u)\|_E = \|\eta\|_{\mathscr{H}_T} + \|u\|_{\mathscr{L}_T}$) into itself, by appeal to the properties (i) and (ii) of proposition 1 and the fact that if w is in $\mathscr{H}_T$ or $\mathscr{L}_T$ then $\int_0^t w(x,\tau)\,d\tau$ is as well (cf. Hardy, Littlewood & Polya (6), p. 148 or Yosida (11), ch. 5, §5, the Bochner integral reducing to an ordinary integral in these circumstances). A solution to the pair of integral equations with the regularity stated in the proposition then corresponds to a fixed point of **A** in E. The contraction mapping principle will be used to show that by taking the parameter T small enough, the mapping **A** has a fixed point in E. To this end the following estimates are established.

Let (η_1, u_1) and (η_2, u_2) be any two elements of the product space. By proceeding as in the proof of theorem 1, there appears

$$\begin{aligned}\|\mathbf{A}(\eta_1,u_1)-\mathbf{A}(\eta_2,u_2)\|_E &= \left\|\int_0^t K*\{(u_1-u_2)(1+\eta_2)+(\eta_1-\eta_2)u_1\}\,d\tau\right\|_{\mathscr{H}_T}\\ &\quad+\left\|\int_0^t [(\eta_2-\eta_1)_x+\tfrac{1}{2}K*\{(u_1+u_2)(u_1-u_2)\}]\,d\tau\right\|_{\mathscr{L}_T}\\ &\leqslant 3T\|u_1-u_2\|_{\mathscr{L}_T}(1+\|\eta_2\|_{\mathscr{H}_T})+3T\|u_1\|_{\mathscr{L}_T}\|\eta_1-\eta_2\|_{\mathscr{H}_T}\\ &\quad+T\|\eta_1-\eta_2\|_{\mathscr{H}_T}+T(\|u_1\|_{\mathscr{L}_T}+\|u_2\|_{\mathscr{L}_T})\|u_1-u_2\|_{\mathscr{L}_T}\\ &\leqslant 3T[1+\|\eta_2\|_{\mathscr{H}_T}+\|u_1\|_{\mathscr{L}_T}+\|u_2\|_{\mathscr{L}_T}]\|(\eta_1,u_1)-(\eta_2,u_2)\|_E.\end{aligned}$$

Suppose now that both (η_1, u_1) and (η_2, u_2) lie in the closed ball B_R of radius R about 0 in E. It then follows that

$$\|\mathbf{A}(\eta_1,u_1)-\mathbf{A}(\eta_2,u_2)\|_E \leqslant T(6R+3)\|(\eta_1,u_1)-(\eta_2,u_2)\|_E.$$

Thus, if $\Theta = T(6R+3)$, then

$$\|\mathbf{A}(\eta_1,u_1)-\mathbf{A}(\eta_2,u_2)\|_E \leqslant \Theta\|(\eta_1,u_1)-(\eta_2,u_2)\|_E. \tag{8}$$

If $\Theta < 1$ and **A** maps B_R into itself, the hypotheses of the contraction mapping theorem would be satisfied and the desired conclusion would then follow. By application of (8) to (η, u) in B_R and $(0, 0)$,

$$\begin{aligned}\|\mathbf{A}(\eta,u)\|_E &= \|\mathbf{A}(\eta,u)-\mathbf{A}(0,0)+(f,g)\|_E\\ &\leqslant \|\mathbf{A}(\eta,u)-\mathbf{A}(0,0)\|_E+\|(f,g)\|_E\\ &\leqslant \Theta\|(\eta,u)\|_E+\|f\|_1+\|g\|\\ &\leqslant \Theta R+b.\end{aligned}$$

Thus if
$$b \leqslant (1-\Theta)R, \tag{9}$$
then **A** maps B_R to itself. It remains only to choose T and R so that $\Theta < 1$ and (9) holds. One choice that satisfies the two criteria is $R = 2b$ and $T = \frac{1}{2}(6R+3)^{-1}$. This yields $\Theta = \frac{1}{2}$ and therefore assures that (9) is valid. The proof of the proposition is now complete.

A completely analogous proof can be made in the case where the initial data f and g are known only to be smooth and bounded functions, not necessarily tending to zero at infinity. If $f \in C_b^1$ and $g \in C_b$, the operator **A** in the last proof is viewed as a mapping of the space $\mathscr{C}_T^1 \times \mathscr{C}_T$ and (iii) of proposition 1 and the inequality (7) above are used instead of (i) and (ii) of proposition 1. Indeed if more generally $f \in C_b^{s+1}$ and $g \in C_b^s$ then by considering **A** as a mapping of the space $\mathscr{C}_T^{s+1} \times \mathscr{C}_T^s$ one can establish existence of a fixed point of $\mathbf{A}$ in $\mathscr{C}_T^{s+1} \times \mathscr{C}_T^s$ (i.e. a solution with the optimal regularity) provided T is sufficiently small. These remarks are summarized as a corollary to the proof of the last proposition.

COROLLARY 1. *Let $s \geqslant 0$ and suppose $f \in C_b^{s+1}$ and $g \in C_b^s$. Let $b' = \|f\|_{C_b^{s+1}} + \|g\|_{C_b^s}$. Then there is a $T = T(s, b')$ such that the integral equations* (2) *have a solution in $\mathscr{C}_T^{s+1} \times \mathscr{C}_T^s$.*

Similar results hold good for data in higher order Sobolev spaces. Again the same contractive mapping argument applies to the operator $\mathbf{A}$ viewed as a mapping in an appropriate product space.

COROLLARY 2. *Let s be a non-negative integer and suppose $f \in H^{s+1}$* and $g \in H^s$. *Let $b'' = \|g\|_s + \|f\|_{s+1}$. Then there exists $T = T(s, b'')$ such that the integral equations* (2) *have a solution (η, u) in $\mathscr{H}_T^{s+1} \times \mathscr{H}_T^s$.*

5. *Regularity of solutions.* A more subtle point than that dealt with in the last two corollaries is now examined. The question under consideration is as follows. Suppose that (η, u) is the solution of (2) in $\mathscr{H}_T \times \mathscr{L}_T$ corresponding to the initial data (f, g) and suppose further that $f \in H^1 \cap C_b^{s+1}$ and $g \in L_2 \cap C_b^s$. Can it be inferred that $(\eta, u) \in \mathscr{C}_T^{s+1} \times \mathscr{C}_T^s$? For T_0 small enough, Proposition 3 and Corollary 1 assure that $\mathbf{A}$ is contractive in suitable balls about zero in both $\mathscr{H}_{T_0} \times \mathscr{L}_{T_0}$ and $\mathscr{C}_{T_0}^{s+1} \times \mathscr{C}_{T_0}^s$. It follows therefore that at least on the time interval $[0, T_0]$ there is a solution which lies in $(\mathscr{H}_{T_0} \cap \mathscr{C}_{T_0}^{s+1}) \times (\mathscr{L}_{T_0} \cap \mathscr{C}_{T_0}^s)$. This solution must coincide with (η, u) by Theorem 1 and so the conjectured regularity obtains at least on $[0, T_0]$, but as yet there is no guarantee that T_0 can be taken as large as T.

The argument above may be iterated. More precisely, the contraction mapping argument may be applied again to (2) with initial data $f_1(x) = \eta(x, T_0)$ and $g_1(x) = u(x, T_0)$. The conclusion is that the additional regularity for (η, u) will continue to hold on the time interval $[T_0, T_1]$ where $T_1 > T_0$. Continuing this line of reasoning leads to a sequence $\{T_j\}_{j=1}^{\infty}$, with $T_{j+1} > T_j$, of times over which the additional regularity of (η, u) may be imputed. The problem is that $T_{j+1} - T_j$ may shrink to zero very quickly as j grows. According to Proposition 3 and Corollary 1 the time step $T_{j+1} - T_j$ depends on $b_j = \|f_j\|_1 + \|g_j\|$ and on $b_j' = \|f_j\|_{C_b^{s+1}} + \|g_j\|_{C_b^s}$, where $f_j = \eta(\cdot, T_j)$ and $g_j = u(\cdot, T_j)$. The numbers b_j are known to be bounded so long as $T_j \leqslant T$ since $(\eta, u) \in \mathscr{H}_T \times \mathscr{L}_T$. If there were available *a priori* bounds on $\|\eta\|_{\mathscr{C}_T^{s+1}}$ and $\|u\|_{\mathscr{C}_T^s}$, then b_j' would be bounded independent of j and a positive lower bound could be inferred on $T_{j+1} - T_j$. It then follows that the above outlined contractive mapping argument would establish the conjectured regularity on $[0, T]$, in a finite number of steps. The derivation of such bounds will therefore prove most of the following result.

THEOREM 2. *Let $(\eta, u) \in \mathscr{H}_T \times \mathscr{L}_T$ be the solution of* (2) *on $[0, T]$ corresponding to the initial data (f, g). Suppose $f \in H^1 \cap C_b^{s+1}$ and $g \in L_2 \cap C_b^s$ for some integer $s \geqslant 0$. Then (η, u) also lies in $\mathscr{C}_T^{s+1, \infty} \times \mathscr{C}_T^{s, \infty}$.*

Proof. In the first place, to show $(\eta, u) \in \mathscr{C}_T^{s+1} \times \mathscr{C}_T^s$ it suffices to derive a uniform bound on $\|\eta\|_{C_b^{s+1}}$ and $\|u\|_{C_b^s}$ for t in $[0, T]$, as has already been remarked. The proof of these bounds may be made by induction on s, the case $s = 0$ being in fact entirely typical of the general calculation.

Therefore assume first that $f \in H^1 \cap C_b^1$ and $g \in L_2 \cap C_b$. Differentiation of the integral expression for η in (2) leads to

$$\eta_x = f' + 3\int_0^t u(1+\eta)\,d\tau - 3\int_0^t L * \{u(1+\eta)\}\,d\tau, \tag{10}$$

where

$$L(z) = \frac{\sqrt{3}}{2}\exp(-\sqrt{3}\,|z|). \tag{11}$$

Since $\eta \in \mathscr{H}_T$, $\eta \in \mathscr{C}_T$ so η is bounded over $\mathbb{R} \times [0, T]$. The relation (10) implies therefore that

$$\|\eta_x\|_{C_b} \leqslant \|f'\|_{C_b} + 3T\|u\|_{C_b}(1 + \|\eta\|_{\mathscr{H}_T}) + 3T\|L * \{u(1+\eta)\}\|_{C_b}. \tag{12}$$

The Schwarz inequality gives

$$|L * \{u(1+\eta)\}| \leqslant \|L\|\,\|u(1+\eta)\| \leqslant \frac{3^{\frac{1}{4}}}{2}1(+\|\eta\|_{\mathscr{H}_T})\,\|u\|_{\mathscr{L}_T}$$

valid for any x in $\mathbb{R}$ and t in $[0, T]$. Letting C_i, $i = 1, 2, \ldots$, denote various t-independent constants, (12) is more plainly written as

$$\|\eta_x\|_{C_b} \leqslant C_1 + C_2\|u\|_{C_b}. \tag{13}$$

Estimating in the second equation in (2),

$$\|u\|_{C_b} \leqslant \|g\|_{C_b} + \int_0^t \|\eta_x\|_{C_b}\,d\tau + \frac{1}{2}\int_0^t \|K\|\,\|u^2\|\,d\tau.$$

But $\|u^2\| \leqslant \|u\|_{C_b}\|u\|_{\mathscr{L}_T}$ as long as u is bounded and $t \leqslant T$, so in sum

$$\|u\|_{C_b} \leqslant C_3 + \int_0^t \|\eta_x\|_{C_b}\,d\tau + C_4\int_0^t \|u\|_{C_b}\,d\tau. \tag{14}$$

Using (13) in (14), there appears

$$\|u\|_{C_b} \leqslant C_5 + C_6\int_0^t \|u\|_{C_b}\,d\tau,$$

where C_5 and C_6 are constants independent of $t \leqslant T$, but dependent on $f, g, \|u\|_{\mathscr{L}_T}, \|\eta\|_{\mathscr{H}_T}$ and T. Gronwall's lemma now allows the conclusion

$$\|u\|_{C_b} \leqslant C_5\,e^{C_6 t} \leqslant C_5\,e^{C_6 T}$$

valid so long as t is in $[0, T]$. This is the required bound on $\|u\|_{C_b}$. A bound for $\|\eta\|_{C_b}$ on $[0, T]$ now follows from (13).

The situation for general s now follows inductively. Suppose the theorem is valid for $s \leqslant k$ where $k \geqslant 0$. The result for $s = k+1$ may be established as follows. Differentiating the integral equation (2) and using the properties of the kernels K and L, there appears

$$\left.\begin{aligned} &\eta_{(k+2)} = f^{(k+2)} + 3(\eta - f)_{(k)} + 3\int_0^t \{u(1+\eta)\}\,d\tau, \\ &\text{for } k = 0, 1, 2, \ldots, \\ &u_{(k+1)} = g^{(k+1)} + 3(u-g)_{(k-1)} + \int_0^t \{3\eta_{(k)} - \eta_{(k+2)} + \tfrac{3}{2}(u^2)_{(k)}\}\,d\tau, \\ &\text{for } k = 1, 2, \ldots, \text{ and} \\ &u_x = g' - \int_0^t \eta_{xx}\,d\tau + \frac{3}{2}\int_0^t \{u^2 - L*(u^2)\}\,d\tau. \end{aligned}\right\} \quad (15)$$

Using Leibnitz's rule to perform the differentiations of products, the induction hypothesis, the hypothesis on the initial data and, in the case $k = 0$, the mapping properties of L, the supremum over $\mathbb{R}$ of most of the terms on the right-hand sides can be bounded outright. There remains

$$\left.\begin{aligned} \|\eta_{(k+1)}\|_{C_b} &\leqslant C_7 + T\|u_{(k)}\|_{C_b}(1 + \|\eta\|_{\mathscr{H}_T}) \leqslant C_7 + C_8\|u_{(k)}\|_{C_b}, \\ \|u_{(k)}\|_{C_b} &\leqslant C_9 + \int_0^t \|\eta_{(k+1)}\|_{C_b}\,d\tau. \end{aligned}\right\} \quad (16)$$

A further application of Gronwall's lemma to (16) gives the desired bounds for $s = k$ and completes the induction.

It is now concluded that $(\eta, u) \in \mathscr{C}_T^{s+1} \times \mathscr{C}_T^s$. The differentiability in the temporal variable is easier to establish. Since, for example,

$$\eta = f + \int_0^t K * \{u(1+\eta)\}\,d\tau,$$

the fundamental theorem of calculus guarantees η_t exists and that

$$\eta_t = K * \{u(1+\eta)\}. \quad (17)$$

Since $u(1+\eta) \in \mathscr{C}_T^s$, $\eta_t \in \mathscr{C}_T^{s+1}$ from (iii) of Proposition 1. Similarly

$$u_t = -\eta_x + \tfrac{1}{2}K * u^2. \quad (18)$$

Again because $\eta \in \mathscr{C}_T^{s+1}$ and $u^2 \in \mathscr{C}_T^s$, $u_t \in \mathscr{C}_T^s$. Differentiating (17) and (18) with respect to t, one may show inductively that $\partial_t^{m+1}\eta$ and $\partial_t^m u$ lie in $\mathscr{C}_T^{s+1}$ and $\mathscr{C}_T^s$ respectively, for each m. This completes the proof of the theorem.

Remarks. The solutions may in fact be confirmed to be *analytic* in their t variation. The proof proceeds by majorizing the formal power series expansion of η and u in their temporal variable, but this aspect is passed over here.

If $f \in C_b^3$ and $g \in C_b^2$, then it follows from the simple properties of the kernel K (cf. (10) and Benjamin *et al.* (2), p. 60) that (η, u) is a classical solution to the initial-value problem (1) (i.e. all the derivatives expressed in the differential equation exist and are continuous).

Theorem 2 may be based on the existence of a solution pair (η, u) in $\mathscr{C}_T^1 \times \mathscr{C}_T$ (rather than $\mathscr{H}_T \times \mathscr{L}_T$) as well.

A closely analogous argument can be used to establish additional L_2 regularity of solutions of (2) in $\mathscr{H}_T \times \mathscr{L}_T$ corresponding to the initial data belonging to more restrictive Sobolev classes. This result is stated as a corollary to the proof of Theorem 2.

COROLLARY 3. *Let $(\eta, u) \in \mathscr{H}_T \times \mathscr{L}_T$ be the solution of (2) on $[0, T]$ corresponding to the initial data (f, g). Suppose $f \in H^{s+1}$ and $g \in H^s$ for some integer $s \geqslant 1$. Then*

$$(\eta, u) \in \mathscr{H}_T^{s+1,\infty} \times \mathscr{H}_T^{s,\infty}.$$

6. *Global existence.* The stage is now set for an attack on the global existence problem for the initial-value problem under study. According to the previously derived results, for any given time interval $[0, T]$ existence and regularity on $[0, T]$ will follow as soon as bounds for $\|\eta\|_1$ and $\|u\|$ on $[0, T]$ are obtained. The principal tool is the invariance of the following non-linear functional of the solution pair (η, u):

$$E(t) = E(\eta, u, t) = \int_{-\infty}^{\infty} \{(1+\eta)\, u^2 + \eta^2 + \tfrac{1}{3}\eta_x^2\}\, dx. \tag{19}$$

This invariance is demonstrated in the following lemma.

LEMMA 1. *Let (η, u) be a solution of (2) with $\eta \in \mathscr{H}_T \cap \mathscr{C}_T^3$ and $u \in \mathscr{L}_T \cap \mathscr{C}_T^2$. Let $E(t)$ be defined as in (19). Then $E(t) = E(0)$ for all t in $[0, T]$.*

Proof. Form the approximation

$$E_R(t) = \int_{-R}^{R} \{(1+\eta)\, u^2 + \eta^2 + \tfrac{1}{3}\eta_x^2\}\, dx.$$

Differentiate E_R with respect to t:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_R(t) &= \int_{-R}^{R} \{u u_t(1+\eta) + \tfrac{1}{2}\eta_t u^2 + \eta\eta_t + \tfrac{1}{3}\eta_x\eta_{xt}\}\, dx \\ &= \int_{-R}^{R} \{u_t(u(1+\eta) - \tfrac{1}{3}\eta_{xt}) + \eta_t(\eta + \tfrac{1}{2}u^2 - \tfrac{1}{3}(u_t+\eta_x)_x) + \tfrac{1}{3}(u_t\,\eta_t)_x + \tfrac{1}{3}(\eta_t\eta_x)_x\}\, dx.\end{aligned}$$

The present hypotheses and the remarks at the conclusion of section 5 imply that (η, u) is a classical solution of the initial-value problem (1). Hence the differential equations (1) may be used to eliminate u_t and η_t outside the parentheses in the first and second terms respectively under the integral. When this is done, there appears

$$\frac{1}{2}\frac{d}{dt}E_R(t) = \int_{-R}^{R} \partial_x\{-(\eta + \tfrac{1}{2}u^2 - \tfrac{1}{3}(u_t+\eta_x)_x)\,(u(1+\eta) - \tfrac{1}{3}\eta_{xt}) + \tfrac{1}{3}u_t\eta_t + \tfrac{1}{3}\eta_t\eta_x\}\,dx. \tag{20}$$

Now because $\eta \in \mathscr{H}_T \cap \mathscr{C}_T^3$ and $u \in \mathscr{L}_T \cap \mathscr{C}_T^2$ are solutions of (2), it follows from Theorem 2 that $\eta_t \in \mathscr{H}_T \cap \mathscr{C}_T^3$ and $u_t \in \mathscr{L}_T \cap \mathscr{C}_T^2$. It is then a consequence of Proposition 2 that each of η, η_x, η_{xx}, η_t, η_{xt}, u, u_x, u_t converge to zero at $\pm\infty$ uniformly for t in $[0, T]$ as it happens. Integrating (20) over $[0, t]$ for $t \leqslant T$, there appears

$$E_R(t) = E_R(0) - 2\int_0^t [(\eta + \tfrac{1}{2}u^2 - \tfrac{1}{3}u_{xt} - \tfrac{1}{3}\eta_{xx})\,(u + u\eta - \tfrac{1}{3}\eta_{xt}) - \tfrac{1}{3}\eta_t\eta_x - \tfrac{1}{3}\eta_t u_t]_{x=-R}^{x=R}\, d\tau.$$

Letting $R \to +\infty$ gives simply $E(t) = E(0)$, valid for t in $[0, T]$. This is the required invariance result.

THEOREM 3. *Let $f \in H^1 \cap C_b^3$, $g \in L_2 \cap C_b^2$ and suppose that for all $x \in \mathbb{R}$,*

$$f(x) > -1 \quad and \quad E(0) = \int_{-\infty}^{\infty} \{(1+f)g^2 + f^2 + \tfrac{1}{3}f'^2\}\,dx < \frac{2}{\sqrt{3}}. \tag{21}$$

Then there exists a unique classical solution (η, u) to the initial-value problem (1) which, with its temporal derivatives of all orders, lies in $\mathscr{H}_\infty \times \mathscr{L}_\infty$ and in $\mathscr{C}_T^3 \times \mathscr{C}_T^2$ for each $T > 0$.

Proof. From Proposition 3 and Theorem 2, it is inferred that there is a classical solution pair (η, u) in $\mathscr{H}_T \times \mathscr{L}_T$ for $T = T(b_1)$ where b_1 is an upper bound for $\|f\|_1 + \|g\|$. Since $\eta \in \mathscr{H}_T$, $\|\eta\|_1$ is a continuous function of time so that at least for some small time interval, say $[0, t_0]$, $1 + \eta(x, t) > 0$. Hence on $[0, t_0]$

$$\begin{aligned} \eta^2(x,t) &= \int_{-\infty}^{x} \eta\eta_x\,dx' - \int_{x}^{\infty} \eta\eta_x\,dx' \leqslant \int_{-\infty}^{\infty} |\eta\eta_x|\,dx' \\ &\leqslant \frac{\sqrt{3}}{2}\int_{-\infty}^{\infty} (\eta^2 + \tfrac{1}{3}\eta_x^2)\,dx' \leqslant \frac{\sqrt{3}}{2}E(t) = \frac{\sqrt{3}}{2}E(0) = \alpha^2 < 1 \end{aligned}$$

because of the last lemma and the assumptions (21). Thus on $[0, t_0]$ at least,

$$\sup_{x\in\mathbb{R}} |\eta| \leqslant \alpha < 1. \tag{22}$$

Since $\inf_{x\in\mathbb{R}} \eta(x, t_0) \geqslant -\alpha > -1$, this argument may be repeated with the general conclusion

$$1 + \eta(x,t) \geqslant 1 - \alpha > 0 \tag{23}$$

as long as the solutions continue to exist, independent of $t \geqslant 0$. Hence,

$$\|u\|^2 + \|\eta\|_1^2 = \int_{-\infty}^{\infty} (u^2 + \eta^2 + \eta_x^2)\,dx \leqslant \beta E(t) = \beta E(0) = \tfrac{1}{2}b_1^2, \tag{24}$$

where $\beta = \max(3, (1-\alpha)^{-1})$ and the last equality is meant to define b_1. Thus it appears that $\|u\| + \|\eta\|_1$ is bounded above by b_1, independent of t. Hence the contraction mapping argument of Proposition 3 may be repeated, taking $\eta(x, T)$ and $u(x, T)$ for initial data, to extend (η, u) to the interval $[0, 2T(b_1)]$. Continuing in this manner, a global solution to the integral equation is thereby defined, which in consequence of the bound (22), lies in $\mathscr{H}_\infty \times \mathscr{L}_\infty$. The fact that the temporal derivatives of (η, u) lie also in $\mathscr{H}_\infty \times \mathscr{L}_\infty$ now follows by a direct inductive argument based on successively differentiating the integral equation with respect to t. The further regularity is a consequence of Theorem 2.

Remarks. (*a*) Additional regularity of the initial data, in either C_b^s or H^k, yields additional regularity of solutions by way of either Theorem 2 or its corollary, respectively.

(*b*) Existence of global solutions consequent on weaker assumptions on the data can be established. This will follow from the conclusions concerning continuous dependence of solutions on the initial data, to be discussed in the next section.

(*c*) The fact that $\eta \in \mathscr{H}_\infty$ implies that η is bounded for all time with a bound dependent only on the initial conditions. This is a satisfying result when interpreted in the underlying physical situation, for it says that the wave height is bounded, for all time, solely in terms of the initial state of the system.

7. *Continuous dependence on the initial data.* When the initial data satisfies the hypotheses of Theorem 3 it is a simple matter to adapt the uniqueness proof to demonstrate that solutions depend continuously on the initial data. Let (η_1, u_1) and (η_2, u_2) be two solution pairs corresponding to initial data (f_1, g_1) and (f_2, g_2) respectively. Let $f = f_1 - f_2$ and $g = g_1 - g_2$ and as before $\gamma = \eta_1 - \eta_2$ and $w = u_1 - u_2$. Proceeding exactly as in the uniqueness proof, the following integral inequality is derived:

$$\|\gamma\|_1 + \|w\| \leqslant \|f\|_1 + \|g\| + C\int_0^t (\|\gamma\|_1 + \|w\|)\, d\tau,$$

where the constant C can be bounded independently of time in terms of

$$E(\eta_j, u_j) = E(f_j, g_j), \quad j = 1, 2.$$

It then follows that for t in $[0, T]$,

$$\|\gamma\|_1 + \|w\| \leqslant (\|f\|_1 + \|g\|)\, e^{CT}. \tag{25}$$

This shows that on time intervals of finite length the solutions of (1) depend continuously in $\mathscr{H}_T \times \mathscr{L}_T$ on the data measured in $H^1 \times L_2$ (and in fact the continuity is uniform on bounded subsets of $H^1 \times L_2$). The solutions also depend continuously on perturbations in more restrictive function classes. This may be established by an adaptation of the inequalities derived in the proof of Theorem 2, yielding the following result.

THEOREM 4. *Let s be a non-negative integer and let T be a positive number. Solutions of the initial-value problem* (1) *depend continuously in $(\mathscr{H}_T \cap \mathscr{C}_T^{s+1}) \times (\mathscr{L}_T \cap \mathscr{C}_T^{s})$ (respectively $\mathscr{H}_T^{s+1} \times \mathscr{H}_T^{s}$) on perturbations of the initial data in $(H^1 \cap C_b^{s+1}) \times (L_2 \cap C^s)$ (respectively $H^{s+1} \times H^s$). In all cases the continuity is uniform on bounded subsets of the space from which the data is selected.*

This result may be used to establish existence of weak solutions to the initial-value problem (1). Taking for instance data (f, g) in $H^1 \times L_2$ satisfying (21), let $\{(f_k, g_k)\}$ be a sequence of smooth functions, say, satisfying the hypotheses of Theorem 3, which converge to (f, g) in $H^1 \times L_2$. The classical solutions $\{(\eta_k, u_k)\}$ associated with the data $\{(f_k, g_k)\}$ are seen, by use of the result of Theorem 4, to form a Cauchy sequence in $\mathscr{H}_T \times \mathscr{L}_T$. Let (η, u) denote their limit. By direct appeal to the differential equations (1) satisfied by (η_k, u_k), or by use of the integral equation, (η, u) is seen to be a (distributional) solution to the initial-value problem. Moreover, by taking the limit of $E(\eta_k, u_k)$ as $k \to \infty$, it appears that $E(\eta, u)$ is invariant with time and hence by the arguments of Theorem 3, $(\eta, u) \in \mathscr{H}_\infty \times \mathscr{L}_\infty$. Uniqueness follows from Theorem 1.

PROPOSITION 4. *Corresponding to data* (f, g) *in* $H^1 \times L_2$, *there exists a unique distributional solution* (η, u), *which lies in* $\mathscr{H}_\infty \times \mathscr{L}_\infty$, *to the initial-value problem* (1).

7. *Other model equations.* The analysis in the previous sections goes through, with only minor modifications, for the family of model equations

$$\left.\begin{aligned} \eta_t + u_x + (u\eta)_x - (\tfrac{1}{2}y^2 - \tfrac{1}{6})\,\eta_{xxt} &= 0,\\ u_t + \eta_x + uu_x - (\tfrac{1}{2}y^2 - \tfrac{1}{6})\,u_{xxt} - (y^2 - \tfrac{2}{3})\,\eta_{xxx} &= 0,\end{aligned}\right\} \tag{26}$$

provided only that $y^2 > \frac{2}{3}$. This restriction is needed to guarantee that the associated energy invariant

$$\int_{-\infty}^{\infty} \{(1+\eta)\,u^2 + \eta^2 + (y^2 - \tfrac{2}{3})\,\eta_x^2\}\,dx$$

is composed of positive terms. In the model (26) η is the wave height and u the velocity at a height y above the channel bed.

The single scalar equation

$$\tfrac{1}{3}N_{xxtt} + N_{xx} - N_{tt} + (\tfrac{1}{2}N^2)_{xx} + \left(\left(\int_x^\infty N_t\,dx'\right)^2\right)_{xx} = 0 \tag{27}$$

derived by Byatt-Smith (5) is amenable to an existence theory, by a contraction mapping argument as presented here, over a time interval which is inversely proportional to the wave amplitude. Introduce the auxiliary dependent variable

$$M = \int_x^\infty N_t\,dx'.$$

Then (27) is equivalent, for waves with zero curvature at infinity, to the coupled system of integral equations

$$\left.\begin{aligned} N &= f - \int_0^t M_x\,d\tau,\\ M &= g + \int_0^t K * \{N + \tfrac{1}{2}N^2 + M^2\}\,d\tau,\end{aligned}\right\} \tag{28}$$

where f and g are the initial values of N and M, respectively, and the kernel K is as in (3).

Interesting non-linear and dispersive phenomena do just occur on the time scale of the inverse of the wave amplitude, but it would be preferable if solutions could be controlled on longer time scales in order to be able to follow the evolution of the non-linear and dispersive effects. The analogy of the conserved functional (19) for (28) is

$$F(t) = F(N, M, t) = \int_{-\infty}^{\infty} \{(1 + \tfrac{1}{3}N)\,N^2 + M^2 + \tfrac{1}{3}M_x^2\}\,dx. \tag{29}$$

If it is assumed that $N > -3$ then the results derived in the previous sections may be carried over to prove existence, uniqueness and continuous dependence on initial data for the initial-value problem for (27). Unfortunately the invariance of the functional F cannot provide such a result and we have been unable to discover a class of initial data for which the assumption $N > -3$ can be established *a priori*. However, as N represents the wave height in the model (27), the equations have long since ceased to be valid once the channel runs dry, so there is implied the practical constraint $N > -1$. Even so, there is still not then implied an upper bound on the wave height independent of time.

Another way in which the present theory might be used in investigations of other similar model equations is to view the alternative models as perturbations of the present model equations (1). This would be in the same spirit as the analysis in (3) for models for one-way propagation of waves. For example, reverting to variables which make explicit the ϵ order of the individual terms and defining $\zeta = (1-\frac{1}{2}\epsilon\partial_x^2)\eta$, then the results of this paper give an existence theory for a perturbation of Boussinesq's equations:

$$\zeta_t + u_x + \epsilon(u\zeta)_x = \tfrac{1}{3}\epsilon^2\partial_x\{u(\tilde{L}*\zeta)\},$$

$$u_t + \zeta_x + \epsilon u u_x - \tfrac{1}{3}\epsilon u_{xxt} = 0,$$

where the convolution kernel $\tilde{L}$ has Fourier transform $k^2(1+\frac{1}{3}\epsilon k^2)^{-1}$. One possible way in which an existence theory for the exact Boussinesq equations might be developed would be to use the function pair ζ, u as the starting point for a contraction mapping.

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