Proceedings of the International Congress of Mathematicians
Helsinki, 1978

Model Equations for Waves in Nonlinear Dispersive Systems

Jerry Bona

The present discussion focuses on models for unidirectional wave propagation in which nonlinear, dispersive and dissipative effects are simulated realistically and in such a way that shock formation and other singular behaviour is avoided. While comparatively narrow, the range of discussion nevertheless covers a number of interesting and challenging scientific issues, several of which still remain open. The model equations take one of the following forms:

$$u_t+f(u)_x+Hu_t=0, \tag{1a}$$

or

$$u_t+f(u)_x-Hu_x=0. \tag{1b}$$

Here $u=u(x,t)\colon \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$ and subscripts denote partial differentiation. The function $f\colon \boldsymbol{R}\to\boldsymbol{R}$ represents nonlinear effects in the physical system being modeled, while H is a linear operator representing dispersive effects, and dissipative effects when they are considered. The best known model in the form (1b) is the KdV equation

$$u_t+u_x+uu_x+u_{xxx}=0 \tag{2}$$

which was introduced by Korteweg and de Vries [17] and has in recent years been the subject of prolific study.

The derivations of approximate equations such as (1a, b) differ from one modeling situation to another. Nevertheless, one may appreciate in general why such models arise, at a certain level of approximation, without relying on the details of a particular application (cf. Benjamin [3]; Benjamin, Bona and Mahony [6]).

Upon linearization of the full equations of motion around a rest state, a dispersion relation is determined for plane simple-harmonic waves which relates frequency

$\omega = kc(k)$ to wave-number k. For example, two-dimensional surface waves in water of uniform depth have the dispersion relation

$$c(k) = \omega(k)/k = \{\tanh(k)/k\}^{1/2} \tag{3}$$

in suitably scaled coordinates. Here the phase velocity $c(k)$ has a maximum $c(0)$, corresponding to the limit of large wavelengths. In general the propagation of infinitesimal waves in such a system will be governed by an equation of the form

$$u_t + Mu = 0, \tag{4}$$

where $\widehat{Mu}(k) = ikc(k)\hat{u}(k)$ and the circumflexes denote Fourier transforms. Note that if $c(k)$ has a nonzero imaginary part, then (4) will contain a dissipative term. In many applications attention is restricted to a long-wave régime $k \ll 1$, and it is then justified to approximate $c(k)$ near $k=0$ in order to obtain a more tractable model equation. In the case of water waves as mentioned above, two relevant models are

$$u_t + u_x + \tfrac{1}{6} u_{xxx} = 0 \quad \text{and} \quad u_t + u_x - \tfrac{1}{6} u_{xxt} = 0, \tag{5}$$

which correspond respectively to the dispersion relations $c(k) = 1 - \frac{1}{6}k^2$ and $c(k) = 1/(1 + \frac{1}{6}k^2)$ approximating (3).

If the effects of dispersion due to finite wavelength are ignored and attention is concentrated solely on the effects of nonlinearity, then it is a general attribute of the systems in question that waves propagate along characteristics which depend on the value of the dependent variable: thus

$$\left.\frac{dx}{dt}\right|_{u=\text{constant}} = g(u).$$

This property is equivalent to

$$u_t + f(u)_x = 0, \tag{6}$$

where $f' = g$. If the régime of interest includes in its characterization an assumption that the waves be of small amplitude, then it is justified to use a simpler model obtained by approximating g for small values of its argument. If a linear approximation to g is presumed to be adequate over the range of amplitudes in question, then we take $g(u) = 1 + u$, say, and so obtain

$$u_t + u_x + uu_x = 0. \tag{7}$$

If the nonlinear, dispersive and dissipative effects are of a similar order of smallness, then normally the interaction between these effects is of a yet higher order of smallness. Accordingly, it is warranted simply to add the extra terms appearing respectively in (5) and (7), so to obtain the model equation (1b).

It is noteworthy that nonlinear, dispersive and dissipative effects are generally small corrections to the basic one-way propagator $u_t + u_x = 0$, which is just a factor, governing propagation in the $+x$-direction, of the one-dimensional wave equation.

Thus, if $g(u)=1+g_1(u)$, where $g_1(u)=O(u)$ as $u\to 0$, and if $c(k)=1+c_1(k)$, where typically $c_1(k)=O(k^2)$ as $k\to 0$, then (1b) may be written as

$$u_t+u_x+f_1(u)_x-Lu_x = 0, \tag{8}$$

where $f_1'=g_1$ and $\widehat{Lv}(k)=c_1(k)\hat{v}(k)$. Provided u and k are required to be small, then f_1 and L are of higher order of smallness than the leading terms u_t and u_x. In such a situation, the basic level of approximation will be unaltered if the approximate relation $u_x=-u_t$ is utilized to alter the higher order terms. Hence the equation

$$u_t+u_x+f_1(u)_x+Lu_t = 0 \tag{9}$$

of the form (1a) may be inferred as a model for the unidirectional propagation of small-amplitude long waves. References to specific examples where (1a, b) have been derived as models may be found in the review article of Jeffrey and Kakutani **[15]** and in the two collections of articles on nonlinear waves edited respectively by S. Leibovich and R. Seebass **[19]** and A. Newell **[21]**.

Some care is necessary in the use of the approximations outlined in the preceding discussion. The stated hypotheses are invariably pivotal to the derivation of these equations as rational models, and they should therefore be respected in using the models to gain insight into a physical situation. In the particular case of irrotational surface waves on shallow water, if the independent variables x and t and the dependent variable u, which represents the height of the wave above the undisturbed depth, are scaled so that u and its derivatives are order one, there appear the two model equations

$$u_t+u_x+\varepsilon uu_x+\delta^2 u_{xxx} = 0 \tag{10a}$$

and

$$u_t+u_x+\varepsilon uu_x-\delta^2 u_{xxt} = 0, \tag{10b}$$

corresponding to the different approximations to the dispersion relation given in (5). The parameter ε is a measure of the amplitude of the waves and δ^{-1} is a measure of their wavelength. It is appropriate to assume both ε and δ are small, in this scaling, and that ε and δ^2 are of the same magnitude. In the literature on water waves this is sometimes expressed by demanding that the Stokes number $S=\varepsilon\delta^{-2}$ is order one. The assumption concerning the Stokes number being valid, an order-one change of the dependent variable gives the special case $\varepsilon=\delta^2$. In this scaling it is apparent that the nonlinear and dispersive terms represent small corrections to the basic propagator $u_t+u_x=0$, the smallness of the corrections being measured explicitly by ε. Needless to say, the zero on the right-hand side of (10a, b) represents an approximation to terms that are of order ε^2.

On time scales of order ε^{-1}, the nonlinear and dispersive corrections can accumulate and have an order-one influence on the wave profile. Equally, on time scales of order ε^{-2}, the higher order terms not included in (10a, b) can have an order-one effect on the wave profile. Hence, on such time scales, the models may have become unreliable and predictions made on longer time scales should be viewed with caution.

At first sight it might appear somewhat contradictory that two different models purport to describe the same physical phenomena, as in (10a, b). However, it has been shown by Bona, Pritchard and Scott **[11]** that if u and v denote respectively the solution of (10a) and (10b) corresponding to the same order-one initial wave profile, then $u-v$ is of order ε over the time scale ε^{-1}. Numerical studies indicate that the difference $u-v$ grows linearly to order one on the time scale ε^{-2}. Hence it appears that the two models may indeed simultaneously provide accurate predictions at least over time scales where either model may be expected to apply. A further conclusion is that expediency should govern the choice of (10a) or (10b) in a particular situation where a model for small-amplitude long waves is needed. For instance, the inverse scattering methodology and the infinite collection of polynomial conserved densities for (10a) may both be very useful for various theoretical considerations (cf. Miura **[20]** for an account of the inverse-scattering method, and Whitham **[27]** and Segur **[26]** for some applications). Olver **[22]** has shown that (10b) has only the three polynomial conversed densities corresponding to mass, momentum and energy in the original physical problem that is modeled. This and certain other facts indicate that there is also no inverse scattering formalism for (10b), at least as we presently understand such a formalism. On the other hand, (10b) is far easier to handle numerically than (10a).

In the task of comparing the predictions of the models (10a) or (10b) with experimental data, the most natural mathematical formulation is an initial- and boundary-value problem to be explained presently. In this setting, (10b) appears definitely easier to use and quantitative comparisons using (10b) have been made by Bona, Pritchard and Scott **[12]**. Their work supplements earlier comparisons, made using the pure initial-value problem for (10a), by Zabusky and Galvin **[28]** and by Hammack and Segur **[14]**, which showed good qualitative agreement between measured data and theoretical predictions.

The experimental configuration used in all the above-mentioned comparisons was a rectangular channel containing water with a wavemaker at one end. For the experiments reported by Bona et al. **[12]**, the water was initially at rest when the wavemaker was set in motion. At several stations along the channel, temporal records of the passage of the waves generated by the wavemaker were taken. An appropriate mathematical problem was suggested and analyzed by Bona and Bryant **[9]**. In dimensionless but unscaled coordinates, it is

$$\begin{gathered} u_t+u_x+uu_x-u_{xxt} = 0, \\ u(x,0) \equiv 0, \; u(0,t) = g(t), \end{gathered} \tag{11}$$

for $x, t \geqslant 0$. In a numerical scheme for (11), the function g is a discretization of the measurement of the wave taken closest to the wavemaker. The numerical integration of the model (11) will then predict $u(x_0, t)$ for any station x_0 further from the wavemaker than the station at which g is measured. Such a prediction may then be directly compared with the measurement taken at the station x_0 and the model judged on the basis of the discrepancy between the two.

The Stokes number S for these experiments ranged from 1/4 to 30. Dissipative effects proved to be generally of the same importance as nonlinear and dispersive effects, and accordingly had to be incorporated into the model. The proper form of dissipation term in equations of the types (10a) or (10b) has been derived by Kakutani and Matsuuchi **[16]** and it is non-local in character. For the experiments in question, where most of the energy was manifested at one frequency, an *ad hoc,* local form of dissipation represented by a term $-\nu u_{xx}$ in (11) can be justified. This form was used by Bona et al. **[12]**, although comparisons are desirable between measured data and predictions from the model incorporating the proper form of dissipation. The latter need poses an interesting mathematical and numerical challenge which is presently under study.

The agreement between the experimental and numerically predicted values was quite good. For S in the range [1/4, 10], the difference between the measured and computed wave traces was about 8% of the size of the physical wave. The agreement was less striking as S became large. The difference between the measured and the computed wave was 22% of the size of the measured wave when $S=30$. These comparisons are all respective to the L_∞ norm. Even for such large values of S, some of the qualitative properties of the wave profile were still modeled well, although quantitatively the situation had deteriorated.

Turning now to a different aspect, we recall one of the most fascinating properties of many of the equations in (1). It is that, when dissipative effects are ignored, the balance between nonlinearity and dispersion admits the possibility of a special class of waves moving at constant velocity and without change of shape. These waves were called solitary waves by Scott Russell **[25]** who first observed them on the surface of a canal in the early 1830s. Scott Russell subsequently conducted experiments which showed the solitary wave to be a very stable waveform, which could sustain repeated complicated interactions without losing its identity. The existence of such permanent waves was at variance with the surface-wave theory known in the middle of last century. Indeed, one of the main accomplishments of Korteweg and de Vries in the 1890s was to resolve the paradox of the solitary wave, at least at the level of their model equation. Even so, the importance of this class of waves was not recognized until the 1960s when computer studies by Kruskal and Zabusky **[18]** of the KdV equation (2) showed that an initial profile of elevation broke up into a sequence of solitary waves and very little else. An analogous result for the sine-Gordon equation had been obtained earlier by Perring and Skyrme **[23]**. The celebrated inverse-scattering theory for (2), first discovered by Gardner, Greene, Kruskal and Miura **[13]**, subsequently established this result, and a host of others including the fact that solitary-wave solutions of (2) emerge from interaction with each other with only a phase shift. This type of exact result is true for a number of other wave equations admitting an inverse-scattering theory, including the sine-Gordon equation.

Recently a class of equations of the form given in (1) has been shown to possess solitary-wave solutions (cf. Benjamin, Bona and Bose [5]; Bona and Bose [8]), even when the operator H is not a differential operator. Moreover, numerical studies show that while the tidy situation regarding interaction of solitary waves for the KdV equation (2) does not in general obtain, the solitary wave nevertheless plays a distinguished role in the long-term evolution of an initial profile of elevation.

To take a concrete example, consider the model equation (10b). It has been proven by Benjamin [2] and Bona [7] that the solitary-wave solution of both KdV and (10b) is stable in the following sense. Let φ denote a solitary-wave profile and let ψ be a perturbation of φ, say in the norm defined by

$$\|w\|^2 = \int_R [w^2(x)+w_x^2(x)]\,dx.$$

Let $\eta>0$ be given. Then there exists a $\delta>0$ such that if $\|\varphi-\psi\|<\delta$, then $d(\varphi, u)<\eta$ for all $t>0$, where u is the solution, of (10b) say, with initial profile ψ and

$$d(w, v) = \inf_{y\in R} \|w(\cdot)-v(\cdot+y)\|$$

is a pseudo-metric that compares the shape of two functions. One may think of d as being defined on the product of the quotient space H^1/τ with itself, where H^1 is the space of measurable functions $f\colon \boldsymbol{R}\to\boldsymbol{R}$ such that $\|f\|<\infty$ and τ is the translation group in $\boldsymbol{R}$. Moreover, Abdulloev, Bogolubsky and Makhanov [1] and Bona, Pritchard and Scott [10] have produced numerical results indicating that when a pair of solitary-wave solutions of (10b) interact, the bulk of the mass emerges as a slightly different pair of solitary waves, shifted in phase, with a very small dispersive tail lagging behind. Finally, numerical experiments indicate than an initial wave of elevation evolving under (10b) breaks up into a finite number of solitary waves followed by a dispersive tail. Similar numerical results hold good for other models of the type given in (1).

Exactly what we should make of all this is still unclear. What is it in common to the models (1a) and (1b) that causes waves to evolve into solitary waves? Whatever this may be, it is probably more fundamental and at the same time less powerful than the inverse-scattering theory. A satisfactory answer to this question might have implications for the more complex models such as the various versions of the Boussinesq equations and ultimately the full equations of motion for various physical systems.

A final point deserves mention. Equations (1) have natural multi-dimensional versions which are of interest. The case of a system of two equations, for example, can serve as a model for the two-way propagation of one-dimensional waves. An existence, uniqueness and regularity theory for such systems has been given by Saut **[24]** in the case (1b) and by Benjamin and Bona **[4]** in the case (1a). However, the qualitative properties of solutions of such systems are still largely unknown.

References

1. Kh. O. Abdulloev, I. L. Bogolubsky and V. G. Makhankov, *One more example of inelastic soliton interaction*, Phys. Letters **56A** (1976), 427—428.

2. T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London **328** (1972), 153—183.

3. *Lectures on nonlinear wave motion*, Lectures in Appl. Math. vol. 15., Amer. Math. Soc. Providence, R. I., 1974.

4. T. B. Benjamin and J. L. Bona, *A class of nonlinear evolutionary equations with globally well-behaved solutions*, (to appear).

5. T. B. Benjamin, J. L. Bona and D. K. Bose, *Solitary-wave solutions for some model equations for waves in nonlinear dispersive media*, Lecture Notes in Math., vol. 503, Springer-Verlag, Berlin, 1976.

6. T. B. Benjamin, J. L. Bona and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London A **272** (1972), 47—78.

7. J. L. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London A **344** (1975), 363—374.

8. J. L. Bona and D. K. Bose, *Solitary-wave solutions for unidirectional wave equations having general forms of nonlinearity and dispersion*, Fluid Mechanics Research Institute report number 98, 1978.

9. J. L. Bona and P. J. Bryant, *A mathematical model for long waves generated by wavemakers in nonlinear dispersive systems*, Proc. Cambridge Philos. Soc. **73** (1973), 391—405.

10. J. L. Bona, W. G. Pritchard and L. R. Scott, *Solitary-wave interaction*, Fluid Mechanics Research Institute report number 94, 1978.

11. *A comparison of solutions of model equations for long waves* (to appear).

12. *A comparison of laboratory experiments with a model equation for water waves* (to appear).

13. C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Method for solving the Korteweg—de Vries equation*, Phys. Rev. Letters **19** (1967), 1095—1097.

14. J. L. Hammack and H. Segur, *The Korteweg—de Vries equation and water waves* Part 2, *comparison with experiment*, J. Fluid Mech. **65**, (1974), 289—314.

15. A. Jeffrey and T. Kakutani, *Weak nonlinear dispersive waves: a discussion centred around the Korteweg—de Vries equation*, SIAM Rev. **14** (1972), 582—643.

16. T. Kakutani and K. Matsuuchi, *Effect of viscosity on long gravity waves*, J. Phys. Soc. Japan **39** (1975), 237—246.

17. D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave*, Phil. Mag. **39** (1895), 422—443.

18. M. D. Kruskal and N. J. Zabusky, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Letters, **15** (1965), 240—243.

19. S. Leibovich and R. Seebass (editors), *Nonlinear waves*, Cornell Univ. Press, Ithaca, New York, 1974.

20. R. M. Miura, *The Korteweg-de Vries equation : a survey of results*, SIAM Rev. **18** (1976), 412—459.

21. A Newell (editor), Lectures in Appl. Math. vol. 15, Amer. Math. Soc., Providence, R. I., 1974.

22. P. J. Olver, *Euler operators and conservation laws of the BBM equation* Math. Proc. Camb. Phil. Soc. **85** (1979), 143—160.

23. J. K. Perring and T. H. R. Skyrme, *A model unified field equation*, Nuclear Phys. **31** (1962), 550—555.

24. J. C. Saut, *Sur certaines généralisations de l'équation de Korteweg—de Vries* I Comptes Rendus Acad. Sci. Paris, Serie A, **280** (1975), 653—656.

25. J. Scott Russell, *Report on waves*, 14th Meeting of the British Association for the Advancement of Science, John Murray, London, 1844.

26. H. Segur, *The Korteweg—de Vries equation and water waves*, 1. *Solutions of the equation*. J. Fluid Mech. **59** (1973), 721—736.

27. G. B. Whitham, *Nonlinear dispersive waves*. Proc. Roy. Soc. London A **283** (1965), 238—261.

28. N. J. Zabusky and C. J. Galvin, *Shallow-water waves, the Korteweg—de Vries equation and solitons*, J. Fluid Mech. **47** (1971), 811—824.

University of Chicago
Chicago, Illinois 60637, U.S.A.