SOLITARY WAVES

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The first observation of a solitary wave appears to have been made by Scott Russell in the third decade of the last century. Standing on the banks of the Edinburgh-Glasgow canal, Russell witnessed a moving barge come suddenly to rest, upon collision with a partially submerged obstacle. The abrupt cessation of motion created a long-crested wave, with an amplitude of some forty cm., which went rolling off down the canal. Giving chase on horseback, Russell observed that the wave propagated essentially without change of shape or of speed.

Fascinated, Russell went on to conduct a series of laboratory experiments on this phenomenon. The outcome of his investigation was reported in 1844, and published in 1845 in a wide-ranging article in which the term "solitary wave" was coined. Russell's work posed interesting theoretical questions which have subsequently been addressed many times.

Several lines of inquiry relating to solitary waves are now discernible. There is a large physics, engineering and geophysical sciences literature featuring solitary waves and related phenomena. There is the mathematical theory for the so-called "full equations of motion" (for surface water waves, these are the two-dimensional Euler equations with the awkward free-surface boundary conditions). There is as well a mathematical theory revolving around various model equations. Finally, numerical simulation of solitary waves and of their various in teractive properties has veen very fruitful in the last two decades. Here the discussion will focus on the third line of inquiry, which dates at least from Boussinesq (1872). For more complete references to these and other aspects of solitary waves, one may consult Bona (1980).

As a first step, the essentials of the derivation of these model equations is recalled, following the general lines laid down by Benjamin et.al. (1972). The derivation does not point to a unique model equation, at a given level of approximation. One of the representative evolution equation is the famous Korteweg-de Vries equation (1895),

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1}$$

where subscripts denote partial differentiation. The variables are dimensionless but unscaled. The dependent variable u is proportional to the displacement of the medium, while x and t are, respectively, proportional to distance in the direction of propagation and elapsed time.

To foster confidence in these model equations, comparisons of numerical simulation of their solutions with laboratory data are essential. Various comparisons of a qualitative nature have been made (cf. the references in Bona, Pritchard and Scott, 1980). A direct quantitative comparison was undertaken, and reported in the last-quoted reference. There an initial — and boundary — value problem for an alternate model was used:

$$u_t + u_x + uu_x - au_{xx} - u_{xxt} = 0, \tag{2}$$

for $x,t \geqslant 0$, with

$$u(x,0) = f(x), \quad \text{for } x \geqslant 0,$$

and

$$u(0,t) = h(t), \quad \text{for } t \geqslant 0.$$

The constant a is positive and the term $-au_{xx}$ is added to account approximately for dissipation.

In the range where these models are formally valid, they worked quite well. If $\overline{u}$ denotes the wave profile predicted by the model and u the measured wave profile, it is found that the relative error

$$\frac{\sup |u-\overline{u}|}{\sup |u|}$$

is about 8%. Even when the model is forced to predict in a regime which is probably outside its formal range of validity the qualitative agreement is not bad.

Taking these results as evidence that such models do embody interesting physical phenomena, attention is now directed to a closer inspection of the mathematical properties of solutions.

As hinted earlier, many of these models equations have similarity solutions representing solitary waves. For example, the KdV equation in the form(1) has the exact solution

$$S_C(x,t) = 3C\operatorname{sech}^2(\tfrac{1}{2}C^{\frac{1}{2}}\left[x-(C+1)t\right]), \qquad (3)$$

where $C > 0$ determines the amplitude and speed of the wave These solutions will be referred to as solitary-wave solutions of the relevant equations.

Two quite striking properties of the special solutions in (3) came to light in the mid-1960's, as a consequence of the inverse-scattering theory for the KdV equation (see Miura, 1976, for a nice summary of this theory). First is their property of exact interaction with each other. More precisely stated, there is an exact solution u of the KdV equation which has the asymptotic forms

$$u(x,t) \sim S_C(x+b,t) + S_D(x,t), \quad \text{as } t \to -\infty,$$

and

$$u(x,t) \sim S_C(x+c,t) + S_D(x+d,t), \text{ as } t \to +\infty,$$

where the "phases", b,c and d are just constants. For $t << 0$, u thus looks like a fair of widely separated and independently propagating solitary waves, and similarly if $t >> 0$. As t increases from $-\infty$, the larger solitary wave, having a greater speed of propagation, (cf.(3)) overtakes the smaller solitary wave. There follows a nonlinear interaction, after which the same two solitary waves are found emerging from the interaction and going their independent ways, with only phase shifts as souvenirs of the interaction.

Even more surprising is the resolution of wave profiles into solitary waves. Consider the pure initial-value problem for (1), namely to solve (1) for $x \in \mathbb{R}$, $t \geqslant 0$, subject to the auxillary condition

$$u(x,0) = f(x), \tag{4}$$

where f is a given reasonably smooth function decaying to 0 at infinity appropriately, along with its first few derivatives. Physically, one may think of f as describing a given initial wave profile at a given instant of time. Suppose $f \geqslant 0$ for convenience. Let $u(x,t)$ be the (unique) solution of KdV satisfying (4). Then for $t >> 0$, u has the form of a finite sequence of independently propagating solitary waves, ordered by increasing amplitude, and very little else. Interpreted practically, this means that a reasonably arbitrary disturbance will sort itself into a finite number of pulses.

These exact results have analogues in a few other model equations of physical interest (eg. one version of the Boussinesq equations and the sine-Gordon equation). However, there seem to be a host of equations which, while perhaps not having these striking properties that obtain exactly for the KdV equation, nevertheless manifest similar behavior.

To take a concrete example, consider the equation (2) with a=0. If is known that this equation does not have an inverse-scattering theory. Nor does it possess infinitely many polynomial conservation laws, as KdV does. And, it does not appear that the solitary-wave solutions of (2), which are similar to the form exhibited in (3), interact exactly either. Nonetheless, solitary-wave solutions of (2) "almost" interact exactly. Additionally the remarkable property of resolution into solitary waves appears to be valid for solutions of equation (2) (still with a=0 of course). These results were obtained recently (Bona, Pritchard and Scott, 1980) by numerical integration of (2). Why this should be the case, for (2) and for a number of other equations, is an interesting open question.

Finally it deserves mention that solitary wave solutions have been shown to exist for a more general class of model equations, having the form

$$u_t + u_x + f(u)_x + Lu_x = 0.$$

Here f is related to nonlinear effects suffered by the waves while L embodies an approximation to the dispersive effects inherent in the system. (Cf. Benjamin, Bona and Bose, 1976 and Bona and Bose, 1978). Non-constructive methods of functional analysis are used in the last-cited works, and so closed formulae for these waves are generally not available. However, preliminary numerical simulations of some of these equations show that, generally, there are classes of initial data that evolve into a sequence of what appear to be solitary-wave solutions of the particular equation. The issues plainly need further investigation.

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