SOLITARY WAVES AND OTHER PHENOMENA ASSOCIATED WITH MODEL EQUATIONS FOR LONG WAVES

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The first recorded observation of a solitary wave appears to have been made by Scott Russell in the third decade of the last century. Standing on the banks of the Edinburgh-Glasgow canal, Russell witnessed a moving barge come suddenly to rest, upon collision with a partially submerged obstacle. The abrupt cessation of motion created a long-crested wave, with an amplitude of some eighteen inches, which went rolling off down the canal. Giving chase on horseback, Russell observed that this wave propagated without much change of shape, or of speed.

Fascinated, Russell went on to conduct a series of laboratory experiments on this phenomenon. The outcome of his investigation was reported in 1844 and published in 1845 in a wide-ranging article in which the term 'solitary wave' was introduced. It is a simple exercise to see that the then existent shallow water theory (cf. Airy, 1845 and Rayleigh, 1876), formally valid for waves of extreme length and of amplitude small compared to the depth, could not explain what Russell had discovered. Consequently, Russell's work posed interesting theoretical questions which have subsequently been addressed many times.

Stokes (1849) considered the question, and concluded that solitary waves could not exist. More precisely, he claimed that progressive long waves cannot propagate without change of form. Despite his erroneous conclusion, Stokes isolated an important parameter, herein referred to as the Stokes number and denoted S, defined as $a\lambda^2/h^3$, where a is a typical wave amplitude, λ is the length-scale of the waves in question, and h is the undisturbed depth. (Throughout this paper, h will be assumed to be constant.) Stokes was later to change his mind about the existence of solitary waves, but seems not to have appreciated the full significance of the dimensionless combination S. Though, it deserves note that he did remark that Airy's shallow-water theory needed large values of S.

Boussinesq (1872) and Rayleigh (1876) both made attempts to describe the solitary wave. Rayleigh considered the solitary wave in an Eulerian frame of reference moving at a velocity that brings the wave to rest. He then proceeded directly to an expansion of the surface elevation in terms of its maximum deviation a from the undisturbed free surface, obtaining the approximate formula

$$a \cdot \mathrm{sech}^2[(b/2)(x - ct)], \tag{1}$$

for the solitary wave profile. Here

$$b^2 = 3a/[h^2(a + h)] \text{ and } c^2 = g(a + h), \tag{2}$$

where h is again the undisturbed depth of the liquid and g is the acceleration due to gravity. Boussinesq followed quite a different line of argument, deriving from the Euler equations an approximate evolution equation, which now bears his name. This evolution equation has a particular solution in the form (1) which may be taken therefore to represent a solitary wave, though the constants a, b, and c are related in a different way than in (2). Boussinesq's and Rayleigh's approximate solitary waves agree exactly if terms of order a^2 are neglected.

Apparently unaware of Boussinesq's work, Korteweg and de Vries (1895) also derived an approximate evolution equation from the Euler equations. Their equation was derived assuming that the waves being considered had an amplitude which is small and a wavelength which is large compared to the undisturbed depth. Additionally, they recognized the importance of presuming the Stokes number to be neither too large nor too small. They referred explicitly to Stokes' paper, cited above, in fact. Going a step beyond Boussinesq's theory, Korteweg and de Vries restricted attention to waves moving in only one direction. Thus they obtained a differential equation which is first-order in the temporal variable instead of a perturbation of the linear wave equation $\eta_{tt} = \eta_{xx}$, such as Boussinesq obtained. Their equation also possesses exact solutions of the form (1), with yet another relationship between the constants a, b, and c. The Korteweg-de Vries version of the solitary wave agrees with Boussinesq's and Rayleigh's version to within terms of order a^2. It is amusing to note that Scott Russell had already empirically found the relationship between amplitude a and speed of propagation c implied in these three approximations to the solitary wave.

Solitary waves continued to attract attention in the ensuing decades, both for their own sake and in conjunction with various other shallow-water phenomena. We point, as examples, to the work of Daily and Stephan (1952), Friedrichs and Hyers (1954), Keady and Pritchard (1974), Keller (1948), Lavrentieff (1943, 1947), Munk (1949), Sverdrup and Munk (1946), and Weinstein (1926). However, the model equations, at the level of approximation envisioned by Boussinesq and by Korteweg and de Vries, attracted comparatively little notice, though the papers of Keulegan and Patterson (1940) and Ursell (1953) deserve note as being exceptional in this aspect. Keulegan and Patterson filled in many of the details in Boussinesq's arguments while Ursell clarified the approximations involved in the Boussinesq equation, including particularly the role played by the Stokes number. (Accordingly, this dimensionless combination is sometimes referred to as the Ursell number.)

Recently progress has been made on the purely mathematical side of the problem of solitary waves. Beale (1977) has refined the existence theory given by Friedrichs and Hyers (1954) for small-amplitude solitary-wave solutions of the Euler equations. And very recently, Amick and Toland (1979) have shown that solitary-wave solutions of the Euler equations exist for all amplitudes, up to a wave of greatest height. Also deserving particular mention are the careful numerical calculations of Longuet-Higgins (1974), Cokelet (1977), and Longuet-Higgins and his collaborators Fenton (1976), Fox (1977, 1978), and Byatt-Smith (1976). These computations have turned up some interesting aspects of larger-amplitude solitary waves. These exciting developments, and many others including the theory of internal solitary waves and periodic permanent waves, will not be featured in what follows. Instead, interest will be focused on model equations, exemplified by the Korteweg-de Vries equation.

In 1960, Gardner and Morikawa derived the Korteweg-de Vries equation as a model for magnetohydrodynamic waves in a cold collisionless plasma. On the heels of this discovery, Kruskal and Zabusky found the Korteweg-de Vries equation as a continuum limit of a system of masses and springs considered by Fermi, Pasta, and Ulam (1955). Their extensive numerical simulations of solutions of the Korteweg-de Vries equation revealed some startling properties of this equation (cf. Zabusky and Kruskal, 1965, Kruskal, 1963, and Zabusky, 1963). These discoveries led to a sustained effort to analyze this equation, which continues to the present. Gardner, Greene, Miura, and Kruskal (1967) found an explicit procedure for solving the KdV equation, as the Korteweg-de Vries equation will be referred to henceforth. With this procedure came a host of exact solutions, asymptotic properties of solutions, and the like. This procedure was put into a form which revealed its essential structure by Lax (1968). It is fair to say that this discovery, the so-called inverse-scattering theory for the KdV equation has opened up a new and rich area of research which has, by now, reached industrial proportions.

Since Gardner and Morikawa's discovery, there have been quite a number of instances in which the KdV equation, or a near relative, has been derived as a model equation at a certain rudimentary level of approximation. Some of the applications of this equation may be found discussed in the review articles of Benjamin (1974), Jeffrey and Kakutani (1972), and Scott, Chu, and McLaughlin (1973). The full equations for the various physical situations where KdV appears as a model are generally of quite disparate character. Consequently, one would like an explanation of the ubiquitousness of the KdV model. An attempt to explain this state of affairs was put forth by Benjamin, Bona, and Mahony (1972), and it will suit our later purposes to briefly review this material.

The starting point of the analysis is the recollection of the assumptions underlying the passage from the full equations (e.g. the Euler equations) to a KdV-type model. The motions considered must be basically one-dimensional, and the waves must propagate only in one direction. Moreover, it is typical that the amplitude of the wave must be small and the wavelength long, but not independently so. Rather, there must be a balance between these two quantities, which in the case of surface water waves in a channel is reflected in the Stokes number being of order one. We shall consider the modeling of such waves, based on the just-mentioned properties along with two other fairly general principles.

A simple model for one-way propagation of one-dimensional waves is given by the conservation law

$$\eta_t + \eta_x = 0. \qquad (3)$$

The variables here are the wave amplitude η, the spatial coordinate x and the temporal coordinate t, which are all dimensionless, of course. Indeed, this model is not too bad, over short time scales. However, over longer time scales, nonlinear, dispersive and dissipative effects can accumulate and render this simple model invalid.

First consider nonlinear effects. The model (3) is equivalent to the characteristic equation

$$\left.\frac{dx}{dt}\right|_{\eta = \text{constant}} = 1, \tag{4}$$

which simply states that the disturbance propagates without change of shape at speed one. In many systems, it is found that, at a higher order of approximation, the disturbance propagates along characteristics which depend on the amplitude of the wave. Thus in place of (4) there is a relation of the form

$$\left.\frac{dx}{dt}\right|_{\eta = \text{constant}} = f(\eta). \tag{5}$$

The function f will be determined by the specific physical problem being investigated. For infinitesimal amplitudes, and in the absence of dispersive and dissipative effects, it is expected that (5) should reduce to (4), at least over certain time intervals. Consequently, f is inferred to have the form

$$f(z) = 1 + g(z),$$

where $g(0) = 0$. If in fact g is a twice continuously differentiable function, then

$$f(z) = 1 + g'(0)z + O(z^2),$$

as $z \longrightarrow 0$. Assuming that $g'(0)$ is non-zero, which is not always the case in interesting examples, and neglecting the higher-order term, on the basis of the assumption that the model is only concerned with small values of η (small amplitudes), the approximation

$$\left.\frac{dx}{dt}\right|_{\eta = \text{constant}} \cong 1 + \gamma\eta \tag{6}$$

emerges. Here $\gamma = g'(0)$. If η is rescaled to be order one, then γ becomes a small parameter. The characteristic equation (6) is formally equivalent to the partial differential equation

$$\eta_t + \eta_x + \gamma\eta\eta_x = 0. \tag{7}$$

More generally, the characteristic equation (5) is formally equivalent to

$$\eta_t + F(\eta)_x = 0,$$

where F is any antiderivative of f. In particular, if $f = 1 + g$, the last equation becomes

$$\eta_t + \eta_x + G(\eta)_x = 0, \tag{8}$$

where $G' = g$. Thus (8), and in particular (7) when g has a non-trivial linear term, may be viewed as a modification of the basic one-way propagator (3) that accounts for the non-linear effects of small, but finite, amplitudes.

The effects of finite amplitudes are now ignored, and a correction for frequency dispersion is addressed. As we are then dealing with infinitesimal waves, the linearized equations of motion are appropriate. If simple-wave solutions of the form $\exp(i[kx - \omega t])$ are sought, then it is generally found, in homogeneous media, that ω is determined as a function of the wave number k. This correspondence between ω and k is called the (linearized) dispersion relation of the system in question. For example, if the two-dimensional Euler equations for surface water waves are linearized, the relation between wave number and frequency, determined as just outlined, is

$$\omega = \omega(k) = [k \cdot \tanh(k)]^{1/2}, \tag{9}$$

in dimensionless variables. If the phase speed $c(k) = \omega(k)/k$ is considered, then it is found typically that c is an even function of k, with a maximum value at $k = 0$, and decreasing monotonically to zero as $k \to \infty$. This corresponds to the fact that, for infinitesimal waves, the longer waves travel faster, a familiar fact in surface water-wave theory.

The relationship between wave number and frequency may be generalized from simple waves to more complex, though still infinitesimal, profiles by use of Fourier's principle.

More precisely, let $\varphi(x)$ be a given initial wave profile. The function φ may be formally decomposed as

$$\varphi(x) = \int_{-\infty}^{\infty} \psi(k) e^{ikx} dk,$$

where ψ is the Fourier transform $\hat{\varphi}$ of φ. An individual wave with wave number k propagates with speed c(k), as determined above. Hence φ, which is the sum over all wave numbers of simple waves, with the appropriate weighting $\psi(k)$, propagates as

$$\eta(x,t) = \int_{-\infty}^{\infty} \psi(k) e^{ik(x - ct)} dk. \qquad (10)$$

Thus, one determines that η evolves according to the equation

$$\eta_t + (L\eta)_x = 0, \qquad (11)$$

where L is the linear operator defined by

$$\widehat{Lv}(k) = c(k)\hat{v}(k), \qquad (12)$$

the circumflexes denoting Fourier transforms again.

So far, this linearized theory of uniform plane waves has introduced no further approximation. If it is supposed that the waves in question are long, then ψ will be negligible outside a small interval about zero. Consequently, it is appropriate to consider convenient approximations to the phase speed c(k) which are valid for small k. Supposing c to be several times continuously differentiable, and keeping in mind the other properties of c thus far presumed (cf. figure 1), we may write

$$c(k) = c(0) - \beta k^2 + O(k^4), \qquad (13)$$

as $k \to 0$, where $\beta = -c''(0)/2 > 0$. As in the consideration of the nonlinear effects, it is expected that (11) reduce to (3) in the limit of waves of extreme length, and so in this scaling c(0) = 1. It is convenient to define $\alpha(k) = c(k) - 1$. Then (11) is written as

$$\eta_t + \eta_x + (M\eta)_x = 0, \qquad (14)$$

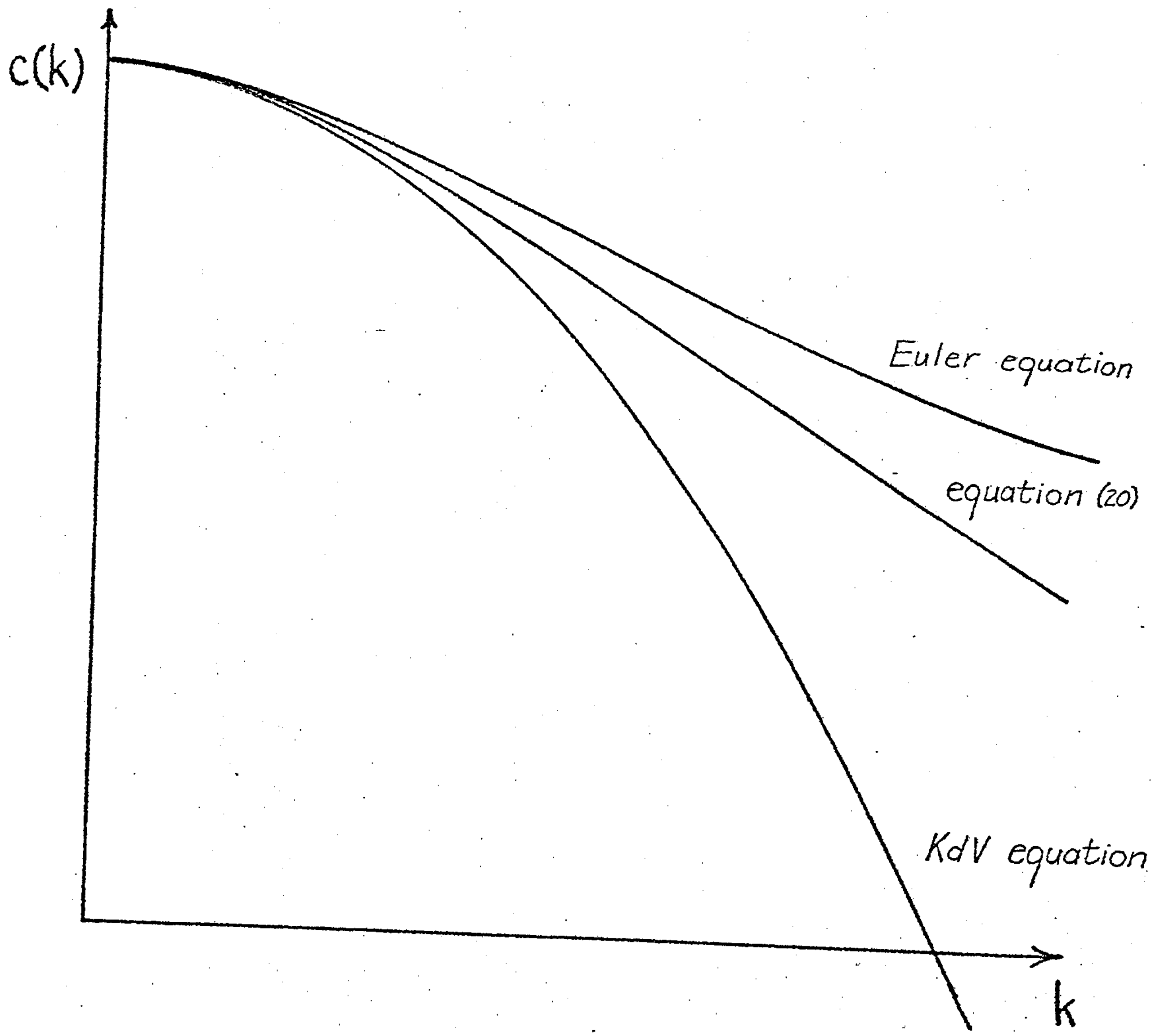

Figure 1. Linear dispersion relations for the two-dimensional Euler equations, the KdV equation and the alternative equation (20).

where

$$\widehat{Mv}(k) = \alpha(k)\hat{v}(k). \tag{15}$$

If the higher order term in (13) is dropped, so that c(k) is approximated by an osculating parabola, (cf. again figure 1) then in this approximation, (14) becomes

$$\eta_t + \eta_x + \beta\eta_{xxx} = 0. \tag{16}$$

If x is rescaled so that k is of order one in the range of interest, then β becomes a small parameter. Thus (14), and particularly (16), provides a correction to the one-way propagator (3) that accounts for the small effects of frequency dispersion suffered by long, but finite length, waves. Note incidentally, that if c(k) has a non-zero imaginary part, then (14) will include a damping term. This possibility is explicitly ignored for the present.

If the corrections for nonlinearity and dispersion are simply combined additively, there appears

$$\eta_t + \eta_x + \gamma\eta\eta_x + \beta\eta_{xxx} = 0, \tag{17}$$

a version of the KdV equation. A number of things are entailed in combining (7) and (16) in this simple way to obtain (17). First there is the implicit assumption that the two correction terms are of comparable order. If the changes of scale, referred to above, are pursued, so that η and its derivatives are of order one and γ and β are both small parameters, then the latter assumption means that γ and β are of comparable size. Or, what is the same, $S = \gamma/\beta$ is of order one. The small parameter γ measures the local size of nonlinear effects, whilst β measures the local size of dispersive effects. The quotient γ/β is a measure of the relative strength of nonlinear and dispersive effects. In the example of surface water waves, the ratio γ/β is exactly the Stokes number, referred to previously. Second, there is implied that the direct interaction between nonlinearity and dispersion is of smaller order than the terms which have been retained. In the various particular cases where a uniform approximation has been made to more complete equations of motion, these considerations may be confirmed. Indeed, the foregoing makes it clear that they must be confirmed to place an equation such as (17) on a sound formal footing. Note also that the zero on the right-hand side of (17) represents terms of formal order γ^2, $\gamma\beta$, and β^2, which have been ignored.

More generally, the model equation

$$\eta_t + \eta_x + G(\eta)_x + (M\eta)_x = 0, \qquad (18)$$

may be inferred from the foregoing discussion, without the necessity of the additional assumptions made on g in (6), nor the assumptions on c(k) made in (13). (The operator M is defined in (15)). As these assumptions are not always valid in physically interesting modeling situations, the more general equations are not just of academic interest. A class of models in the form (18) will be touched on later.

It is a point of interest that equation (17) may be cast in other forms by use of the lowest order relation (3). That is, to lowest order, $\eta_x = -\eta_t + O(\gamma,\beta)$, and this relation may be used to alter the higher-order terms without affecting the formal level of approximation. Thus, the following eight equations may be obtained, on exactly the same formal basis as (17).

$$\eta_t + \eta_x + \gamma\cdot\begin{bmatrix}\eta\eta_x\\ -\eta\eta_t\end{bmatrix} + \beta\cdot\begin{bmatrix}\eta_{xxx}\\ -\eta_{xxt}\\ \eta_{xtt}\\ -\eta_{ttt}\end{bmatrix} = 0. \qquad (19)$$

It needs to be recognized that whilst the eight equations represented in (19) are <u>formally</u> equivalent, as far as predictions regarding long waves of small amplitudes are concerned, there may be criteria of modeling, mathematics and convenience that indicate some of these equations over others. A detailed analysis of all eight of the equations represented in (19) will not be entered upon here. Some additional commentary on this abundance of model equations is given in the appendix. For the present, two of these equations will be singled out for further study. These are the classical KdV equation and the alternative equation

$$\eta_t + \eta_x + \gamma\eta\eta_x - \beta\eta_{xxt} = 0, \tag{20}$$

proposed by Peregrine (1964) and Benjamin _et al_ (1972). By assumption, the Stokes number $S = \gamma/\beta$ is of order one. To simplify the further discussion, suppose in fact that $S = 1$, and let ϵ denote the single small parameter $\gamma = \beta$. (This may always be made the case by an order-one change of variables.) Then the two equations are written,

$$\eta_t + \eta_x + \epsilon\eta\eta_x + \epsilon\eta_{xxx} = O(\epsilon^2), \tag{21}$$

and

$$\eta_t + \eta_x + \epsilon\eta\eta_x - \epsilon\eta_{xxt} = O(\epsilon^2), \tag{22}$$

where the order of the neglected terms has been indicated explicitly. Recall that in this scaling, η and its derivatives are all order one.

The nonlinear and dispersive terms in (21) and (22) may formally contribute an effect of order one on time scales of order $1/\epsilon$. Thus $1/\epsilon$ is the time scale over which significant modification of the wave profile may be expected to take place. Similarly, the neglected terms may contribute an order-one effect on time scales of order $1/\epsilon^2$. Hence predictions, using these models, must be viewed with caution on time scales approaching or exceeding $1/\epsilon^2$. With these scales in mind, the following result, quoted somewhat informally from Bona, Pritchard, and Scott (1980c), is illuminating. In the following, the pure initial-value problem, relative to (21) and (22), will be considered. That is, it will be supposed that the wave profile η is known everywhere at a given instant of time, say $t = 0$. Interest is then focused on the subsequent evolution of the wave, for $t > 0$. Mathematically, this amounts to asking for a solution of (21) or (22), defined for all x, and $t \geq 0$, and obeying the specification

$$\eta(x,0) = \varphi(x), \text{ for } x \in \mathbb{R}. \tag{23}$$

The pure initial-value problem for these equations is of considerable theoretical importance, though other ways of specifying auxillary data are also of interest, as will appear shortly.

THEOREM. Let φ and its first six derivatives be square-integrable over $\mathbb{R}$. Let $\epsilon > 0$ be given and let η^ϵ and ζ^ϵ be the (unique) solutions of (21) and (22) (with right-hand side set to zero), respectively, corresponding to the initial data φ as in (23). Then for all x and for $0 \leq t \leq 1/\epsilon$,

$$\left|\eta^\epsilon(x,t) - \zeta^\epsilon(x,t)\right| \leq C\epsilon^2 t, \tag{24}$$

where C is a constant not dependent on ϵ or t.

This result needs a little explication. First, a function φ is square-integrable over $\mathbb{R}$ if

$$\int_{-\infty}^{\infty} \varphi^2(x)\,dx < +\infty.$$

The hypotheses on φ, whilst slightly technical, essentially amount to the assumption that the initial wave profile is sufficiently smooth and decays to zero at $\pm\infty$, along with its first few derivatives. This is certainly not a serious practical restriction. The fact that both equations have unique smooth solutions corresponding to the initial value φ has been established in various ways, with various restrictions on φ, by several authors (cf. Benjamin <u>et al</u>, 1972, Bona and Smith, 1975, Kato, 1975, Cohen, 1979, and the references given therein). At the time $t = 1/\epsilon$, when both η^ϵ and ζ^ϵ may have evolved significantly, due to the accumulation of nonlinear and dispersive effects, the two solutions only differ by at most order ϵ. As neglected terms could have accumulated to this order at this time, neither solution is formally accurate to more than this order. Hence, η^{ϵ} and ζ^{ϵ} are the <u>same</u>, to the neglected order, on this time scale. Numerical studies indicate that (24) remains valid for $0 \leq t \leq 1/\epsilon^2$. If this is so, it would show that the two solutions diverge from one another on the time scale $1/\epsilon^2$, over which either model has become potentially suspect. The general conclusion to be drawn, then, is that as far as modeling long waves of small amplitude is concerned, these equations work equally well. Of course, as far as we know at this juncture in the discussion, neither may work very well!

We turn to the question of how well such models work in practice. Indeed, there is a fair amount of evidence in favor of the model equations under discussion, at least as they apply to surface water waves. Comparisions of predications of these models with experimental measurements have been made by Zabusky and Galvin (1971), Hammack (1973), Hammack and Segur (1974), and Weidman and Maxworthy (1978). In these studies, various aspects of the models predictive power were checked. Generally, good qualitative agreement between predictions and what is seen in the laboratory was found.

A direct quantitative comparision has been made recently by Bona, Pritchard, and Scott (1980b). Instead of the pure initial-value problem, an initial- and boundary-value problem, to be explained presently, was used in these latter comparisons.

The experiments were performed in a channel with uniform sides and a flat bottom. At the beginning of each set of measurements, the water in the channel was at rest and a wavemaker at one end of the channel was set in periodic motion. This had the effect of generating plane waves which propagated down the channel. The passage of the waves was measured at several stations in the channel. This type of measurement is experimentally simple and quite accurate.

When the leading wave reached the end of the channel, the experiment ceased. The experimental configuration is sketched in figure 2. For details, the reader is referred to the last-quoted reference.

The mathematical model proposed for this experiment is an initial- and boundary-value problem based on the equation (22): for $x, t \geq 0$,

$$\eta_t + \eta_x + (3/2)\eta\eta_x - \nu\eta_{xx} - (1/6)\eta_{xxt} = 0,$$
$$\eta(x,0) \equiv 0, \text{ and } \eta(0,t) = f(t). \tag{25}$$

Here, the variables are dimensionless, but unscaled. The undisturbed depth h is taken as the unit of length and $(h/g)^{1/2}$ is taken as the unit of time, where g is the acceleration due to gravity. Note that a model dissipative term $-\nu\eta_{xx}$ has been added to the differential equation. Dissipative effects proved to be as important as nonlinear and dispersive effects, and consequently such a term is crucial in obtaining good quantitative agreement. The constant ν was determined by a separate experiment on waves in the linear range. The initial data $\eta(x,0) \equiv 0$ corresponds to the liquid being undisturbed at the start of each experiment. The boundary data $\eta(0,t) = f(t)$ is determined by measurement, taken at the station closest to the wavemaker (cf. figure 2).

The initial-boundary-value problem (25) was analyzed by Bona and Bryant (1973), and shown to be well-posed. A similar theory is available for the KdV equation (cf. Bona and Winther, 1980). The model (25) appeared to be easier to handle numerically, and consequently was preferred.

The comparison of the model with the experimentally obtained data was made as follows. Corresponding to a measurement of $f(t) = \eta(0,t)$, a numerical integration of (25) was performed, over the relevant range of the quarter-plane $x, t \geq 0$. From this integration, $\eta(\bar{x},t)$ was read off at the values $\bar{x}$ corresponding to the other stations in the channel where measurements were taken. Thus, referring to figure 2,

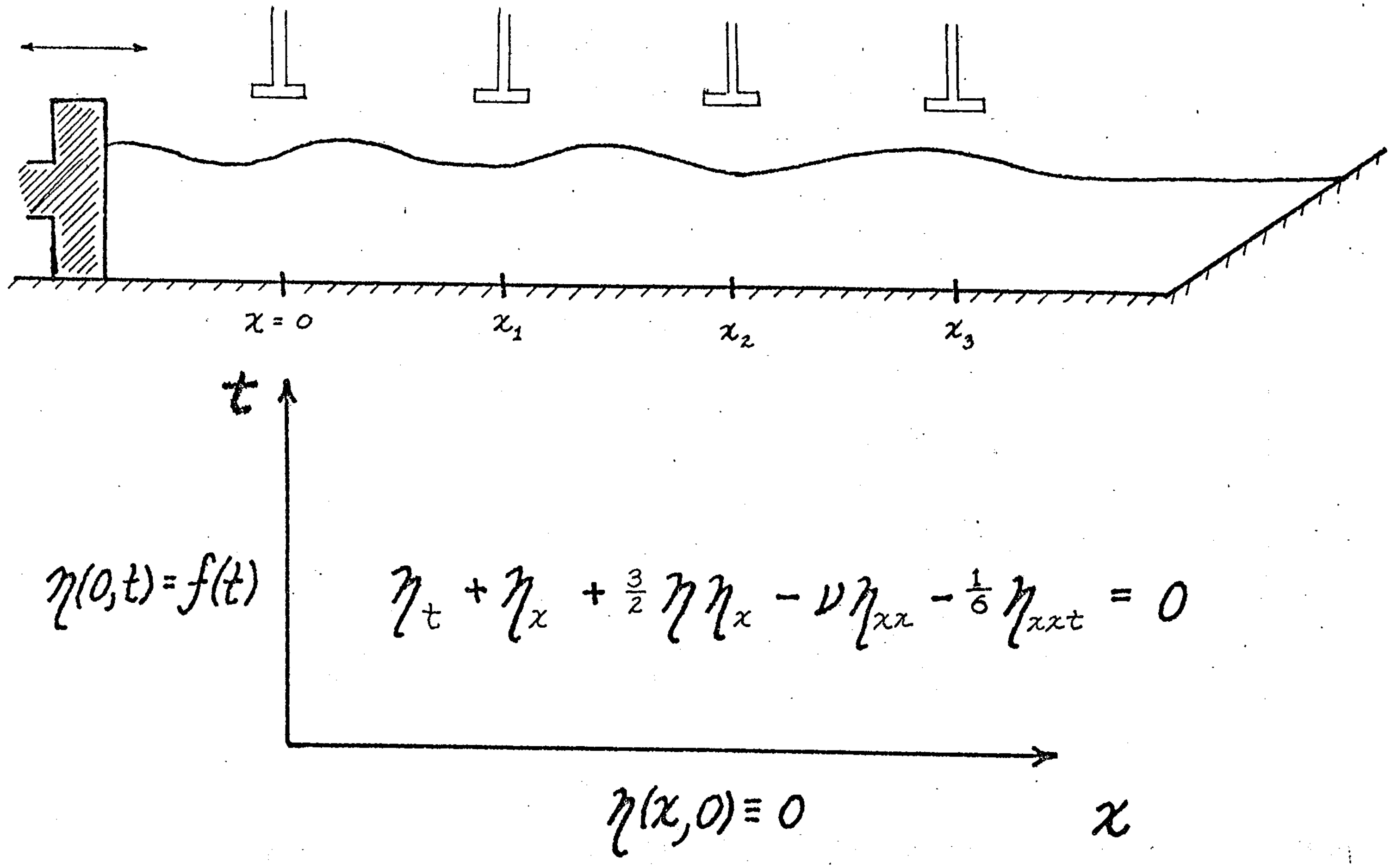

Figure 2. Sketch of the experimental configuration and the proposed mathematical model.

$\eta(x_j,t)$, $j = 1, 2, 3$, were determined numerically from $f(t)$. These predictions were then compared to the actual measurements observed at these stations, and the discrepancy used to judge the model's performance.

In these experiments, the wavelength was kept constant, and the amplitude of the waves varied. Thus a spread of Stokes numbers could be covered. The values of S reported span roughly the range between 1/2 and 35. Quite good agreement between theory and experiment was found, for S between 1 and 10. Figure 3 shows a comparision of numerically computed and experimentally measured data which is typical of this range. For this run, the Stokes number was 4.5 and the relative error was about 8%. For larger values of S, the agreement deteriorated somewhat. However, by tinkering with the model somewhat, the agreement could be improved considerably. This alteration consisted mainly of fitting the dispersion relation a little more accurately. Figures 4 and 5 show a comparision as in figure 3, except that S = 26.3. In figure 4, the comparision is made with the model (25), while in figure 5 the comparision is made using the improved match to the dispersion relation.

We will take the foregoing to suggest that the model equations under discussion do have an interest as predictors of physical phenomena. Consequently, various theoretical questions concerning their solutions are of potential interest. For the considerations that follow, the small parameters appropriate for modeling long waves are not relevant. They will, therefore, be scaled out, and reference will be to the tidy forms

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0, \qquad (26)$$

and

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0, \qquad (27)$$

henceforth.

As mentioned earlier, the inverse-scattering method for solving the KdV equation has contributed much detailed know-

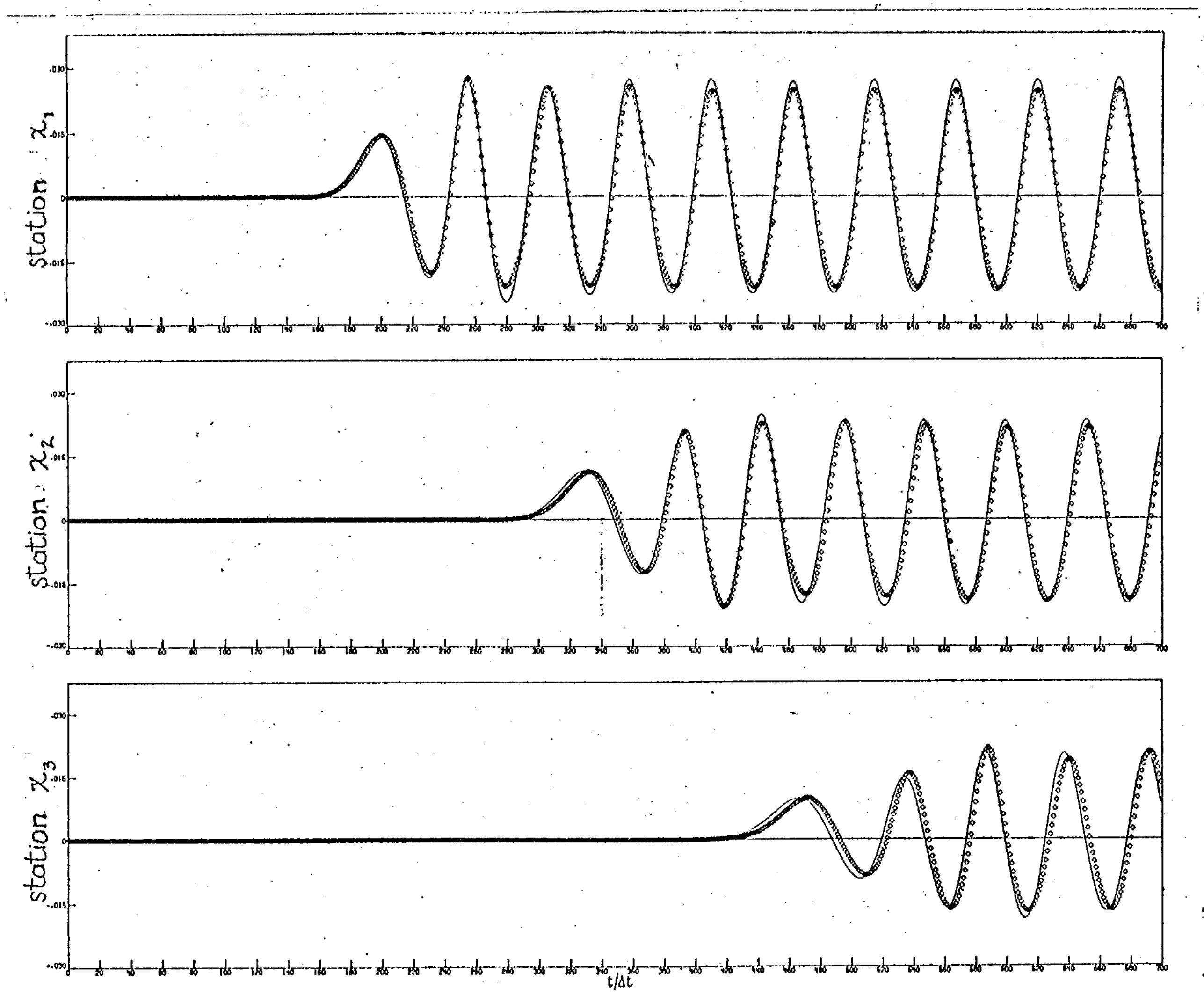

Figure 3. Comparison of experimental data (the ◇'s) and numerical integration of the model (25), for Stokes number 4.5.

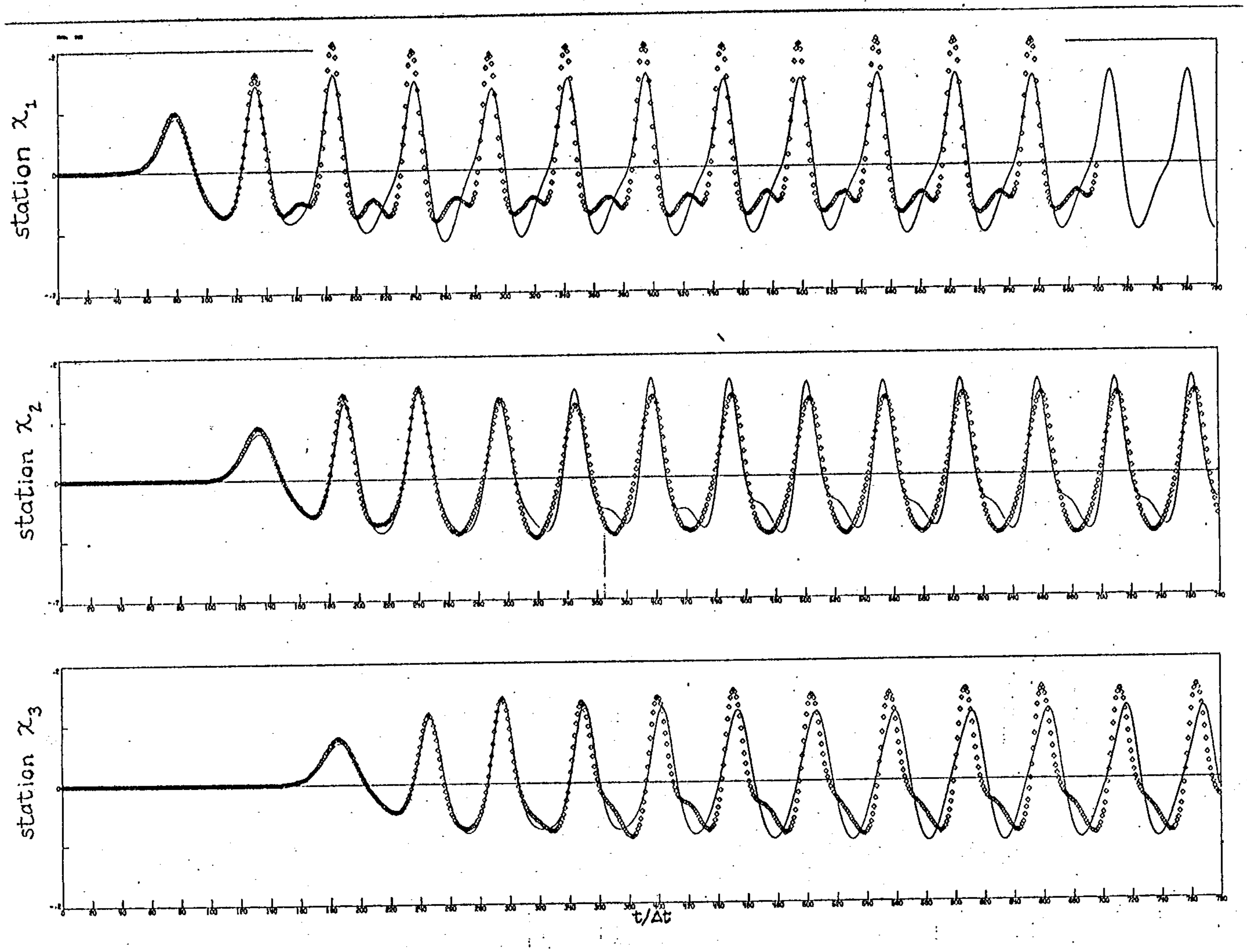

Figure 4. Comparison of experimental data (the ◇'s) and numerical integration of the model (25), for Stokes number 26.3.

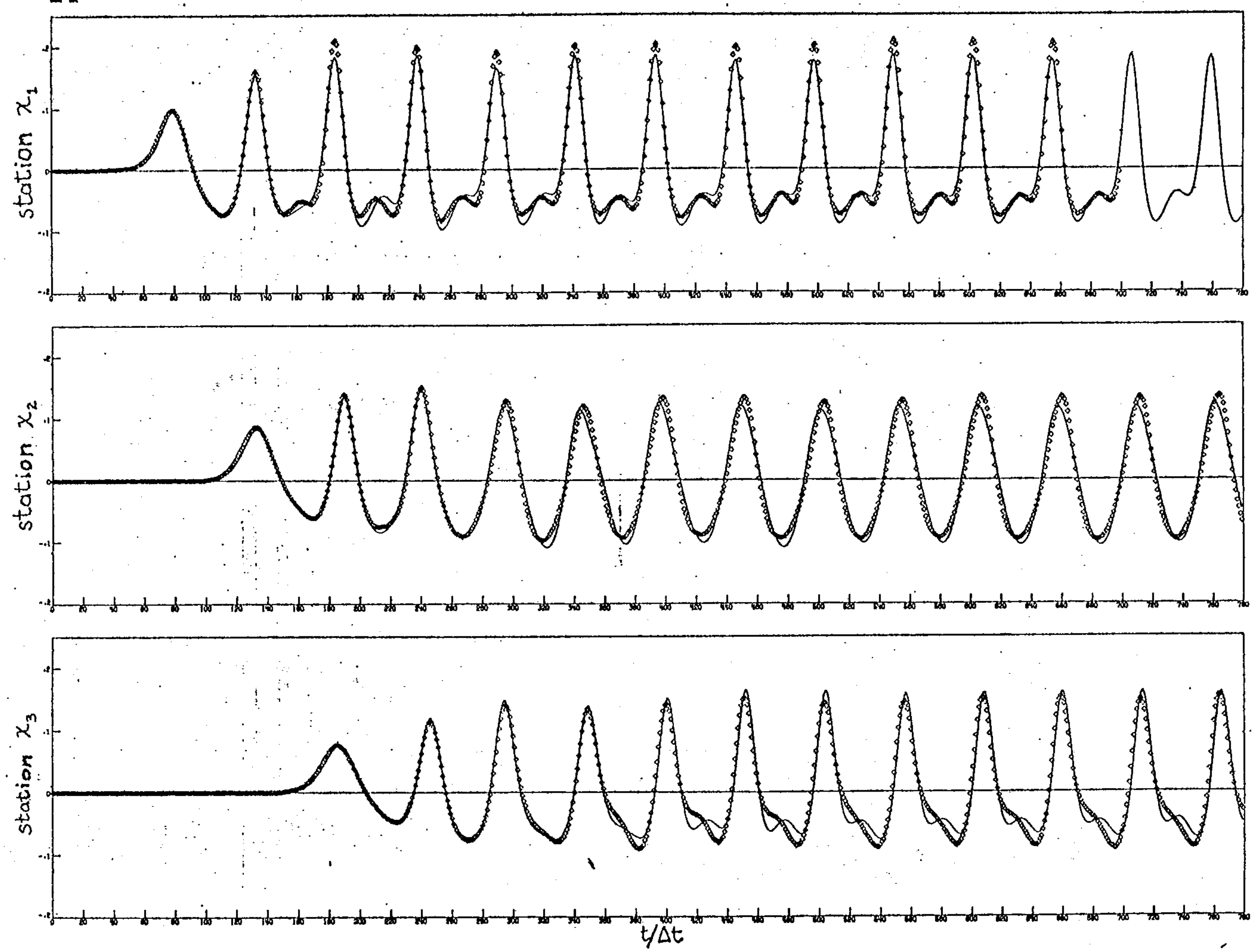

Figure 5. Comparison of experimental data (the ◊'s) and numerical integration of a modified version of (25) for Stokes number 26.3.

ledge concerning solutions of this equation. Two points will be of particular concern here. Before these are addressed, we remind the reader that the KdV equation has an exact solution,

$$\eta(x,t) = 3C\operatorname{sech}^2(\tfrac{1}{2}C^{1/2}[x - (C + 1)t]), \tag{28}$$

where $C > 0$, which is the aforementioned approximation to the solitary wave. This solution will be called a solitary-wave solution of the KdV equation.

The first point is a remarkable property of solutions of the pure initial-value problem for KdV. Let φ be an initial datum for KdV, which is smooth and decays to zero at $\pm\infty$ appropriately. For convenience of the present brief exposition, suppose that $\varphi \geqslant 0$. Then φ evolves into a finite sequence of widely-separated independently-propagating solitary waves, ordered by increasing amplitude, and very little else. The "very little else" is what is commonly called a dispersive tail. A dispersive tail is an oscillatory waveform, whose maximum amplitude tends to zero as t becomes large, while at the same time lengthening and developing more oscillations (cf. figure 7). This result is surely unexpected. For nonlinear evolution equations, one does not expect particular solutions to play a distinguished role in the evolution of a general class of initial data.

The second point is the exact interaction of solitary waves. Suppose two unequal solitary waves are started off widely separated, with the larger crest to the left of the smaller crest. As this initial profile evolves under the KdV equation, the larger solitary wave will overtake the smaller one, because of its greater speed of propagation. There follows a nonlinar interaction between the two waves. After this stage, the two waves separate cleanly from each other, unaltered except for a phase shift. Of course, the larger wave is now to the right of the smaller wave. There is absolutely no residue from this interaction except that the larger wave has been shifted forward of the position it would have occupied in the absence of any interaction and the smaller wave has similarly been retarded.

The alternative model equation (27) also has solitary-wave solutions, similar in form to those of the KdV equation:

$$\eta(x,t) = 3C\operatorname{sech}^2\left(\frac{1}{2}\left(\frac{C}{1+C}\right)^{1/2}[x - (C+1)t]\right). \qquad (29)$$

One would naturally like to know whether the interesting properties of solutions of the KdV equation, just described, hold for the alternative equation. Results of Olver (1979), McLeod and Olver (1979), and Tsujishita (1979) indicate that equation (27) does not have an associated inverse-scattering formalism for solutions, at least as such a formalism is currently understood.

Turning first to the interaction of the solitary-wave solutions of (27), it was reported by Eilbeck and McGuire (1977) that these waves interacted exactly, to numerical accuracy, just as for the KdV equation. However, Abdulloev, Bogolubsky, and Makhanov (1976) observed a small "rarefaction wave" forming behind the interacting solitary-wave solutions of (27) in their numerical experiments (see also Alexander and Morris, 1979, who have reported some related numerical computations). Abdulloev et al's findings have been confirmed by Bona, Pritchard, and Scott (1980a). A sample of their results is shown in figures 6 and 7. Figure 7 depicts the same wave profiles as figure 6, except that the vertical scale has been magnified one hundred times, so that the small dispersive tail developing behind the solitary waves emerging from the interaction is clearly visible. Thus it may be that solitary-wave solutions of (27) do not interact exactly. Nevertheless, as Bona et al (1980a) pointed out, the comparatively small size of the dispersive tail shown in figure 7 was entirely typical of a more extended experimental study. This fact leads to the speculation that (27) is not far from having an exact-interaction property for its solitary-wave solutions. But, the sense that the term "not far" is to be given is not presently known.

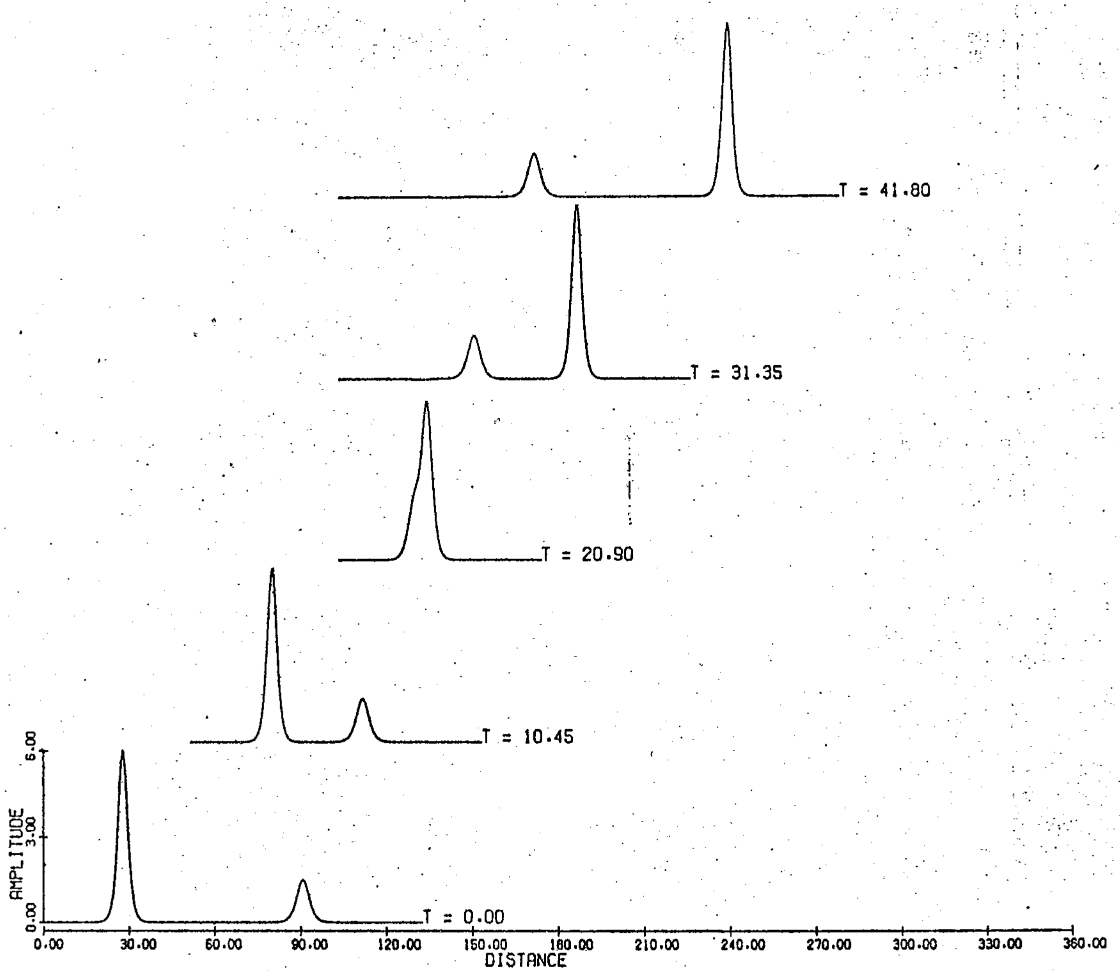

Figure 6. Interaction of solitary-wave solutions of (27).

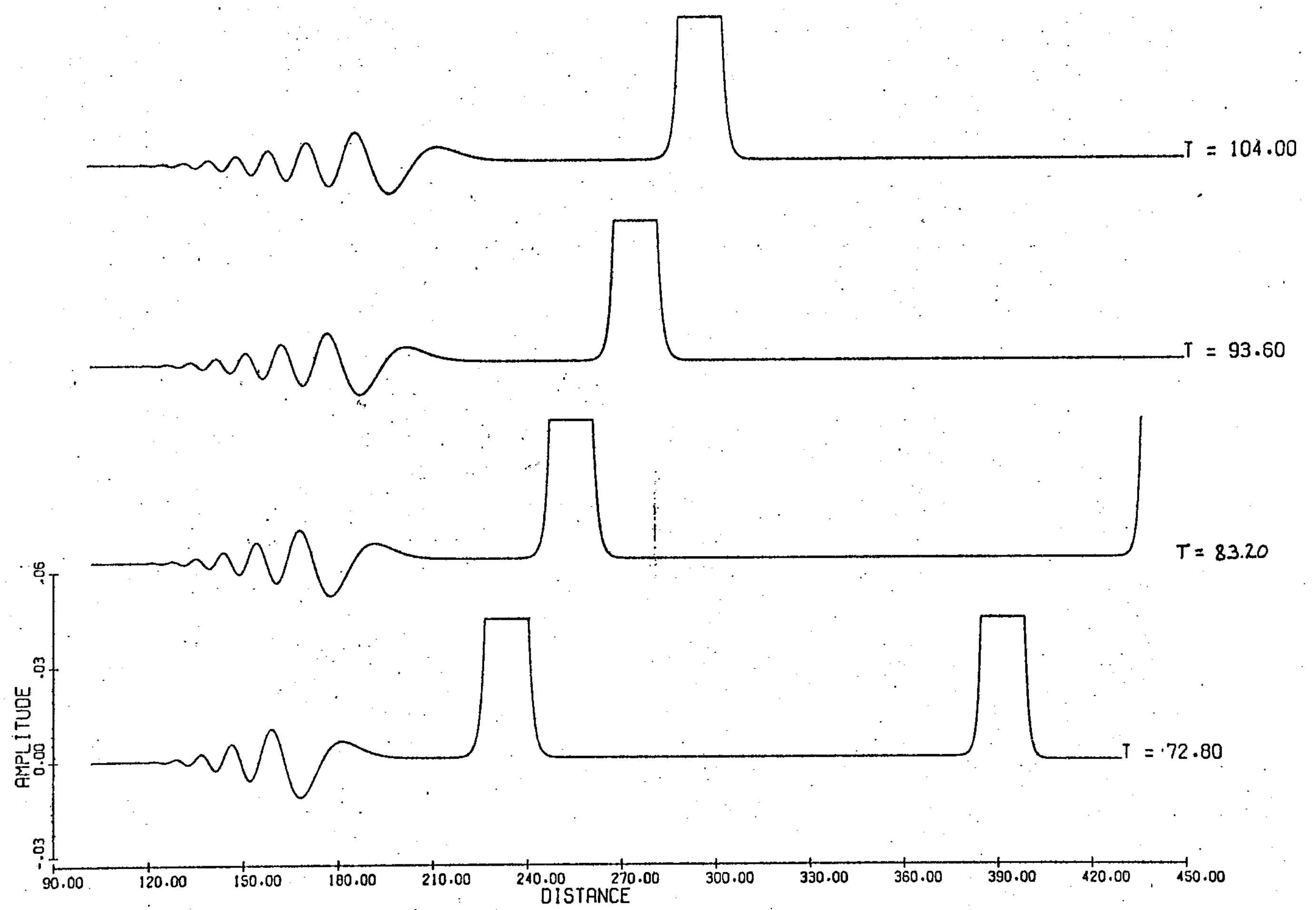

Figure 7. Interaction of solitary-wave solutions of (27), with the vertical scale magnified 100 times.

From the point of view of potential applications of these model equations, the resolution of a fairly general wave profile into a sequence of solitary waves is perhaps more significant than the exact interaction of their solitary-wave solutions. Concerning the former property, the evidence so far available for (27) is the result of numerical experiments conducted by the author in collaboration with W. G. Pritchard and L. R. Scott. On the basis of these, it is tempting to assert that equation (27) does indeed have some sort of resolution property, just as the KdV equation does. An example of the sorting of an initial wave profile into solitary waves, for equation (27), is shown in figure 8.

To some extent, the preceding discussion applies to the wider class of model equations

$$\eta_t + \eta_x + G(\eta)_x + (M\eta)_x = 0, \tag{30}$$

and

$$\eta_t + \eta_x + G(\eta)_x - (M\eta)_t = 0. \tag{31}$$

Equations having the form depicted in (30) were introduced already in (18). The alternate form, given in (31), may be obtained as an approximate equation by the same type of arguments put forward in deducing the model (20) from the KdV equation. A more thorough discussion of this issue may be found in Bona (1979). Specific examples of these sorts of equations have appeared in the literature as approximate models for diverse physical phenomena (cf. Benjamin et al, 1972, Leibovich and Randall, 1972, Pritchard, 1970, Smith, 1972, Whitham, 1974, and Zabusky, 1967).

Some of the general mathematical properties of solutions of these equations have been investigated by Benjamin et al (1972), Benjamin and Bona (1980) and Saut (1979). One outcome of these studies is that the initial-value problem, where $\eta(x,0)$, say, is a prescribed function $\varphi(x)$, is well-posed. For equations in the form (31), the theory is especially satisfactory, including the solutions' continuous

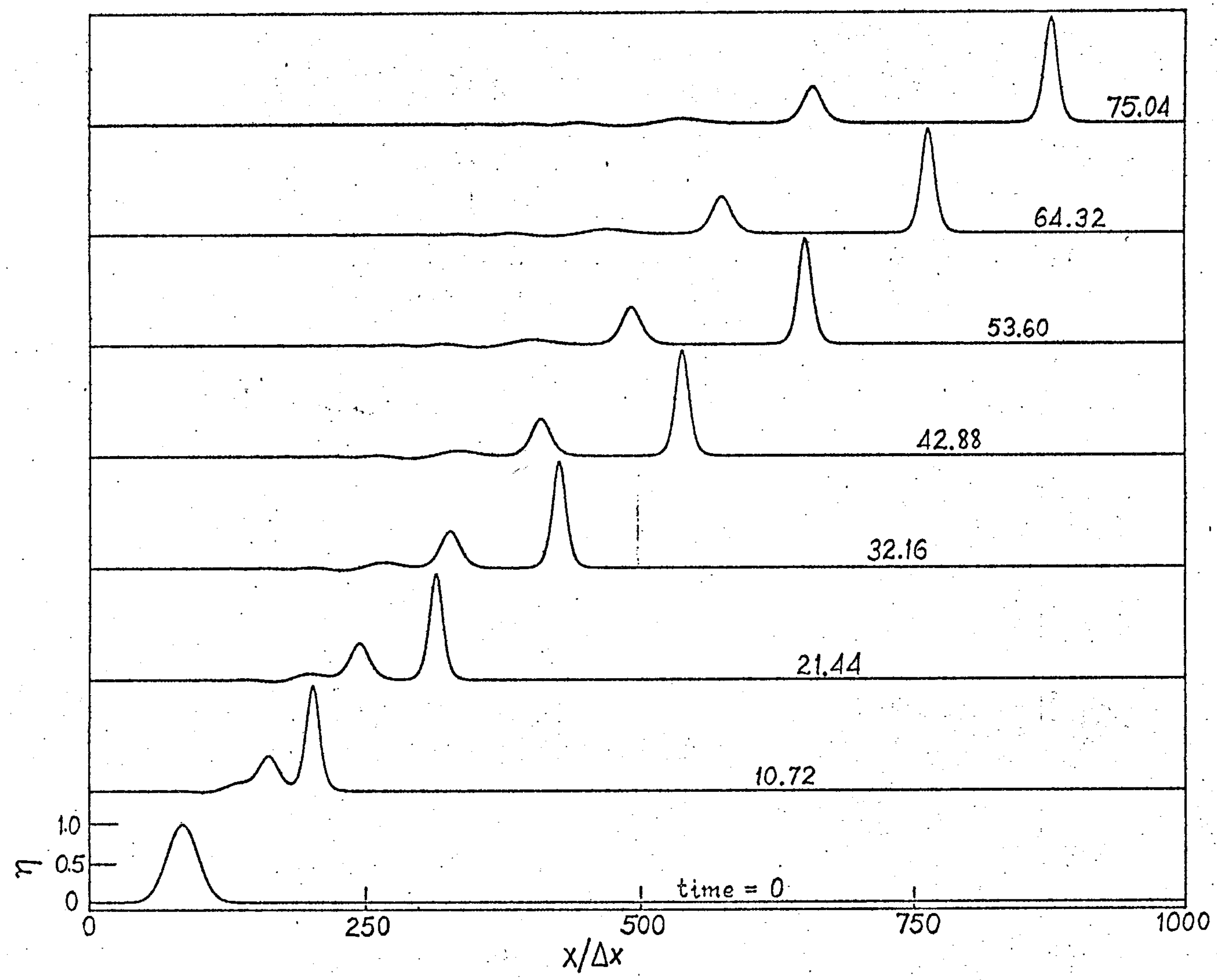

Figure 8. A solution of (27) breaking up into a sequence of solitary waves.

dependence, not just on the data φ, but on G and M as well. This latter property may be viewed as a kind of continuity of the modeling process, in that small errors in the form of the approximation made for nonlinearity or dispersion lead only to small changes in the resulting solutions.

One of the earlier examples in which the generalized form shown in (30) arose in fluid mechanics is the so-called Benjamin-Ono equation. This equation was introduced by Benjamin (1967) as a model for internal waves in a deep and stably stratified ocean. It has $G(u) = u^2$ and M defined as in (15) by

$$\widehat{Mv}(k) = -|k|\hat{v}(k), \tag{32}$$

where, as previously, the circumflex over a function connotes that function's Fourier transform. Benjamin was able to give, in closed form, solitary-wave solutions to his model equation. These played an important role in various interpretive aspects of internal waves treated by Benjamin.

Recently, Hirota and Nakamura (1979) have shown that Benjamin's equation possesses a full-fledged inverse-scattering formalism, despite the non-local character of the operator M in (32). It follows that the kind of special properties discussed above, pertaining to the solutions of the KdV equation, also hold for Benjamin's equation. Hirota and Nakamura's result has been generalized somewhat by Satsuma, Ablowitz, and Kodama (1979) to include a range of equations, of which KdV and Benjamin's equation are particular limiting forms.

In addition, a broad class of these types of equations have been shown to be possessed of solitary-wave solutions. Suppose, for example, that we search for a solution of (31) in the form of a travelling wave,

$$u(x,t) = \psi(x - Ct),$$

where $C > 1$ is a constant. Substituting this form into (31), and integrating once, there appears

$$(C - 1)\psi - CM\psi = G(\psi). \qquad (33)$$

The constant of integration has been found to be zero, by requiring that ψ and $M\psi$ tend to zero at infinity.

The operator $-M$ is typically a positive operator. For instance, in equation (27), $-M = -\partial_x^2$. Since $C > 1$, it follows that the operator,

$$B = (C - 1)I - CM,$$

where I is the identity operator, is invertible. More precisely, in the notation introduced in (15),

$$\widehat{B\psi}(k) = [C - 1 - C\alpha(k)]\hat{\psi}(k),$$

and normally, $-\alpha(k) = 1 - \mathcal{C}(k) \geqslant 0$, corresponding to the aforementioned fact that, for infinitesimal waves, the longer waves travel faster. Hence B^{-1} is formally given by

$$\widehat{B^{-1}\psi}(k) = \frac{1}{[C - 1 - C\alpha(k)]}\hat{\psi}(k),$$

and this formula defines a bounded linear operator, for example, on the space $L_2(\mathbb{R})$ of functions f with finite square integral

$$\int_{-\infty}^{\infty} f^2(x)\, dx < +\infty.$$

Applying B^{-1} to both sides of (33) yields

$$\psi = B^{-1}G(\psi) = A\psi, \qquad (34)$$

say.

By these formal manipulations, the question of the existence of solitary-wave solutions of (31) has been reduced to a question of existence of a fixed point of the nonlinear operator A defined in (34). That is, we seek a function ψ that A maps into itself. Of course, ψ must have the kind of single-crested non-negative profile that is appropriate to a solitary wave. Subject to various technical assumptions, which are ignored here, (34) has been shown to possess the desired kind of solution, and as a consequence, (31) has been inferred to have solitary-wave solutions. These results are set out in more detail in Benjamin (1977), Benjamin et al (1976), and Bona and Bose (1978). Non-constructive methods of functional analysis are used in the last-cited works, and therefore formulae for these solitary-waves are not available. However, preliminary numerical simulations of some of these equations show that, generally, there are classes of initial data that evolve into a sequence of what appear to be solitary-wave solutions of the particular equation. These issues plainly need further investigation.

The decomposition of initial waveforms into solitary waves may have implications for the time scales over which it is reasonable to use these types of approximate models. Reverting to the scaled versions (21) and (22) of the main equations under consideration, suppose that an initial wave profile has sorted itself into solitary waves well before the formal breakdown time $t_o=1/\varepsilon^2$. From the time of sorting onward, the evolution of the bulk of the solution of the equation is quite simple. Since the solitary waves are sorted, meaning their crests are far enough apart that the interaction between them is negligible, they propagate virtually independently and their separation increases with time. If the solution of the equation is still a good approximation to what is happening in the physical system being modeled, as it should be well before t_o, then the physical wave is sensibly a sequence of solitary waves too. It seems possible that the further evolution of the physical waves may be quite simple as well. In consequence, what is predicted by the model equation, on even quite long time scales, may not be qualitatively very different from what one observes. Of course, quantitative agreement will probably be poor, if for no other reason than the accumulation over time of small differences in phase velocity.

APPENDIX

Formula (19) in the main text presents eight separate model equations for the propagation of one-dimensional long waves of small amplitude in certain kinds of nonlinear dispersive media. This may strike the reader as an embarrassment of riches. As mentioned earlier, whilst the eight equations represented in (19) are formally equivalent, there may be criteria for choosing amongst them in any particular context of their potential use. For example, some studies might usefully employ the inverse-scattering theory, and so at present the KdV equation would be preferred. Or, an equation which is easy to handle numerically might be desirable, and thus one or another of these equations might be singled out. If a physically-relevant initial-boundary-value problem is not well-posed for one of the equations, there may be grounds for rejecting the equation, at least for the application in view.

We are not presently in a position to give a definitive theory for all eight of these partial differential equations. However, one aspect does appear to shed light on some of the differences inherent in these models. This is the various linear dispersion relations represented in these equations. The four dispersion relations $\omega = \omega(k)$, and their associated group velocities $\omega'(k)$, represented in (19) are given below in (A1). These dispersion relations all agree for very long waves, as they must. But they are quite disparate away from $k = 0$.

	$c(k) = \dfrac{\omega(k)}{k}$	$\omega'(k) = \dfrac{d\omega}{dk}$
i)	$1 - \gamma k^2,$	$1 - 3\gamma k^2,$
ii)	$\dfrac{1}{1 + \gamma k^2},$	$\dfrac{1 - \gamma k^2}{(1 + \gamma k^2)^2},$
iii)	$\dfrac{(1 + 4\gamma k^2)^{1/2} - 1}{2\gamma k^2},$	$\dfrac{(1 + 4\gamma k^2)^{1/2} - 1}{2\gamma k^2 (1 + 4\gamma k^2)^{1/2}},$
iv)	$\dfrac{3(r(X) - r(-X))}{2X},$	$\dfrac{r(X) + r(-X)}{2(1 + X^2)^{1/2}},$

(A1)

where $X = (3/2)(3\gamma)^{1/2}k,$

$$r(X) = [(1 + X^2)^{1/2} + X]^{1/3},$$

and the positive branch of all the fractional powers is understood. In a wave train, the phase velocity $c(k)$ governs the propagation of individual crests whilst the group velocity governs the propagation of energy, for the respective linear equations from which the dispersion relations are derived (cf. Whitham, 1974). Note especially that in (A1iii) and (A1iv) the dispersion relations have more than one branch. Only that branch which is relevant to the modeling situation envisaged is displayed.

In practical situations, such models are used in regimes where k is not especially small. For example, in the experimental studies of Boczar-Karakiewicz, Bagińska, and Bona (1980), Bona et al (1980b), Hammack (1973), Hammack and Segur (1974), and Zabusky and Galvin (1971) one will find non-trivial amounts of energy in wave numbers k larger than 0.5, in the non-dimensional but unscaled coordinates used in (25). Hence it may be prudent to choose among the possibilities in (19) a dispersion relation that fits closely the dispersion relation for the equations being modeled over a reasonably large range of wave numbers. For surface water waves the dispersion relation for the linearized Euler equations is given in (9). A short calculation shows that, among the possibilities in view, (A1iii) is the best fit to (9), with (A1ii) and (A1iv) both better than (A1i). (To make the comparison directly with the formula in (9), γ must be taken equal to 1/6, corresponding again to the variables introduced near (25).)

Another consideration is that numerical schemes for the integration of these equations naturally introduce short-wave components, with which the model must then contend. If the phase and group velocities are unbounded as in (A1i), there may be difficulties writing a numerical scheme which is stable for reasonable size temporal discretizations. The multi-valued dispersion relations in (A1iii) and (A1iv) may also contribute numerical difficulties. For while the branch written down in (A1) is nicely behaved, the other branches are problematical with regard to the modeling situation being considered herein.

The question of the well-posedness of certain initial- and boundary-value problems associated to the modeling scenario may also be important. Perhaps the least technical of the potentially interesting problems to pose is the pure initial-value problem, as in (23), where the wave profile is supposed to be known everywhere, at a given instant of time. That is,

$$\eta(x,0) = \varphi(x), \text{ for } x \in \mathbb{R}, \tag{A2}$$

where φ is selected from some reasonable class of functions appropriate to wave propagation. This problem is well-posed for the first two dispersion relations in (A1).

At first sight, it may appear that the equations in (A1) with higher than first-order temporal derivatives are ill-suited to the prescription of only the auxillary data in (A2).

To make the problem well-posed, $\eta_t(x,0)$ would generally need to be specified, and this is not normally desirable. This criticism may be circumvented by again using the lowest order approximation for all these equations, given in (3). As an example, it might make sense to consider the problem,

$$\eta_t + \eta_x + \beta\eta\eta_x + \gamma\eta_{xtt} = 0,$$
$$\eta(x,0) = \varphi(x), \quad \eta_t(x,0) = -\varphi'(x). \qquad (A3)$$

For the equations with a third-order temporal derivative, where three pieces of initial data need to be specified, it may be appropriate to set $\eta_{tt}(x,0) = \varphi''(x)$.

These issues will not be pursued here. The main point to be drawn from the discussion in this appendix is that a choice of model equation in a certain context may depend upon considerations which lie partly or wholly outside that context. Even if a model is determined rationally, via formal expansion techniques, matched asymptotics or the like, it may not be unique within its range of approximation. Nor indeed does there have to be a single "correct" model for a given physical situation, at a given level of approximation, as the theorem quoted earlier indicates.

In a way, this point is a negative one. For it suggests that successful model-building is as much an art as a science.

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