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Travelling-wave solutions to the Korteweg-de Vries-Burgers equation

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Synopsis

The existence and certain qualitative properties of travelling-wave solutions to the Korteweg-de Vries-Burgers equation,

$$u_t + uu_x + \delta u_{xxx} - \varepsilon u_{xx} = 0,$$

are established. The limiting behaviour of these waves, when ε tends to zero and when δ tends to zero is examined together with a singular limit wherein both ε and δ tend to zero.

1. Introduction

The Korteweg-de Vries equation,

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{1.1}$$

posed for $x \in \mathbb{R}$ and $t \geqq 0$, has been proposed as a model for small-amplitude, long waves in many different physical systems. It incorporates effects of dispersion and of nonlinear convection and has here been written relative to coordinates travelling to the right at a normalised speed one. This equation, or its near relatives, has been shown to yield good qualitative predictions of various observable phenomena (cf. [**11**], [**21**]). However, to effect quantitative agreement of predictions with experimentally obtained data, dissipation may need to be accounted for, at least approximately. Dissipative mechanisms corresponding to physical systems for which (1.1) may serve as an approximate model are disparate and not always well understood. Accordingly, the equation

$$u_t + uu_x + \delta u_{xxx} - \varepsilon u_{xx} = 0 \tag{1.2}$$

has gained some prominence in studies where the practical need to model dissipation arises (cf. [**5**], [**10**], [**13**], [**14**], [**15**]). Here ε and δ are positive parameters. We shall refer to this model equation as the Korteweg-de Vries-Burgers equation (KdVB equation henceforth, the Korteweg-de Vries equation itself will be abbreviated as KdV equation) since it represents a marriage of the Korteweg-de Vries equation and the classical Burgers equation,

$$u_t + uu_x - \varepsilon u_{xx} = 0. \tag{1.3}$$

A number of theoretical issues concerning (1.2) may have potential bearing on its ultimate applicability as a mathematical model. Here consideration is given to the travelling-wave solutions to (1.2), which are solutions of the form

$$u(x, t) = S(x - ct; \varepsilon, \delta), \tag{1.4}$$

where c is a fixed positive constant. Despite their very special nature, such solutions are understood to play a significant role in the evolution of a large class of initial profiles for both (1.1) and (1.3). A similar situation may well obtain for the KdVB equations, though this is conjectural at present (cf. [**7**]). (Other theoretical issues relating to the KdVB equation have been addressed in [**1**], [**5**], [**6**], [**7**], [**13**], [**16**], and [**19**], for example.)

The travelling-wave solutions of the KdVB equation that are discussed here have already attracted some attention. We point especially to Grad and Hu [**10**] who studied what amounts to our equation (2.6) in the context of weak shocks in a cold plasma. Johnson [**14**] suggested equation (1.2) as an approximate model equation for waves in physical systems in which the weak effects of nonlinearity, dissipation, and dispersion are present. He was especially interested in the travelling-wave solutions and developed formal asymptotic expansions for the particular regimes where $\varepsilon \ll \delta$ and where $\delta \ll \varepsilon$. Interesting and suggestive numerical calculations concerning the KdVB equation with especial focus on the travelling-wave solutions were reported by Canosa and Gazdag [**7**]. Particular aspects of their study will be referred to later. Mention should also be made of the brief but incisive remarks of Whitham [**20**].

The present study goes beyond these earlier works in several respects. First, the rigorous mathematical issues are clarified and set in good order, especially as regards the uniqueness of these special waveforms. Second, supplementing the work of Johnson mentioned above, the behaviour of S in the limits $\varepsilon \downarrow 0$ or $\delta \downarrow 0$ is analysed.

The paper comprises seven sections including the introduction. Section 2 is devoted to establishing the existence and uniqueness of the solutions to (1.2) having the form (1.4), and corresponding to given positive values of c, ε and δ. In Section 3, certain qualitative aspects of these solutions are proved, following on the remarks in the earlier works mentioned above. Sections 4 and 5 delve into the limiting behaviour of $S(\cdot; \varepsilon, \delta)$ as $\varepsilon \downarrow 0$ and as $\delta \downarrow 0$, respectively. In Section 6, the limiting form of $S(\cdot; \varepsilon, \delta)$ as ε and δ tend to zero is addressed, whilst Section 7 contains some brief concluding remarks. A more detailed view of the outcome of the forthcoming analysis is given below.

Note that the function S necessarily satisfies the ordinary differential equation

$$-cS' + SS' + \delta S''' - \varepsilon S'' = 0. \tag{1.5}$$

Existence and uniqueness is established by global analysis of a vector field associated with (1.5). For fixed positive values of ε and δ it is proved that (1.5) has a unique bounded solution $S = S(\xi) = S(\xi; \varepsilon, \delta)$ such that

$$S_L = \lim_{\xi \to -\infty} S(\xi) \quad \text{and} \quad S_R = \lim_{\xi \to +\infty} S(\xi) \tag{1.6a}$$

exist. The function S then also satisfies the additional asymptotic conditions

$$\lim_{|\xi| \to +\infty} S^{(j)}(\xi) = 0, \quad j = 1, 2, 3, \ldots. \tag{1.6b}$$

Here, $S^{(j)}$ denotes the j-th derivative of S with respect to $\xi = x - ct$ and the two asymptotic states S_L and S_R are restricted by the requirements that $c > S_R$ and $S_L + S_R = 2c$.

An analysis of the linearisation of equation (1.5) around the critical points of the associated vector field yields information about the geometry of these solutions. If dissipation dominates, in the precise sense that

$$\varepsilon^2 \geqq 4\gamma\delta \tag{1.7}$$

where $\gamma = c - S_R$, then S resembles the familiar monotone travelling-wave solution of Burgers equation (1.3). If dispersion dominates, in the sense of the negation of (1.7), then S has an oscillatory character as $\xi \to -\infty$ found commonly in the theory of weak undular bores (cf. [**15**], [**18**], [**20**]).

For fixed positive δ, $S(\xi; \varepsilon, \delta)$ converges to a solitary-wave solution,

$$S_R + 3\gamma \operatorname{sech}^2 \left[\left(\frac{\gamma}{4\delta} \right)^{\frac{1}{2}} (x - x_0 - ct) \right], \tag{1.8}$$

of the Korteweg-de Vries equation (1.1), as $\varepsilon \downarrow 0$. Here γ is as above and x_0 specifies the location of the wave's crest at $t = 0$. This convergence is somewhat subtle because of the oscillatory nature of S when $\varepsilon \ll \delta$, and because the KdV equation itself does not admit a travelling-wave solution having distinct limits at $\pm\infty$.

For fixed $\varepsilon > 0$, the limiting form of S as $\delta \downarrow 0$ proves to be the usual travelling-wave solution to Burgers equation with speed of propagation c, namely,

$$S_R + \gamma \left\{ 1 - \tanh \left[\frac{\gamma}{\varepsilon} (x - x_0 - ct) \right] \right\}. \tag{1.9}$$

If ε and δ are both allowed to tend to zero, but in such a way that the quotient δ/ε^2 remains bounded, then S tends to the step function χ given by

$$\chi(\xi) = \begin{cases} S_L, & \text{for } \xi < 0, \\ S_R, & \text{for } \xi > 0. \end{cases}$$

The function $\chi(x - ct)$ is the well-known, weak, travelling-wave solution to the conservation law $u_t + uu_x = 0$.

It is worth noting that the results concerning travelling-wave solutions to the KdVB equation obtained in this paper go over without essential change to the alternate model equation

$$u_t + u_x + uu_x - \delta u_{xxt} - \varepsilon u_{xx} = 0,$$

derived in [**3**] and [**18**], and which was the focus in [**5**] of an extended comparison of numerical predictions and experimental observations in a wave tank.

2. Existence and uniqueness of the travelling wave

Here the fundamental question of the existence and uniqueness of a travelling-wave solution to (1.2) is addressed. It will often be convenient to suppress the dependence of a travelling wave $S = S(\xi; \varepsilon, \delta)$ on ε and δ, by writing simply $S(\xi)$. As was already noted, S satisfies the equation

$$-cS' + SS' + \delta S''' - \varepsilon S'' = 0, \tag{2.1}$$

where $'$ denotes differentiation with respect to $\xi = x - ct$ and $c > 0$ is the speed of propagation of the wave. A search will be initiated for a bounded solution to (2.1) joining states S_L and S_R. That is, S will be determined so that (1.6a) holds. It is useful to normalise S by defining

$$s(\xi) = S(\xi) - S_R. \tag{2.2}$$

It follows immediately that

$$-(c - S_R)s' + ss' + \delta s''' - \varepsilon s'' = 0.$$

If we define $\gamma = c - S_R$, this latter equation has the same form as equation (2.1) with γ playing the role of c. Thus equation (2.1) may be represented as

$$-\gamma s' + ss' + \delta s''' - \varepsilon s'' = 0. \tag{2.3}$$

For s, the limiting conditions at infinity are

$$\lim_{\xi \to -\infty} s(\xi) = s_0 = S_L - S_R \quad \text{and} \quad \lim_{\xi \to +\infty} s(\xi) = 0. \tag{2.4}$$

The conditions

$$\lim_{|\xi| \to +\infty} s^{(j)}(\xi) = 0, \quad j = 1, 2, 3, \ldots \tag{2.5}$$

corresponding to (1.6b) will also be imposed. (In fact, (2.5) follows from (2.3) and (2.4) as we shall see.) Notice that if s, s' and s'' vanish at $+\infty$, then by integrating equation (2.3) over $[y, \infty)$, it is confirmed that (2.3) is equivalent to

$$-\gamma s(y) + \tfrac{1}{2}s^2(y) + \delta s''(y) - \varepsilon s'(y) = 0. \tag{2.6}$$

LEMMA 1. *Let s be a non-constant solution to* (2.3) *satisfying* (2.4) *and* (2.5). *Then $\gamma > 0$ and $s_0 = 2\gamma$.*

Proof. In equation (2.6), let $y \to -\infty$ to obtain

$$-\gamma s_0 + \tfrac{1}{2}s_0^2 = 0.$$

So, either $s_0 = 0$ or $s_0 = 2\gamma$. Now multiply (2.6) by s' and integrate the result over $\mathbb{R}$ to reach the relation

$$\varepsilon \int_{-\infty}^{\infty} s'(y)^2 \, dy = \tfrac{1}{2}\gamma s_0^2 - \tfrac{1}{6}s_0^3, \tag{2.7}$$

in which the left-hand side is inferred to be bounded since the right-hand side is bounded by assumption. Note that if $s_0 = 0$, then $s'(y) = 0$ for all y, so s is a constant function, contrary to hypothesis. Therefore, it must transpire that $s_0 = 2\gamma$, whence, from (2.7),

$$0 < \varepsilon \int_{-\infty}^{\infty} s'(y)^2 \, dy = \tfrac{2}{3}\gamma^3.$$

Thus γ must be positive and the lemma is established.

COROLLARY 2. *Let $u(x,t)=S(x-ct)$, $c>0$, be a travelling-wave solution to (1.2) that satisfies the asymptotic conditions (1.6a) and (1.6b). Then*

$$S_L - S_R = 2\gamma = 2(c - S_R) > 0, \tag{2.8}$$

or equivalently

$$c > S_R \quad \text{and} \quad S_L + S_R = 2c. \tag{2.9}$$

Remark. In what follows, it will always be assumed that (2.9) holds. Thus, for (2.3) it is presumed that $\gamma>0$ and that $s_0 = 2\gamma$.

The following result will be used several times in the subsequent analysis.

LEMMA 3. *Let s be a non-constant solution to (2.6). Suppose $s'(\xi_0)=0$ for some ξ_0 in $\mathbb{R}$. Then ξ_0 is an isolated extreme point of s and*

(i) *$s(\xi_0)$ is a local minimum of s if $s(\xi_0)\in(0,2\gamma)$, while*

(ii) *$s(\xi_0)$ is a local maximum of s if $s(\xi_0)\notin[0,2\gamma]$.*

Moreover, if s has a local maximum at a point ξ_1 where $s(\xi_1)<0$, then $s(\xi)<s(\xi_1)$ for all $\xi\neq\xi_1$.

Proof. If $s'(\xi_0)=0$, then from equation (2.6),

$$\delta s''(\xi_0) = \gamma s(\xi_0) - \tfrac{1}{2}s^2(\xi_0).$$

Therefore, $s''(\xi_0)<0$ if $s(\xi_0)\notin[0,2\gamma]$ and $s''(\xi_0)>0$ if $s(\xi_0)\in(0,2\gamma)$. If $s''(\xi_0)=0$, then s is constant, contrary to hypothesis. Thus ξ_0 is an isolated extreme point and (i) and (ii) are valid.

Suppose s has a local maximum at ξ_1 and $s(\xi_1)<0$. From above, ξ_1 is an isolated local extremum. If the desired conclusion is false, then there is a point ξ_2 closest to ξ_1 where $s(\xi_2)=s(\xi_1)$. Somewhere in the interval between ξ_1 and ξ_2 s must take a local minimum value, which contradicts (i). The proof of the lemma is complete.

Because of the asymptotic conditions in (2.4) and (2.5), equation (2.3) is equivalent to equation (2.6). Following Johnson [**14**], we define an auxiliary dependent variable $r=\delta s'$. Then (2.6) is equivalent to the first-order system

$$\begin{aligned} s' &= \delta^{-1} r, \\ r' &= \gamma s + \varepsilon\delta^{-1} r - \tfrac{1}{2}s^2. \end{aligned} \tag{2.10}$$

The system (2.10) has just the two critical points $(0,0)$ and $(2\gamma,0)$. The eigenvalues of the system (2.10), when linearised about $(0,0)$, are

$$\lambda_\pm = [\varepsilon \pm (\varepsilon^2 + 4\gamma\delta)^{\frac{1}{2}}]/2\delta, \tag{2.11}$$

and the eigenvalues of the system, when linearised about $(2\lambda, 0)$, are

$$\Lambda_\pm = [\varepsilon \pm (\varepsilon^2 - 4\gamma\delta)^{\frac{1}{2}}]/2\delta. \tag{2.12}$$

Thus $(0,0)$ is always a stable saddle point while $(2\gamma,0)$ is a nodal point if $\varepsilon^2 \geqq 4\gamma\delta$ and a spiral point if $\varepsilon^2<4\gamma\delta$. Since $\gamma>0$, Re $(\Lambda_\pm)>0$, regardless of the relative sizes of ε and δ, and so $(2\gamma,0)$ is always unstable.

Let $\mathcal{R}$ be any bounded orbit of the system (2.10). We inquire into its asymptotic states at $+\infty$ and $-\infty$, its ω and α limit sets, respectively.

First note that (2.10) admits no non-constant periodic solutions. For if (s_p, r_p) is a solution to (2.10) that is periodic of period $p>0$, then s_p is a periodic solution to (2.6). Multiply (2.6) by s_p' and integrate the result over a period. By periodicity, all terms integrate to zero except one, and so there remains the identity,

$$\varepsilon \int_0^p s_p'(y)^2 \, dy = 0.$$

It follows that s_p is constant, and so (s_p, r_p) is just one of the two critical points.

Because there are no non-trivial periodic solutions to (2.10) both the ω limit set and the α limit set must contain critical points of the system. (They might also contain orbits connecting critical points, but that is easily excluded here as will presently become apparent.) Since these sets are connected, they must each contain exactly one critical point, and hence the orbit must tend asymptotically to a critical point, both at $+\infty$ and $-\infty$. It follows from (2.10) that the asymptotic conditions (1.6b) pertain to this orbit. The limits at $\pm\infty$ cannot be the same, for this would lead to a solution S to (2.1) with $S_L = S_R$, which possibility is excluded by Corollary 2. Hence any bounded orbit of (2.10) necessarily connects the critical points $(0, 0)$ and $(2\gamma, 0)$. Since $(0, 0)$ is stable and $(2\gamma, 0)$ is unstable, $\mathcal{R}$ must tend to $(0, 0)$ at $+\infty$ and to $(2\gamma, 0)$ at $-\infty$. Moreover, since $(0, 0)$ is a saddle point, the general theory pertaining to such systems implies that there are exactly two semi-orbits of (2.10) that converge to $(0, 0)$ as $\xi \to +\infty$, and they both approach the origin at the angle whose tangent is $\delta\lambda_-$ (cf. [**8**, chapters 13 & 15] or [**12**, Chapter VIII]). That is, up to a translation of the independent variable ξ, there are two solutions $(s(\xi), r(\xi))$ to (2.10) such that $(s(\xi), r(\xi)) \to (0, 0)$ as $\xi \to +\infty$, and for each of these solutions,

$$\lim_{\xi \to +\infty} \left[\frac{r(\xi)}{s(\xi)}\right] = \delta\lambda_-. \tag{2.13}$$

One orbit approaches the origin from the fourth quadrant $Q_4 = \{(s, r): s>0, r<0\}$ in the phase plane whilst the other approaches through the second quadrant $Q_2 = \{(s, r): s<0, r>0\}$. According to the above analysis, the continuation of one or the other of these semi-orbits to all $\xi \in \mathbb{R}$ provides the only possibilities for bounded orbits. Consequently, if it can be shown that exactly one of these orbits is bounded, the following theorem will be established.

THEOREM 4. *Let γ, δ and ε be given positive numbers. There exists a unique bounded orbit $\mathcal{R}$ of the system* (2.10) *corresponding to these values. Moreover, $\mathcal{R} \subseteq \{(s, r): 0<s<3\gamma\}$ and $\mathcal{R}$ tends to $(2\gamma, 0)$ at $-\infty$ and to $(0, 0)$ at $+\infty$.*

Proof. Let $(s(\xi), r(\xi))$ be a solution to (2.10), defined at least for large values of ξ, that corresponds to one of the two semi-orbits that approach the origin. Since both $s(\xi)$ and $r(\xi)$ tend to 0 as $\xi \to +\infty$, it follows from (2.10) that $s'(\xi)$ and $r'(\xi) = \delta s''(\xi)$ both tend to 0 as $\xi \to +\infty$. A straightforward induction confirms that $s^{(j)}(\xi) \to 0$, as $\xi \to +\infty$, for all $j \geqq 0$.

Since $(s(\xi), r(\xi))$ satisfies (2.10), s satisfies (2.6). If equation (2.6) is multiplied by s' and the result integrated over $[y, \infty)$, we obtain

$$\tfrac{1}{2}\gamma s(y)^2 - \tfrac{1}{6}s(y)^3 = \varepsilon \int_y^\infty s'(\xi)^2 \, d\xi + \tfrac{1}{2}s'(y)^2. \tag{2.14}$$

It is known from Lemma 3 that the zeros of s' are isolated. Hence the integral is strictly positive for any value of y for which s is defined for all $\xi \geqq y$. Therefore, for any such y,

$$\tfrac{1}{2}\gamma s(y)^2 - \tfrac{1}{6}s(y)^3 > 0,$$

from which one concludes that $s(y) < 3\gamma$ and that s never vanishes.

Consider the orbit $\mathcal{R}_0$ that approaches $(0, 0)$ from Q_2. For large values of ξ, $s(\xi) < 0$. Since s never vanishes, this orbit must have $s(\xi) < 0$ for all ξ for which the solution is defined. Hence, this orbit cannot converge to $(2\gamma, 0)$ as $\xi \to -\infty$, and in consequence of our previous remarks it cannot be bounded.

The only possibility for a bounded orbit thus lies with the semi-orbit represented, say, by $(s(\xi), r(\xi))$ that approaches $(0, 0)$ from Q_4. Since $s(\xi) > 0$ for large ξ, it follows that $s(\xi) > 0$ for all ξ for which the solution is defined. Thus $s(\xi) \in (0, 3\gamma)$ for all ξ over which s extends.

The system (2.10) is locally Lipschitz and thus it is assured that $\mathcal{R}$ can be extended either over all $\xi \in \mathbb{R}$ or until it becomes unbounded. As $s(\xi)$ is already known to be bounded, if r can be shown to be bounded the theorem would follow.

First it is claimed that $r(\xi) \geqq -\delta\gamma^2/2\varepsilon$, say, for any ξ. Suppose this not to be the case and let $-\mu < -\delta\gamma^2/2\varepsilon$. Then, because $r(\xi) \to 0$ as $\xi \to +\infty$, there is a largest ξ_0 where $r(\xi_0) = -\mu$, and for $\xi > \xi_0$, $r(\xi) > r(\xi_0)$. In particular, $r'(\xi_0) \geqq 0$. But from (2.10).

$$\begin{aligned} r'(\xi_0) &= \varepsilon\delta^{-1}r(\xi_0) + \gamma s(\xi_0) - \tfrac{1}{2}s(\xi_0)^2 \\ &= -\varepsilon\mu\delta^{-1} + \gamma s(\xi_0) - \tfrac{1}{2}s(\xi_0)^2 \leqq -\varepsilon\mu\delta^{-1} + \tfrac{1}{2}\gamma^2 < 0, \end{aligned}$$

a contradiction. A similar argument shows that $r(\xi) < 3\gamma^2\delta/2\varepsilon$, for all ξ. Suppose this not to be true and let ξ_1 be the largest value for which $r(\xi_1) = 3\gamma^2\delta/2\varepsilon$, so that $r(\xi) < r(\xi_1)$ for $\xi > \xi_1$. Of course, $r'(\xi_1) \leqq 0$, but, on the other hand, since $s < 3\gamma$,

$$\begin{aligned} r'(\xi_1) &= \varepsilon\delta^{-1}r(\xi_1) + \gamma s(\xi_1) - \tfrac{1}{2}s(\xi_1)^2 \\ &= \tfrac{3}{2}\gamma^2 + \gamma s(\xi_1) - \tfrac{1}{2}s(\xi_1)^2 > 0, \end{aligned}$$

a contradiction. Thus the orbit $\mathcal{R}$ is bounded, and so defined for all $\xi \in \mathbb{R}$ and convergent to $(2\gamma, 0)$ as $\xi \to -\infty$. It follows as before that $s^{(j)}(\xi) \to 0$ as $\xi \to -\infty$, for all $j \geqq 1$. The theorem is proved.

This theorem is reinterpreted in terms of the problem to which attention was originally directed.

COROLLARY 5. *Let ε, δ and c be given positive constants, and suppose that c, S_R and S_L satisfy* (2.9). *Then, up to translation in the independent variable ξ, there is a unique solution $S(\xi)$ to* (2.1) *satisfying the asymptotic conditions* (1.6a). *Moreover, S also satisfies the additional boundary conditions* (1.6b).

Proof. Let $\gamma = c - S_R$ as before. By Theorem 4, there is a unique (up to translations) solution $(\bar{s}(\xi), \bar{r}(\xi))$ to the system (2.10) such that $\bar{s}(\xi) \to 0$ as $\xi \to +\infty$, and $\bar{s}(\xi)\to 2\gamma$ as $\xi\to-\infty$. Moreover, $\bar{s}^{(j)}(\xi)\to 0$ as $|\xi|\to\infty$, for all $j\geqq 1$. If $\bar{S}(\xi)=\bar{s}(\xi)+S_R$, it follows that $\bar{S}(\xi)$ is a solution to (2.1) satisfying both (1.6a) and (1.6b). If S is any solution to (2.1) satisfying (1.6a), let $s(\xi)=S(\xi)-S_R$ as before. Then s satisfies (2.3) and (2.4) with $s_0=2\gamma$. We may write (2.3) as

$$\frac{d}{d\xi}(-\gamma s+\tfrac{1}{2}s^2+\delta s''+\varepsilon s')=0,$$

whence it follows that there is a constant μ such that

$$-\gamma s+\tfrac{1}{2}s^2+\delta s''-\varepsilon s'=\mu. \tag{2.15}$$

If $\mu=0$, then s is a solution to (2.6) and therefore

$$\{(s(\xi), s'(\xi)) \colon -\infty<\xi<\infty\}$$

is a bounded orbit of the system (2.10). We know there is only one such orbit by Theorem 4, so after a translation of the independent variable, $s\equiv\bar{s}$, whence $S\equiv\bar{S}$.

The corollary will be in hand if it is adduced that μ must be zero. Since $s(\xi)\to 0$ as $\xi\to+\infty$, (2.15) implies that

$$\frac{d}{d\xi}(\delta s'-\varepsilon s)=\delta s''-\varepsilon s'\to\mu,$$

as $\xi\to+\infty$. Suppose $\mu\neq 0$. For sufficiently large ξ, $\delta s''(\xi)-\varepsilon s'(\xi)$ will be bounded away from zero. Hence for large enough ξ, $\delta s'(\xi)-\varepsilon s(\xi)$ is either bounded below by $\nu\xi$ or bounded above by $-\nu\xi$, for some positive constant ν, depending on whether $\mu>0$ or $\mu<0$, respectively. Since $s(\xi)\to 0$ as $\xi\to+\infty$, it follows that $s'(\xi)$ is either bounded below by $\nu\xi/2\delta$ or above by $-\nu\xi/2\delta$, say, for large enough ξ. But if $\xi>y$,

$$s(\xi)-s(y)=\int_y^\xi s'(r)\,dr.$$

By letting $\xi\to+\infty$, we obtain

$$s(y)=-\int_y^\infty s'(r)\,dr.$$

However, the integral is plainly divergent, in the light of the upper or lower bound on s'. We thus reach a contradiction, and so conclude that $\mu=0$. The corollary is now established.

3. Geometry of the travelling wave

A more detailed description of the travelling-wave solution to (1.2) will now be given. It will be shown that if dissipation dominates, in the sense that $\varepsilon^2\geqq 4\gamma\delta$, then S decreases steadily from S_L to S_R as ξ increases. Whereas, if dispersion dominates, in the sense that $4\gamma\delta>\varepsilon^2$, then S is monotone decreasing and convex as $\xi\to+\infty$, but oscillates infinitely often as $\xi\to-\infty$. This geometric information is deduced using a global analysis of the vector field associated with the system (2.10).

THEOREM 6. *Suppose* $\varepsilon^2\geqq 4\gamma\delta$. *Let* $s(\xi)=s(\xi;\varepsilon,\delta)$ *be the unique solution to*

(2.3)–(2.5). *Then, for all* $\xi \in \mathbb{R}$, $0 < s(\xi) < 2\gamma$, *and* $s'(\xi) < 0$. *Moreover, there is a unique inflection point* ρ_0 *of* s *such that* $(\xi - \rho_0)s''(\xi) > 0$, *for* $\xi \neq \rho_0$.

Proof. Let $(s(\xi), r(\xi))$ be a solution to (2.10) tracing the bounded orbit $\mathcal{R}$. Then $(s(\xi), r(\xi)) \to (0, 0)$ through the fourth quadrant Q_4, as $\xi \to +\infty$. Therefore, for large values of ξ, $s(\xi) > 0$ and $r(\xi) < 0$.

We first demonstrate that this latter conclusion is valid for all values of ξ. To this end, recall that it is already established that $\mathcal{R}$ cannot intersect the r-axis. Also, $\mathcal{R}$ cannot exit from Q_4 through the line segment l_0 given by

$$l_0 = \{(s, r) : r = 0, 0 \leqq s \leqq 2\gamma\}. \tag{3.1}$$

For if it did, let ξ_0 be the largest value for which $(s(\xi), r(\xi)) \in l_0$. For $\xi > \xi_0$, it must be the case that $r(\xi) < 0$. Hence, $r'(\xi_0) \leqq 0$. But $r(\xi_0) = 0$, so from (2.10),

$$r'(\xi_0) = \gamma s(\xi_0) - \tfrac{1}{2}s(\xi_0)^2 > 0$$

since $0 < s(\xi_0) < 2\gamma$. This is contrary to our presumption. Now define

$$l = \{(s, r) : r = m(s - 2\gamma), 0 \leqq s \leqq 2\gamma\}, \tag{3.2a}$$

where

$$m = \tfrac{1}{2}[\varepsilon - (\varepsilon^2 - 4\gamma\delta)^{\frac{1}{2}}], \tag{3.2b}$$

and the positive square root of $\varepsilon^2 - 4\gamma\delta$ is understood. Note that $m > 0$, in consequence of the assumption that $\varepsilon^2 \geqq 4\gamma\delta$. It will be shown that $\mathcal{R}$ never intersects l. On the supposition that the contrary holds, let ξ_0 be the largest value such that $(s(\xi), r(\xi)) \in l$. It follows that $0 \leqq s(\xi_0) \leqq 2\gamma$,

$$r(\xi_0) = m(s(\xi_0) - 2\gamma), \quad \text{and} \quad \frac{r'(\xi_0)}{s'(\xi_0)} \leqq m. \tag{3.3}$$

On the other hand, from (2.10),

$$\begin{aligned} m \geqq \frac{r'(\xi_0)}{s'(\xi_0)} &= \frac{\gamma s(\xi_0) - \frac{1}{2}s(\xi_0)^2 + \varepsilon\delta^{-1}r(\xi_0)}{\delta^{-1}r(\xi_0)} \\ &= \varepsilon - \frac{s(\xi_0)(s(\xi_0) - 2\gamma)}{2\delta^{-1}r(\xi_0)} \\ &= \varepsilon - \frac{\delta s(\xi_0)}{2m}. \end{aligned}$$

Hence

$$m^2 - \varepsilon m + \tfrac{1}{2}\delta s(\xi_0) \geqq 0.$$

But, since $s(\xi_0) < 2\gamma$,

$$m^2 - \varepsilon m + \tfrac{1}{2}\delta s(\xi_0) = m^2 - \varepsilon m + \gamma\delta + \delta(\tfrac{1}{2}s(\xi_0) - \gamma) = \tfrac{1}{2}\delta(s(\xi_0) - 2\gamma) < 0,$$

a contradiction.

Thus $\mathcal{R}$ is confined to the subset of Q_4 bounded by the r-axis and the line segments l and l_0, a conclusion with several consequences. First, $r(\xi) < 0$, for all ξ, or what is the same, $s'(\xi) < 0$, for all ξ. Thus s decreases monotonically from 2γ to

0 as ξ increases. Moreover, since $s'(\xi)\to 0$, as $\xi\to+\infty$, there must be points ξ where $s''(\xi)=0$, or equivalently, $r'(\xi)=0$. From (2.10), if $r'(\xi)=0$, then

$$r''(\xi)=s'(\xi)(\gamma-s(\xi)).$$

So if $s(\xi)<\gamma$, $r''(\xi)<0$ and if $s(\xi)>\gamma$, then $r''(\xi)>0$. In the special possibility where $s(\xi)=\gamma$, then $r''(\xi)=0$ and

$$r'''(\xi)=-(s'(\xi))^2<0.$$

Thus, at points ξ where $r'(\xi)=0$ we have the following trichotomy:

(i) If $0<s(\xi)<\gamma$, then ξ is a strict local maximum for r,

(ii) if $\gamma<s(\xi)<2\gamma$, then ξ is a strict local minimum for r, or

(iii) if $s(\xi)=\gamma$, then ξ is a saddle point for r.

Since $r(\xi)<0$ for all ξ, and $r(\xi)\to 0$, as $\xi\to\pm\infty$, r has a minimum value, which is taken on at some point ξ_0, say. It is claimed that this global minimum is the only local minimum that r possesses. If ξ_1 is any local minimum of r, then by (ii), $s(\xi_1)>\gamma$. Suppose $\xi_1\neq\xi_0$. Then between ξ_0 and ξ_1 there must be at least one local maximum value of r, and this would have to occur at a value of $\xi>\gamma$, contradicting (ii). Therefore, if $r'(\xi)=0$ for some $\xi\neq\xi_0$, it must be that $s(\xi)\leq\gamma$. If, on the other hand, $r'(\xi)=0$ at a point ξ where $s(\xi)<\gamma$, then ξ is a strict local maximum of r and $r(\xi)<0$. But $r(\xi)\to 0$ as $\xi\to+\infty$, and so there would have to be a $\bar{\xi}>\xi$ such that r takes a local minimum at $\bar{\xi}$. Since $s(\bar{\xi})<s(\xi)<\gamma$, this contradicts (i). Finally, if $r'(\xi)=0$ and $s(\xi)=\gamma$, then ξ is a saddle point for r, and, from (2.10), $r(\xi)=-\delta\gamma^2/2\varepsilon$. But in the proof of Theorem 4 it was shown that $r(\xi)\geqq-\delta\gamma^2/2\varepsilon$, for all ξ. Consequently, ξ cannot be a saddle point. Hence ξ_0 is the only point where r' or s'' vanishes, and $s(\xi_0)>\gamma$. It follows that $r'(\xi)$ has one sign for $\xi>\xi_0$, and similarly for $\xi<\xi_0$. Since $r(\xi_0)<0$ and $r(\xi)\to 0$ as $\xi\to+\infty$, it must be that $r'(\xi)>0$ for $\xi>\xi_0$. Likewise, $r'(\xi)<0$ for $\xi<\xi_0$ since $r'(\xi)\to 0$ as $\xi\to-\infty$. The theorem is proved.

Attention is now turned to the case in which the dispersion tends to dominate the dissipation.

THEOREM 7. *Suppose* $4\gamma\delta>\varepsilon^2$. *Let* $s(\xi)=s(\xi;\varepsilon,\delta)$ *be the unique (up to spatial translation) solution to* (2.3)–(2.5). *Then* $s(\xi)>0$ *for all* ξ, *and the following hold good.*

(a) *If* $M_0=\sup_\xi s(\xi)$, *then* M_0 *is attained at a unique value* $\xi=z_0$, *and for* $\xi>z_0$, $s'(\xi)<0$.

(b) *There is a* $\xi_0>z_0$ *such that* $(\xi-\xi_0)s''(\xi)>0$, *for all* $\xi>z_0$, $\xi\neq\xi_0$.

(c) *the solution* $s(\xi)$ *has an infinite number of local maxima and minima. These are taken on at points* $\{z_i\}_{i=0}^\infty$ *and* $\{w_i\}_{i=1}^\infty$ *where* $z_i>w_{i+1}>z_{i+1}$, *for all* $i\geqq 0$, *and* $\lim_{i\to\infty} z_i=\lim_{i\to\infty} w_i=-\infty$. *Moreover,*

$$2\gamma<s(z_{i+1})<s(z_i) \quad \textit{and} \quad 2\gamma>s(w_{i+1})>s(w_i),$$

for all i.

Proof. Since $\varepsilon^2-4\gamma\delta<0$, the critical point $(2\gamma,0)$ is an unstable spiral point of the system (2.10). The general theory [**12**, Chap. VIII] asserts that any solution $(s(\xi),r(\xi))$ to (2.10) that converges to $(2\gamma,0)$ as $\xi\to-\infty$ must have the form

$$(s(\xi)-2\gamma,r(\xi))=Ce^{\alpha\xi}(\cos(\beta\xi+\theta_0+o(1)),\sin(\beta\xi+\theta_0+o(1))) \tag{3.4}$$

as $\xi \to -\infty$, where $\alpha = \text{Re}(\Lambda_+) > 0$ and $\beta = \text{Im}(\Lambda_+) \neq 0$, Λ_+ is defined in (2.12), and C and θ_0 are constants that determine and are determined by the particular solution in question. The bounded orbit $\mathcal{R}$ of (2.10) therefore has the form given in (3.4) for large negative ξ, and so necessarily has an infinite number of points where r vanishes. According to Lemma 3, these are all strict local maxima or minima of s. Moreover, between any two strict local maxima (minima) there must be a local minimum (maximum). Hence s has an infinite number of maxima and minima, as $\xi \to -\infty$ and these are intertwined. Because $s(\xi) > 0$, for all ξ, from Theorem 4, a local maximum corresponds to the orbit crossing the s-axis at a point ξ where $s(\xi) > 2\gamma$, and a local minimum corresponds to a point ξ where $0 < s(\xi) < 2\gamma$.

Now, we know that $(s(\xi), r(\xi)) \in Q_4$ for large ξ and, as in the proof of Theorem 6, the orbit cannot intersect the line segment $\{(s, 0)\colon 0 \leqq s \leqq 2\gamma\}$ from Q_4. Thus $\mathcal{R}$ must exit Q_4 at a point z_0 where $s(z_0) > 2\gamma$ and $r(z_0) = 0$, so that z_0 is a strict local maximum of s and $s'(\xi) < 0$ for $\xi > z_0$. By arguing exactly as in the proof of Theorem 6, it is concluded that there is a point $\xi_0 > z_0$ where $s''(\xi_0) = 0$, and $s''(\xi) > 0$, for $\xi > \xi_0$, and $s''(\xi) < 0$, for $\xi < \xi_0$.

Let $z_1 < z_0$ be the nearest local maximum of s to z_0, and let w_1 be the unique intervening local minimum. By continuing in this way, we inductively define $\{z_i\}_{i=0}^{\infty}$, $\{w_i\}_{i=1}^{\infty}$ decreasing sequences such that z_i is a local maximum for s and w_i is a local minimum of s, and $z_i > w_{i+1} > z_{i+1}$, for all $i \geqq 0$. Of course, $s(z_i) > 2\gamma$ and $s(w_i) < 2\gamma$, for all i.

Naturally, $s(z_1) < s(z_0)$. For since all the maxima and minima are strict, the orbit $(s(\xi), r(\xi))$ always crosses the s-axis transversally. Hence $r(\xi) > 0$ for $w_1 < \xi < z_0$ and $r(\xi) < 0$ for $z_1 < \xi < w_1$. If $s(z_1) \geqq s(z_0)$, then the curve $\{(s(\xi), r(\xi))\colon w_1 \geqq \xi \geqq z_1\}$ would have to intersect the curve $\{(s(\xi), r(\xi))\colon \xi \geqq z_0\}$. That is, the orbit $\mathcal{R}$ would intersect itself, which is impossible. Inductively, using at each stage this simple argument, it is determined that $s(z_{i+1}) < s(z_i)$, for all i. Similarly, it is argued that $s(w_{i+1}) > s(w_i)$, for all $i \geqq 1$.

Finally the z_i and w_i can only accumulate at $-\infty$. Otherwise $(s(\xi), r(\xi))$ would take the value $(2\gamma, 0)$ at some finite ξ, which is again impossible because distinct orbits cannot intersect.

Interpreting Theorems 6 and 7 in terms of the original context, equation (2.1), the following result emerges.

COROLLARY 8. *Let ε, δ and c be given positive constants, and suppose c, S_R and S_L satisfy (2.9) so that $\gamma = c - S_R > 0$. Let $S(\xi) = S(\xi; \varepsilon, \delta)$ denote the unique (up to translations in ξ) bounded solution to (2.1) satisfying the asymptotic conditions (1.6).*

(i) *If $\varepsilon^2 \geqq 4\gamma\delta$, then $S'(\xi) < 0$, for all ξ, and S decreases monotonically from S_L to S_R as ξ increases. Moreover, there is a point ξ_0 such that $S''(\xi) > 0$ for all $\xi > \xi_0$ and $S''(\xi) < 0$ for $\xi < \xi_0$.*

(ii) *If $\varepsilon^2 < 4\gamma\delta$, then S takes its maximum value at a point z_0, and then $S(\xi)$ decreases monotonically from $S(z_0)$ to S_R as $\xi > z_0$ increases without bound. There is a point $\xi_0 > z_0$ such that $S''(\xi) > 0$ for $\xi > \xi_0$ and $S''(\xi) < 0$ for $z_0 < \xi < \xi_0$. Moreover, there are intertwined sequences $\{z_i\}_{i=0}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$, with $z_i > w_{i+1} > z_{i+1}$, for all $i \geqq 0$, decreasing to $-\infty$, and such that S takes a local maximum value at z_i and a local minimum value at w_i. Additionally, $S(w_{i+1}) < S_L < S(z_i)$, for $i \geqq 0$.* .

4. Limiting behaviour as ε tends to zero

In this section, the behaviour of the solution $S(\xi; \varepsilon, \delta)$ to (2.1) will be studied in the limit as ε tends to zero whilst δ remains fixed. It will be demonstrated that if $S(\xi; \varepsilon, \delta)$ is suitably normalised, then it converges, uniformly on rays of the form $\{\xi; \xi \geqq a\}$, to the solitary-wave solution to the KdV equation given in (1.8). This complements work of Johnson [**13**], [**14**], who, assuming the existence of S, developed a revealing asymptotic expansion, formally valid for small ε.

As before, we shall work with the dependent variable $s = S - S_R$, which is the unique solution (up to translations in ξ) to (2.3)–(2.5). Since δ is fixed, there is no loss of generality in taking $\varepsilon^2 < 4\gamma\delta$, where γ is the positive constant appearing in (1.3). For each such ε, let $s_\varepsilon(\xi)$ denote $s(\xi + \xi_0; \varepsilon, \delta)$, where ξ_0 is the unique point where $s(\cdot; \varepsilon, \delta)$ takes its maximum value (see Theorem 7). It follows from the previous theory that $s'_\varepsilon(\xi) < 0$, for all $\xi > 0$.

THEOREM 9. *Let $\{s_\varepsilon\}_{\varepsilon>0}$ be the family of travelling-wave solutions to* (2.3)–(2.5) *defined above. Then for any $a \in \mathbb{R}$, and any integer $j \geqq 0$,*

$$\lim_{\varepsilon \downarrow 0} \frac{d^j}{d\xi^j} s_\varepsilon(\xi) = \frac{d^j}{d\xi^j} u_\gamma(\xi),$$

uniformly on $\{\xi; \xi \geqq a\}$, where

$$u_\gamma(\xi) = 3\gamma \operatorname{sech}^2\left[\left(\frac{\gamma}{4\delta}\right)^{\frac{1}{2}} \xi\right]. \tag{4.1}$$

Proof. It has been established that $0 < s_\varepsilon(\xi) < 3\gamma$, independently of $\varepsilon > 0$. Moreover, for all ε with $0 < \varepsilon^2 < 4\gamma\delta$, $s_\varepsilon(0) \geqq 2\gamma$. We next show that the derivatives $s_\varepsilon^{(j)}$ also possess ε-independent bounds. If $r'_\varepsilon = \delta s'_\varepsilon$, as before, then it was shown in the proof of Theorem 4 that

$$|r_\varepsilon(\xi)| \leqq \tfrac{3}{2}\gamma^2\delta\varepsilon^{-1}, \tag{4.2}$$

or, what is the same,

$$|\varepsilon s'_\varepsilon(\xi)| \leqq \tfrac{3}{2}\gamma^2, \tag{4.3}$$

for all ξ. Returning to equation (2.10), written in the form

$$\delta s''_\varepsilon = \gamma s_\varepsilon - \tfrac{1}{2}s_\varepsilon^2 + \varepsilon s'_\varepsilon, \tag{4.4}$$

it is deduced that

$$\sup_{\xi \in \mathbb{R}} |s''_\varepsilon(\xi)| \leqq \frac{3\gamma^2}{\delta}.$$

Thus s''_ε is bounded, independently of ε. Since s_ε and s''_ε are bounded, it follows that s'_ε is as well, independently of ε. By differentiating (4.4), ε-independent bounds on all higher derivatives of s_ε may be deduced inductively.

In consequence of the bounds just derived, the Ascoli-Arzelà theorem, and a diagonalisation process, if $\{\varepsilon_k\}_{k=1}^\infty$ is any sequence tending to zero, then there is an infinitely differentiable function U and an increasing subsequence $\{k_j\}_{j=1}^\infty$ such that if

$$s_j(\xi) = s(\xi + \xi_0; \varepsilon_{k_j}, \delta),$$

then $s_j^{(i)} \to U^{(i)}$, as $j \to \infty$, uniformly on any compact set in $\mathbb{R}$, for all $i \geqq 0$. It is plain that U satisfies $U'(0)=0$ and, from (4.4),

$$\delta U''=\gamma U-\tfrac{1}{2}U^2. \tag{4.5}$$

The bounded real-valued solutions to (4.5) with $U'(0)=0$ are easily determined (cf. Jeffrey and Kakutani [**13**]). The possibilities are that, either $U\equiv 0$, $U\equiv 2\lambda$, U is a periodic function (cnoidal wave), or else U is a solitary-wave solution.

It must be the case that U is a solitary wave, and in fact the solution given in (4.1). For, $s_j(\xi)$ is decreasing, for $\xi\geqq 0$, and so U is non-increasing, for $\xi\geqq 0$. Hence U is not periodic unless it is constant. Thus if it is established that U is non-constant, then U must coincide with a solitary-wave solution to (4.5). First, since $s_j(0)>2\gamma$, for all j, $U(0)\geqq 2\lambda$. Hence U is not the zero function. Multiply (4.4) by s'_ε and integrate the result over $[0,\infty)$. Because $s'_\varepsilon(0)=0$ and $s_\varepsilon(\xi)$, $s'_\varepsilon(\xi)\to 0$, as $\xi\to+\infty$, there appears

$$\varepsilon\int_0^\infty s'_\varepsilon(\xi)^2\,d\xi=\tfrac{1}{2}\gamma s_\xi^2(0)-\tfrac{1}{6}s_\varepsilon^3(0). \tag{4.6}$$

Now, $s'_\varepsilon(\xi)<0$, for $\xi\geqq 0$, and s'_ε is bounded, independently of ε. The following estimate therefore applies:

$$\varepsilon\int_0^\infty |s'_\varepsilon(\xi)|^2\,d\xi\leqq-\varepsilon\sup_{\xi\in\mathbb{R}}|s'_\varepsilon|\int_0^\infty s'_\varepsilon\,d\xi=\varepsilon\sup_{\xi\in\mathbb{R}}|s'_\varepsilon(\xi)|\,s_\varepsilon(0),$$

and the latter converges to 0 as $\varepsilon\downarrow 0$. Hence, from (4.6),

$$\tfrac{1}{2}\gamma s_j^2(0)-\tfrac{1}{6}s_j^3(0)\to 0,$$

as $j\to\infty$. Since $s_j(0)\geqq 2\gamma$, for all j, this forces $s_j(0)\to 3\gamma$. This shows in particular that $U\not\equiv 2\gamma$ and so U must be a solitary-wave solution to (4.5). That is, U must have the form,

$$U(\xi)=a+b\operatorname{sech}^2(d\xi),$$

for appropriate constants a, b and d.

If (4.5) is multiplied by U', then the result may be represented as

$$\frac{d}{d\xi}\left(\tfrac{1}{2}\delta U'(\xi)^2-\tfrac{1}{2}\gamma U(\xi)^2+\tfrac{1}{6}U(\xi)^3\right)=0,$$

or, what is the same,

$$\tfrac{1}{2}\delta U'(\xi)^2=\tfrac{1}{2}\gamma U(\xi)^2-\tfrac{1}{6}U(\xi)^3+B, \tag{4.7}$$

for some constant B. If the latter relation is evaluated at $\xi=0$, then since $U(0)=3\gamma$ and $U'(0)=0$, it follows that B must be zero. The unique non-trivial solution to (4.7), with $B=0$ and normalised so that $U'(0)=0$ is exactly the function $u_\gamma(\xi)$ given in (4.1) (see again [**13**, §3.1]).

In effect, it has been shown that any positive sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 has a subsequence $\{\varepsilon_{k_j}\}_{j=1}^\infty$ such that the associated travelling-wave solutions s_j converge to u_γ, uniformly on compacta, and the same holds good for derivatives. It follows that, as $\varepsilon\downarrow 0$, $s_\varepsilon^{(j)}\to u_\gamma^{(j)}$, for all j, uniformly on compacta. This result may be immediately improved via the following observation. If $K=\{\xi;\ \xi\geqq a\}$ and $\nu>0$ are given, let $\xi_1>\max\{0, a\}$ be such that $u_\gamma(\xi_1)\leqq\nu/2$. Let $\varepsilon_0>0$ be such that if $0<\varepsilon\leqq\varepsilon_0$, then

$$|s_\varepsilon(\xi)-u_\gamma(\xi)|\leqq\frac{\nu}{2},$$

for $a\leqq\xi\leqq\xi_1$. In particular, $s_\varepsilon(\xi_1)\leqq\nu$, so $s_\varepsilon(\xi)\leqq\nu$ for all $\xi\geqq\xi_1$. Hence, for all $\xi\geqq a$,

$$|s_\varepsilon(\xi)-u_\gamma(\xi)|\leqq\nu.$$

Thus, $s_\varepsilon\to u_\gamma$, uniformly on right rays. On viewing the second equation in (2.10) as an ordinary differential equation for r_ε and utilising the asymptotic conditions $r_\varepsilon(\xi)\to 0$ as $\xi\to\pm\infty$, it is inferred that

$$r_\varepsilon(\xi)=-\int_\xi^\infty(\gamma s_\varepsilon(\eta)-\tfrac{1}{2}s_\varepsilon(\eta)^2)e^{-\varepsilon(\xi-\eta)/\delta}\,d\eta.$$

It follows from this formula and the uniform convergence of s_ε on right rays that $\varepsilon(s_\varepsilon'-u_\gamma')\to 0$, uniformly on right rays. From (4.4) we then see that $s_\varepsilon''\to u_\gamma''$, uniformly on right rays. It may then be inferred that $s_\varepsilon'\to u_\gamma'$ in the same way. The fact that $s_\varepsilon^{(j)}\to u_\gamma^{(j)}$, uniformly on right rays, now follows inductively by repeated differentiation of (4.4) and (4.5). This concludes the proof of the theorem.

Remarks. According to the calculation in (2.7) and following,

$$\varepsilon\int_{-\infty}^{\infty}[s_\varepsilon'(y)]^2\,dy=\tfrac{2}{3}\gamma^2.$$

Thus we see explicitly that the L_2-norm of s' blows up as $\varepsilon\downarrow 0$. Moreover, as above, for small ε,

$$\varepsilon\int_0^\infty[s_\varepsilon'(y)]^2\,dy\leqq-\varepsilon\sup_{\xi\in\mathbb{R}}|s_\varepsilon(\xi)|\int_0^\infty s_\varepsilon'(y)\,dy=\varepsilon s_\varepsilon(0)^2\to 0,$$

as $\varepsilon\to 0$. Hence the oscillation of s_ε near $-\infty$ is sufficient that the square integral of s_ε' is not bounded, as $\varepsilon\downarrow 0$. Indeed, the asymptotic form of $\delta s_\varepsilon'(\xi)=r_\varepsilon(\xi)$, as $\xi\to-\infty$, is given in (3.4), From (3.4) and (2.12) it follows that as $\varepsilon\downarrow 0$, then $\alpha\downarrow 0$ whilst β converges to $(\gamma/\delta)^{\frac{1}{2}}$. Thus one cannot expect s_ε to converge to u_γ, as $\varepsilon\downarrow 0$, uniformly on the whole line, or even in Sobolev spaces on the line.

Theorem 9 is now reinterpreted in the context of solutions to (2.1) satisfying the boundary conditions (1.6).

COROLLARY 10. *Let c and δ be given positive constants and suppose that S_L and S_R satisfy* (2.9) *so that $\gamma=c-S_R>0$. Let $\varepsilon^2<4\gamma\delta$ and let $S_\varepsilon(\xi)=S(\xi+\xi_0;\varepsilon,\delta)$ be the solution to* (2.1) *satisfying the boundary conditions* (1.6), *normalised by ξ_0 so that S_ε takes its maximum value at $\xi=0$. Then, for all $j\geqq 0$, $S_\varepsilon^{(j)}$ converges to the j-th derivative of the function*

$$U_\gamma(\xi)=S_R+3\gamma\,\mathrm{sech}^2\left[\left(\frac{\gamma}{4\delta}\right)^{\frac{1}{2}}\xi\right],$$

uniformly on rays of the form $\{\xi\colon \xi\geqq a\}$.

5. Limiting behaviour as δ tends to zero

In this section, the behaviour of the solution $S(\xi;\varepsilon,\delta)$ to (2.1) will be studied in the limit as δ tends to zero whilst ε is held fixed. It will be shown that if $S(\xi;\varepsilon,\delta)$ is suitably normalised, then it converges, uniformly on $\mathbb{R}$, to the travelling-wave solution to Burgers equation given in (1.9).

As has been the case throughout, we shall work with the variable $s=S-S_R$, which is the unique solution (up to translations in ξ) to (2.3)–(2.5). Since ε is fixed and positive, there is no loss of generality in supposing that $4\gamma\delta<\varepsilon^2$, where γ is defined as before. Because of Theorem 6, it is known that $s(\xi;\varepsilon,\delta)$ is strictly decreasing in the variable ξ, taking the limit 2γ at $-\infty$ and 0 at $+\infty$. Because of these properties, there is a unique value ξ_0 such that $s(\xi_0;\varepsilon,\delta)=\gamma$. Define $s_\delta(\xi)=s(\xi+\xi_0;\varepsilon,\delta)$, so suppressing the dependence of s on the fixed parameter ε. Then s_δ satisfies (2.3)–(2.5), $s_\delta'(\xi)<0$, for all ξ, and $s_\delta(0)=\gamma$, for all $\delta>0$.

THEOREM 11. *Let* $\{s_\delta\}_{\delta>0}$ *be the family of bounded solutions to* (2.3)–(2.5) *defined above. Then, for any integer* $j\geqq 0$,

$$\lim_{\delta\downarrow 0}\frac{d^j}{d\xi^j}s_\delta(\xi)=\frac{d^j}{d\xi^j}v_\gamma(\xi),$$

uniformly for $\xi\in\mathbb{R}$, *where*

$$v_\gamma(\xi)=\gamma\left[1-\tanh\left(\frac{\gamma}{2\varepsilon}\xi\right)\right]. \tag{5.1}$$

Proof. The proof is similar to that of Theorem 9. First it is shown that s_δ, and all its derivatives, are bounded, independently for δ. For s_δ itself, it is known from the results of §2 that $0<s_\delta<2\gamma$, for all $\delta>0$. In the proof of Theorem 4, it was also shown that $r_\delta=\delta s_\delta'$ satisfies $-\delta\gamma^2/2\varepsilon\leqq r_\delta\leqq 3\gamma^2\delta/2\varepsilon$. Hence, for all ξ

$$-\frac{\gamma^2}{2\varepsilon}\leqq s_\delta'(\xi)\leqq\frac{3\gamma^2}{2\varepsilon},$$

and so s_δ' is bounded also, independently of $\delta>0$. Now view equation (2.3) as a first-order equation for s_δ''. Since $s_\delta''(\xi)\to 0$, as $\xi\to\pm\infty$, it follows that

$$s_\delta''(\xi)=-\int_\xi^\infty[\gamma s_\delta'(y)-s_\delta(y)s_\delta'(y)]\delta^{-1}e^{-\varepsilon(y-\xi)/\delta}\,dy. \tag{5.2}$$

Since s_δ and s_δ' are bounded, independently of $\delta>0$,

$$|s_\delta''(\xi)|\leqq\varepsilon^{-1}\sup_{y\geqq\xi}|\gamma s_\delta'(y)-s_\delta(y)s_\delta'(y)|,$$

and so s_δ'' is also bounded, independently of $\delta>0$. If equation (2.3) is differentiated once, there appears,

$$-\gamma s_\delta''+(s_\delta')^2+s_\delta s_\delta''++\delta s_\delta''''-\varepsilon s_\delta'''=0. \tag{5.3}$$

By viewing (5.3) as a first-order equation for s_δ''', we obtain,

$$s_\delta'''(\xi)=-\int_\xi^\infty\{\gamma s_\delta''(y)-s_\delta'(y)^2-s_\delta(y)s_\delta''(y)\}\delta^{-1}e^{-\varepsilon(y-\xi)/\delta}\,dy.$$

Since s_δ, s'_δ and s''_δ are now known to possess δ-independent bounds, it follows that s'''_δ also possesses such a bound. On continuing inductively in this fashion, it is inferred that $s_\delta^{(j)}$ is bounded, independently of $\delta>0$, for all j.

Now by applying the Ascoli-Arzelà theorem repeatedly, and a Cantor diagonalisation, it is deduced that any sequence $\delta_n \downarrow 0$ has a subsequence $\{\delta_k\}_{k=1}^\infty$ such that if

$$s_k(\xi)=s_{\delta_k}(\xi),$$

then there is a function V for which $s_k^{(j)}\to V^{(j)}$, uniformly on compact subsets of $\mathbb{R}$, for all integers $j\geqq 0$. It follows that the C^∞-function V has $0\leqq V(\xi)\leqq 2\gamma$ and $V'(\xi)\leqq 0$ for all ξ, and that V satisfies equation (2.6) with δ set to zero, namely

$$-\gamma V+\tfrac{1}{2}V^2-\varepsilon V'=0. \tag{5.4}$$

This is just the equation for the travelling-wave solution to Burgers equation. Since $V'\leqq 0$, V is monotone non-increasing and bounded. It therefore has limits at $\pm\infty$, and from (5.4) these could only be 0 or 2γ. If they were both 0 or 2γ, then V would be constantly equal to this value. But $V(0)=\gamma$ since $s_\delta(0)=\gamma$ because of our normalisation, so V cannot be identically 0 or identically 2γ. As V decreases, it cannot take the value 0 at $-\infty$ and 2γ at $+\infty$. It follows that $V(\xi)\to 0$ as $\xi\to+\infty$ and $V(\xi)\to 2\gamma$ as $\xi\to-\infty$. By integrating equation (5.4), subject to the just-derived conditions we obtain

$$V(\xi)=\frac{2\gamma}{1+Ke^{\gamma\xi/\varepsilon}},$$

for some constant K. Further, since $V(0)=\gamma$, it follows that $K=1$, whence, after a short manipulation, it appears that

$$V(\xi)=\gamma\left[1-\tanh\left(\frac{\gamma}{2\varepsilon}\xi\right)\right],$$

as in (5.1).

The fact that s_k and its derivatives converges to V and its corresponding derivatives, uniformly on $\mathbb{R}$, now follows in exactly the way we extended the convergence in Theorem 9 to be uniform on right rays. As before, since the limiting function $V=v_\gamma$ is unique, it is concluded that $s_\delta^{(j)}\to v_\gamma^{(j)}$, as $\delta\downarrow 0$, for each $j\geqq 0$. The proof of the theorem is complete.

Theorem 11 has an immediate consequence as regards the travelling-wave solution to (1.2).

COROLLARY 12. *Let c and ε be given positive constants and suppose that S_L and S_R satisfy* (2.9) *so that $\gamma=c-S_R>0$. Let $4\gamma\delta<\varepsilon^2$ and let $S_\delta(\xi)=S(\xi+\xi_0;\varepsilon,\delta)$ be the solution to* (2.1) *satisfying* (1.6), *normalised by ξ_0 so that $S_\delta(0)=c$. Then for each $j\geqq 0$, $S_\delta^{(j)}(\xi)$ converges to the j-th derivative of the function*

$$V_\gamma(\xi)=S_R+\gamma\left[1-\tanh\left(\frac{\gamma}{2\varepsilon}\xi\right)\right],$$

as $\delta\downarrow 0$, uniformly for $\xi\in\mathbb{R}$.

6. Convergence to a shock-wave profile

The limiting form of $S(\xi;\varepsilon,\delta)$ as ε and δ simultaneously tend to zero is examined here. In the case where the ratio δ/ε^2 remains bounded, it transpires that S converges to a shock-wave profile (weak solution to the conservation law $u_t+u_x=0$, cf. **[9]**),

$$\psi(\xi)=\varphi(\xi)+S_R, \tag{6.1}$$

where

$$\varphi(\xi)=\begin{cases}2\gamma, & \text{for } \xi<0,\\ \gamma, & \text{for } \xi=0,\\ 0, & \text{for } 0<\xi.\end{cases} \tag{6.2}$$

As before, let $s=S-S_R$ be the unique (up to translations) solution to (2.3)–(2.5) and set

$$s_{\varepsilon,\delta}(\xi)=s(\xi+\xi_0;\,\varepsilon,\delta),$$

where $s(\xi_0;\varepsilon,\delta)=\gamma$ and $s(\xi;\varepsilon,\delta)<\gamma$ for $\xi>\xi_0$. The general theory developed in Sections 2 and 3 assures that ξ_0 is well defined for any given positive values of ε and δ.

THEOREM 13. *Let $\alpha>0$ and $M>0$ be given. Then for any integer $j\geqq 0$,*

$$\lim_{\substack{\varepsilon,\delta\to 0\\ \delta/\varepsilon^2\leqq M}}\frac{d^j}{d\xi^j}s_{\varepsilon,\delta}(\xi)=\frac{d^j}{d\xi^j}\varphi(\xi), \tag{6.3}$$

uniformly for $|\xi|\geqq\alpha$.

Proof. Define a new dependent variable T by

$$T(\xi)=s_{\varepsilon,\delta}(\varepsilon\xi). \tag{6.4}$$

The dependence of T on ε and δ is suppressed. A short calculation using (2.6) reveals that

$$-\gamma T+\tfrac{1}{2}T^2+\mu T''-T=0, \tag{6.5}$$

where $\mu=\delta/\varepsilon^2$. The uniqueness result in Theorem 4 assures that $T(\xi)=T_\mu(\xi)=s(\xi+\xi_0;1,\mu)$, where the normalisation ξ_0 is such that $T_\mu(0)=\gamma$ and $T_\mu(\xi)<\gamma$ for $\xi>0$.

The relation (6.3) will be established by an argument involving sequences. To this end consider sequences $\{\varepsilon_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ of positive parameters such that

$$\varepsilon_n\to 0,\quad \delta_n\to 0,\quad \text{as}\quad n\to\infty,$$

and

$$\mu_n=\frac{\delta_n}{\varepsilon_n^2}\leqq M,\quad \text{for all } n. \tag{6.6}$$

Take any subsequence $\{n_k\}_{k=1}^\infty$ of the positive integers for which $\nu_k=\mu_{n_k}$ tends to a limit, say ν, as $k\to\infty$. Naturally we must have $0\leqq\nu\leqq M$. Let T_k denote T_{ν_k} and let $v_\gamma(\xi)$ denote the function $\gamma[1-\tanh(\gamma\xi/2)]$, as before.

Two cases are distinguished now, namely $\nu=0$ and $\nu>0$. If $\nu=0$, then Theorem 11, with $\varepsilon=1$, applies and it is thereby concluded that for $j\geqq 0$,

$$\frac{d^j}{d\xi^j}T_k(\xi)\to\frac{d^j}{d\xi^j}v_\gamma(\xi),$$

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