Physica 24D (1987) 343–366
North-Holland, Amsterdam

SUFFICIENT CONDITIONS FOR STABILITY OF SOLITARY-WAVE SOLUTIONS OF MODEL EQUATIONS FOR LONG WAVES

J.P. ALBERT
Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA

J.L. BONA
Department of Mathematics, The University of Chicago, Chicago, IL 60637, USA

and

D.B. HENRY
Instituto de Matemática, Universidade de São Paulo, São Paulo, Brazil

Received 15 April 1986
Revised manuscript received 12 May 1986

We consider solitary-wave solutions of model equations for long waves that feature a general form of linear dispersion. Sufficient conditions for the non-linear stability of such solutions are derived. These conditions are shown to obtain for the Korteweg–de Vries equation and certain of its generalizations such as the Benjamin–Ono equation and the intermediate long-wave equation.

1. Introduction

Considered herein is a class of model equations for the propagation of long waves in nonlinear, dispersive media. These equations have the form

$$u_t + u_x + u^p u_x + Lu_x = 0 \tag{1.1}$$

or

$$u_t + u_x + u^p u_x - Lu_t = 0, \tag{1.2}$$

where x, t are real variables, subscripts denote partial differentiation, $p \geq 1$ is an integer, and L is an operator formally defined by

$$\widehat{Lv}(k) = \alpha(k)\hat{v}(k). \tag{1.3}$$

Here the circumflex over a function denotes the function's Fourier transform, and the symbol α of L will be restricted presently. The well known Korteweg–de Vries equation,

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1.4}$$

which arose first in the study of small-amplitude, long-wavelength, water waves [1] falls within the category of models to be considered, but other equations of the form (1.1) or (1.2) are also in view. Examples that are specifically treated here are the Benjamin–Ono equation [2, 3], which is of type (1.1) with $p = 1$ and $\alpha(k) = |k|$; the intermediate long-wave equation [4], also of type (1.1), with $p = 1$ and $\alpha(k) = k\coth(kH) - H^{-1}$, where H is a positive constant; and a generalization of eq. (1.4) in which the nonlinear term uu_x is replaced by $u^p u_x$ where $p > 1$. (Consideration is also given to variants of the above equations which are of type (1.2).) Many other instances of equations of these types have been derived as models for wave phenomena in various settings. As explained by Benjamin et al. [5], the value of the parameter p appearing in the nonlinear terms of eqs. (1.1) and (1.2) is related to nonlinear effects suffered by the waves being modelled, while the form of the symbol α is related to dispersive and, possibly, dissipative effects.

An important aspect of equations of the form (1.1) and (1.2) is their solitary-wave solutions. For a given $C > 0$ a solitary wave of speed $1 + C$ is a solution u_C of (1.1) or (1.2) having the form $u_C(x,t) = \varphi_C(x - (1 + C)t)$, where φ_C is typically a smooth positive function having a unique maximum, symmetric about its maximum, and decaying monotonically to zero away from its maximum. For example, the solitary-wave solutions of (1.3) have the form

$$\varphi_C(y) = 3C\,\mathrm{sech}^2\left(C^{1/2}y/2\right), \tag{1.5}$$

as noted by Korteweg and de Vries [1], whilst for the Benjamin–Ono equation

$$\varphi_C(y) = 4C/(1 + C^2y^2), \tag{1.6}$$

as determined by Benjamin [2]. For a broad class of symbols α the existence of such similarity solutions has been established [6–9]. These solitary waves are interesting because they are known to play a distinguished role in the large-time asymptotics of whole classes of solutions of certain equations of type (1.1) (see ref. [10] and the extensive list of references given therein). The situation regarding equations of type (1.2) appears to be similar to that obtaining for those of type (1.1), as attested by various numerical experiments (cf. refs. [11–13]). However, the beautiful analytical theory pertaining to certain equations of the form given in (1.1) has no known counterpart for equations of type (1.2).

Laboratory and field observations suggest that this special sort of plane wave of permanent form generally comprises a very stable phenomenon (cf. refs. [14–17]). It is our purpose here to examine the question of stability of such travelling-wave solutions when considered as solutions of the initial-value problem for (1.1) and (1.2). Sufficient conditions will be presented that insure the stability of solitary-wave solutions of (1.1) or (1.2) to small perturbations in the waveform. These conditions will be shown to be effective in a variety of circumstances.

The results obtained here are an outgrowth of the pioneering paper of Benjamin [18] (see also refs. [19, 20]) in which the stability of the solutions (1.5) of (1.4) was demonstrated. His result will obtain as an easy corollary of our general criterion, as will the recent theories of Bennett et al. [21] pertaining to the solutions (1.6) of the Benjamin–Ono equation and of Weinstein [22] pertaining to solitary-wave solutions of (1.1) with $L = -\partial_x^2$ and $p < 4$.

The paper is organized as follows. In section 2, notation is introduced and certain mathematical preliminaries are presented. Sufficient conditions for the stability of a given solitary wave are derived in section 3, whilst the examples to which the theory has been applied are given in section 4. The concluding section 5 sets forth a number of open problems in this area.

2. The initial-value problem

In this portion of the paper notation is given and a brief review provided of some aspects of the initial-value problem for (1.1).

If X is a Banach space, its norm will be denoted by $\|\cdot\|_X$, or in some cases by special abbreviations introduced below. Similarly the inner product in a Hilbert space H is written $(\ ,\ )_H$. For $1 \le p < \infty$, L_p is the (equivalence classes of) measurable functions f: $\mathbf{R} \to \mathbf{R}$ which are pth power absolutely integrable, with its usual norm

$$|f|_p = \left\{\int_{-\infty}^{+\infty} |f(x)|^p \,\mathrm{d}x\right\}^{1/p},$$

and L_∞ is the class of essentially bounded functions f: $\mathbf{R} \to \mathbf{R}$ with the norm

$$|f|_\infty = \underset{x\in\mathbf{R}}{\text{essential supremum}}\, |f(x)|.$$

The norm of a function f in the Hilbert space L_2 will be singled out and denoted simply by $\|f\|$. For $s \geq 0$ the space H^s is the linear subspace of L_2 consisting of those functions f for which

$$\|f\|_s = \left\{ \int_{-\infty}^{+\infty} (1+k^2)^s |\hat{f}(k)|^2 \,\mathrm{d}k \right\}^{1/2} < \infty. \tag{2.1}$$

If $s < 0$, H^s is the collection of tempered distributions whose Fourier transforms are measurable functions for which the above integral is finite. In a natural way, H^{-s} is the dual of H^s under the pairing

$$\langle f, U \rangle_s = \int_{-\infty}^{+\infty} \hat{f}(k) \hat{U}(k) \,\mathrm{d}k.$$

We let $H^\infty = \bigcap_{s>0} H^s$. The elements of H^∞ are infinitely differentiable functions all of whose derivatives lie in L_2.

The spaces L_p $(2 \leq p < \infty)$ and H^s $(s \geq 0)$ are related by the Sobolev Imbedding Theorem (see e.g., ref. [23], p. 124). One consequence of this theorem, which will be used later, is the inequality

$$|f|_p \leq A_p \|f\|_{1/2}, \tag{2.2}$$

which holds for f in $H^{1/2}$ and $2 \leq p < \infty$, with A_p independent of f.

For a detailed development of the theory of the Sobolev spaces H^s, the reader may consult refs. [23–25].

If X is any Banach space and J a closed interval in $\mathbf{R}$, let $C_b(J; X)$ denote the bounded continuous functions u: $J \to X$. This is again a Banach space with the norm

$$\|u\|_{C_b(J;\,X)} = \sup_{t\in J} \|u(t)\|_X.$$

In case J is bounded, the subscript b for "bounded" will be dropped. For $T > 0$ the symbol $C(0, T; X)$ wil be employed as an abbreviation for $C([0, T]; X)$.

In the remainder of the paper, when Sobolev or L_p norms or inner products are taken of functions of two variables x and t, they will always be assumed to be taken with respect to the x variable only.

If X and Y are Banach spaces, the space of bounded linear operators from X to Y is denoted $B(X, Y)$ and its standard norm is $\|\cdot\|_{X,Y}$. The set of closed linear operators from X to Y is denoted $C(X, Y)$. These spaces will appear here only when $X = Y = L_2$ or $X = Y = L_2 \times L_2$. When $X = Y$, we abbreviate $B(X, X)$ by $B(X)$ and $C(X, X)$ by $C(X)$. The norm in $B(L_2)$ is singled out and written as $\|\cdot\|_{2,2}$.

Let u be a solution of (1.1) or (1.2). It will be supposed that at some instant t_0, u resembles closely a solitary-wave solution $\varphi = \varphi_C$. In this circumstance interest will be focused on showing that u resembles a solitary wave for all time. As the theory presented here takes no note of whether time runs forward or backward, it suffices to take $t_0 = 0$ and to show that u lies close to φ for all $t \geq 0$. Attention is thus directed to the initial-value problems

$$\begin{aligned} &u_t + u_x + u^p u_x - L u_x = 0, \\ &u(x,0) = \psi(x), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &u_t + u_x + u^p u_x + L u_t = 0, \\ &u(x,0) = \psi(x). \end{aligned} \tag{2.4}$$

The proposition in view is that, for either problem, if $\psi - \varphi$ is small in some suitable sense, then $u(\cdot, t) - \varphi(\cdot - (C+1)t)$ is small, for all $t \geq 0$.

As pointed out already in ref. [5] the last-stated proposition is false without further qualification. The bulk of the solution u that emanates from ψ may travel at a different speed than φ, and consequently $u(\cdot,t)-\varphi(\cdot-(C+1)t)$ will not be small for all t. If X is a given Banach space of functions defined on $\mathbf{R}$, a pseudo-metric based on the norm on X is defined as follows. For f and g in X, let

$$d_X(f,g)=\inf_{y\in\mathbf{R}}\|f(\cdot)-g(\cdot+y)\|_X.$$

(In case $X=H^s$ for some s, $d_X(f,g)$ will also be written as $d_s(f,g)$.) A correct version of the preceding proposition is stated as follows: for all $t>0$, $d_X(u(\cdot,t),\ \varphi(\cdot-(C+1)t))$ is small provided it is small enough at $t=0$.

In discussing the well-posedness of the initial-value problems (2.3) and (2.4), we will use the terminology of Kato [26]. In general, if X is a Banach space, U an open subset of X, and f a continuous mapping from U into another Banach space Y containing X, then the initial-value problem

$$\mathrm{d}u/\mathrm{d}t=f(u),\qquad u(0)=\psi, \tag{2.5}$$

will be said to be *locally well-posed in* U if the following two statements are true.

(i) For each $\psi\in U$ there exists a real number $T>0$ and a unique function $u\in C(0,T;\ X)$ satisfying (2.5). (It follows that $\mathrm{d}u/\mathrm{d}t$ is a member of $C(0,T;\ Y)$ for each t in $[0,T]$.)

(ii) The map that assigns to the initial data ψ in X the solution u in $C(0,T;\ X)$ guaranteed in (i) is continuous.

If T can be taken arbitrarily large in (i) and (ii), we say that problem (2.5) is *globally well-posed in* U. For a large class of initial-value problems, local well-posedness combined with the information that solutions u are bounded in X on bounded time intervals implies global well-posedness.

Henceforth it will be assumed that the initial-value problems (2.3) and (2.4) are locally well posed in some H^s-neighborhood of a given solitary wave φ if s is greater than some fixed s_0, and that solutions can be continued in time as long as they remain bounded in H^m, where m is another index with $s_0\geq m\geq 0$. Results of this nature are known to hold at least under certain conditions on the exponent p and the symbol α (cf. refs. [5, 27, 28]).

As a final notational point, the letters A, A_1, A_2, etc. will be used throughout to denote various constants, and different occurrences of the same letter will not necessarily represent the same constant. The end of a proof will be marked with the symbol ■.

3. Analysis of perturbations of solitary waves

It will be assumed in this section that the symbol α of L satisfies the inequalities

$$0\leq\alpha(k)\leq A_1(1+|k|^{2m}), \tag{3.1}$$

for all $k\in\mathbf{R}$, and

$$\alpha(k)\geq A_2|k|^{2m}, \tag{3.2}$$

for all large values of k, where A_1 and A_2 are positive constants and $m\geq 1/2$. Adding precision to our remarks in section 2, it is assumed that the initial-value problems (2.3) and (2.4) are locally well posed in an H^s-neighborhood of a given solitary wave, for all $s>s_0$, where $s_0\geq m$, and that the local theory extends to arbitrary time intervals in the presence of bounds on the H^m-norm of the solution. As the outcome of our analysis is a demonstration that a solution starting near to φ has a translation that remains near φ in H^m norm, over any time interval for which it exists, the triangle inequality insures that such solutions remain bounded in appropriate Sobolev spaces. In consequence of this preview, we are justified in making the stronger assumption that (2.3) and (2.4) are globally well posed in an H^s neighborhood of φ, and that this holds for any s larger than the constant s_0. The value of s_0 will be left unspecified until the theory is applied to concrete classes of equations in section 4.

Define functionals V, V_1 and M by the formulas

$$V(f)=\int_{-\infty}^{+\infty}f^2(x)\,\mathrm{d}x,$$

$$V_1(f) = \int_{-\infty}^{+\infty} [f^2(x) + f(x)Lf(x)]\,dx,$$

$$M(f) = \int_{-\infty}^{+\infty} \left[f(x)Lf(x) - \frac{2}{(p+1)(p+2)} f^{p+2}(x) \right] dx.$$

Then V, V_1, and M are continuous maps from H^m to $\mathbf{R}$. To see this, notice that (3.1) implies that L is a bounded linear operator from H^s to H^{s-2m}, for any real s. In particular, L maps H^m continuously into H^{-m}. Therefore, the map defined by

$$f \to \int_{-\infty}^{+\infty} f(Lf)\,dx = \langle f, Lf \rangle_m$$

is continuous from H^m to $\mathbf{R}$. For $p \geq 0$, the estimate

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} (f^{p+2} - g^{p+2})\,dx \right| \\ &\leq A \int_{-\infty}^{\infty} |f-g| (|f|^{p+1} + |g|^{p+1})\,dx \\ &\leq A \|f-g\| \left(\int_{-\infty}^{\infty} (|f| + |g|)^{2p+2}\,dx \right)^{1/2} \\ &\leq A \|f-g\|_m (\|f\|_m + \|g_m\|)^{p+1} \end{aligned}$$

follows from Hölder's inequality and (2.2). Taking $p = 0$, we deduce that $V(f) = \|f\|^2$ is continuous on H^m. It then follows that $V_1(f) = \|f\|^2 + \langle f, Lf \rangle_m$ is continuous on H^m. Finally, using the just preceding estimate for arbitrary p, the continuity of the map

$$f \to \int_{-\infty}^{\infty} f^{p+2}(x)\,dx$$

is inferred, and therefrom the continuity of M is assured.

The importance of the functionals V, V_1, and M is that they define invariants of the motion generated by (1.1) or (1.2).

Lemma 1. Suppose $\psi \in H^s$ where $s > s_0$ and let $u(x, t)$ be the corresponding solution of (2.3). Then for any $t \geq 0$,

$$V(u(\cdot, t)) = V(\psi) \quad \text{and} \quad M(u(\cdot, t)) = M(\psi).$$

Similarly, if u solves (2.4), then for any $t \geq 0$,

$$V_1(u(\cdot, t)) = V_1(\psi) \quad \text{and} \quad M(u(\cdot, t) = M(\psi).$$

Proof. We may assume without loss of generality that $\psi \in H^\infty$. The result for $\psi \in H^s$ follows from the result for smooth ψ by the continuity of V, V_1, and M as functionals on H^s and the well-posedness properties of problems (2.3) and (2.4).

If $\psi \in H^\infty$ and u solves (2.3), then u and Lu are infinitely differentiable functions of x and t, all of whose derivatives lie in L_2. Accordingly, the following formal computations are rigorously justified:

$$\begin{aligned} \frac{d}{dt}(V(u)) &= 2\int_{-\infty}^{\infty} u(x,t) u_t(x,t)\,dx \\ &= -2\int_{-\infty}^{\infty} u(u_x + u^p u_x - Lu_x)\,dx \\ &= -2\int_{-\infty}^{\infty} \partial_x \left(\tfrac{1}{2}u^2 + \frac{1}{p+2}u^{p+2} - \tfrac{1}{2}uLu \right) dx \\ &= 0, \end{aligned}$$

since L is symmetric and u and Lu tend to zero as $x \to \pm\infty$. Similarly, it appears that

$$\begin{aligned} \frac{d}{dt}M(u) &= 2\int_{-\infty}^{\infty} \left(u_t Lu - u_t \frac{u^{p+1}}{p+1} \right) dx \\ &= -2\int_{-\infty}^{\infty} \left(Lu - \frac{u^{p+1}}{p+1} \right) \times (u_x + u^p u_x - Lu_x)\,dx \\ &= \int_{-\infty}^{\infty} \partial_x \left(uLu - \frac{2}{(p+1)(p+2)}u^{p+2} - \frac{u^{2p+2}}{(p+1)^2} + (Lu)^2 \right) dx \\ &\quad -2\int_{-\infty}^{\infty} \left(u^p u_x Lu + \frac{u^{p+1}}{p+1} Lu_x \right) dx. \end{aligned}$$

The first integral on the right-hand side vanishes since u and Lu tend to zero as $x \to \pm\infty$, while the second is seen to be zero after integration by parts.

Next suppose that $\psi \in H^{\infty}$ and that u solves (2.4) with initial data ψ. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(u) = 2\int_{-\infty}^{\infty} (uu_t + uLu_t)\,\mathrm{d}x$$

$$= -2\int_{-\infty}^{\infty} u(u_x + u^p u_x)\,\mathrm{d}x$$

$$= -2\int_{-\infty}^{\infty} \partial_x\left(\tfrac{1}{2}u^2 + \frac{u^{p+2}}{p+2}\right)\mathrm{d}x = 0.$$

To prove that $M(u)$ is independent of t in the case where u solves (2.4), introduce the function

$$w = -(I+L)^{-1}\left(u + \frac{1}{1+p}u^{p+1}\right).$$

(Here $(I+L)^{-1}$ denotes the Fourier multiplier operator with symbol $1/[1+\alpha(k)]$.) Because u and u^{p+1} are in H^{∞}, w is also; moreover,

$$\partial w/\partial x = -(I+L)^{-1}(u_x + u^p u_x)$$
$$= (I+L)^{-1}(u_t + Lu_t) = u_t.$$

Hence the following calculation is decisive:

$$\frac{\mathrm{d}}{\mathrm{d}t} M(u) = 2\int_{-\infty}^{\infty}\left(uLu_t - u_t\frac{u^{p+1}}{p+1}\right)\mathrm{d}x$$

$$= 2\int_{-\infty}^{\infty}\left(uLw_x - u_t\frac{u^{p+1}}{p+1}\right)\mathrm{d}x$$

$$= -2\int_{-\infty}^{\infty}\left((Lw)u_x + u_t\frac{u^{p+1}}{p+1}\right)\mathrm{d}x$$

$$= 2\int_{-\infty}^{\infty}\left(Lw(w_x + u^p u_x + Lw_x) + \frac{u^{p+1}}{p+1}(u_x + u^p u_x + Lw_x)\right)\mathrm{d}x$$

$$= \int_{-\infty}^{\infty}\left(wLw + (Lw)^2 + \frac{2u^{p+2}}{(p+1)(p+2)} + \frac{u^{2p+2}}{(p+1)^2}\right)\mathrm{d}x$$

$$+2\int_{-\infty}^{\infty}\left(\frac{u^{p+1}}{p+1}Lw_x + Lwu^p u_x\right)\mathrm{d}x.$$

As before, the first integral on the right-hand side integrates to zero, and the second vanishes after an integration by parts. This establishes the lemma. ■

The main result of this section is now in view. For the present, consideration is restricted to equations of the type depicted in (1.1); the theory for equations of type (1.2) is similar and will be dealt with later.

Choose a fixed value of $C > 0$, and let $\varphi = \varphi_C$ be a solitary-wave solution of (1.1) with speed $1 + C$. Thus φ satisfies the relation

$$L\varphi' + (C - \varphi^p)\varphi' = 0. \tag{3.3}$$

Define an operator $\mathscr{L}_{\varphi}$ by

$$\mathscr{L}_{\varphi} f(x) = Lf(x) + [C - \varphi^p(x)]f(x).$$

If, as will be assumed henceforth, $\varphi \in H^r$, where $r > 1/2$, then a straightforward induction making use of (3.3) and the fact that $m > 0$ in assumption (3.1) shows that indeed $\varphi \in H^{\infty}$. This in turn implies that $\mathscr{L}_{\varphi}$ is a self-adjoint, closed operator on L_2 with domain H^{2m}. Notice that, according to (3.3), $\mathscr{L}_{\varphi}$ has 0 as an eigenvalue, with corresponding eigenfunction $\varphi'(x)$.

Theorem 1. Suppose that $\mathscr{L}_{\varphi}$ has the following three properties:

(P_1) the eigenvalue 0 of $\mathscr{L}_{\varphi}$ is simple;

(P_2) the intersection of $\operatorname{spec}(\mathscr{L}_{\varphi})$ with the negative real axis consists of a single, simple eigenvalue α; and

(P_3) if ψ_{α} is any eigenfunction corresponding to the eigenvalue α, the following inequality holds:

$$\left(1 + \frac{\beta}{|\alpha|}\right)\left(\frac{(\varphi, \psi_{\alpha})}{\|\varphi\| \cdot \|\psi_{\alpha}\|}\right)^2 > 1, \tag{3.4}$$

where $\beta = \inf\{\lambda \in \operatorname{spec}(\mathscr{L}_{\varphi}): \lambda > 0\}$.
Under these hypotheses, φ is a stable solution of (1.1) in the following sense. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\psi \in H^{s_0}$ and $\|\psi - \varphi\|_m < \delta$, then the solution u of (2.3) satisfies $d_m(u, \varphi) < \varepsilon$, for all $t > 0$.

Proof. Define

$$h(x,t)=u(x,t)-\varphi(x+a(t)), \tag{3.5}$$

where $a(t)$ will be chosen presently. Whatever the choice of a, one has

$$d_m(u,\varphi)\le \|h(\cdot,t)\|_m. \tag{3.6}$$

Hence if $a(t)$ can be chosen so that the right-hand side of (3.6) is small for all $t\ge 0$, the stability of φ stated in the theorem will be established.

In the previous work to which this study is a direct successor [18, 19, 21], the crucial step was to follow an idea of Boussinesq [29] and use the functional

$$\Delta M=M(u(\cdot,t))-M(\varphi)$$

as a Lyapunov function in the analysis of u. Since $M(u(\cdot,t))$ does not depend on t, and since φ is fixed, ΔM depends only on the initial data, $u(x,0)=\psi(x)$. In conjunction with the use of ΔM as Lyapunov function, a side condition that plays a considerable role in the analysis has been imposed in the previous theories, namely, that $\Delta V=0$, where

$$\Delta V=V(u(\cdot,t))-V(\varphi).$$

This slightly odd side condition is dispensed with later in the proof by calling upon a simple property of the solitary-wave solutions of the particular equations that have been studied heretofore.

Here this latter step will be obviated by choosing as the prospective Lyapunov function not ΔM, but

$$\Lambda=\Delta M+C\Delta V.$$

As both M and V are time independent when evaluated on smooth solutions of (1.1), $\Lambda=\Lambda(\psi)$ depends only on the choice of the initial data ψ of u.

It is useful to express Λ in terms of φ and h, using (3.5) to eliminate u. By virtue of (3.3), it appears that

$$\Lambda=\int_{-\infty}^{+\infty}\left[hLh+(C-\varphi^p)h^2\right]\mathrm{d}x + \int_{-\infty}^{+\infty}R(h,\varphi)\,\mathrm{d}x, \tag{3.7}$$

where

$$R(h,\varphi)=\frac{-2}{(p+1)(p+2)}\Big\{(\varphi+h)^{p+2} - \left[\varphi^{p+2}+(p+2)\varphi^{p+1}h + \tfrac{1}{2}(p+2)(p+1)\varphi^p h^2\right]\Big\}$$

(the argument of φ is $x+a(t)$ here and throughout, unless explicitly noted to the contrary). From Taylor's theorem, it follows that

$$|R(h,\varphi)|\le A|h|^3(|\varphi|+|h|)^{p-1}, \tag{3.8}$$

where A is independent of h.

As will become apparent presently, finding a suitable lower bound for Λ is the crux of our analysis. This task would be very simple if the function $C-\varphi^p$ was positive. Unfortunately, it is always the case that $\max_x\{\varphi^p(x)\}>(p+1)C$, so the second term in the first integrand on the right-hand side of (3.7) is not everywhere positive. (To see that the last assertion holds, integrate (3.3) over $\mathbf{R}$ and use the fact that

$$\int_{-\infty}^{+\infty}L\varphi(x)\,\mathrm{d}x=\alpha(0)\hat\varphi(0) =\alpha(0)\int_{-\infty}^{+\infty}\varphi(x)\,\mathrm{d}x\ge 0.$$

One then has that

$$\int_{-\infty}^{+\infty}\left(C-\frac{1}{p+1}\varphi^p(x)\right)\varphi(x)\,\mathrm{d}x\le 0,$$

from which the assertion follows since $\varphi>0$.) Therefore, the quadratic form

$$\int_{-\infty}^{+\infty}\left[h(Lh)+(C-\varphi^p)h^2\right]\mathrm{d}x$$

is not positive definite, and obtaining a lower bound on Λ may present difficulties. It will be seen below that a successful estimate for Λ will depend on a propitious choice for the function $a(t)$, and on the properties (P_1)–(P_3) of the operator $\mathscr{L}_\varphi$.

Fix $t \geq 0$ and introduce temporarily the notation $\mathscr{L}_a$ for the operator

$$\mathscr{L}_a f(x) = Lf(x) + [C - \varphi^p(x+a)] f(x),$$

so emphasizing the role of the as yet unspecified parameter $a = a(t)$. Formula (3.7) for Λ can be rewritten in the form

$$\Lambda = \int_{-\infty}^{+\infty} h(\mathscr{L}_a h)\,dx + \int_{-\infty}^{+\infty} R(\varphi, h)\,dx.$$

Clearly properties (P_1)–(P_3), assumed to hold for $\mathscr{L}_\varphi$, also hold for $\mathscr{L}_a$ whatever be a.

Lemma 3. Suppose $f \in H^m$ and $a \in \mathbf{R}$ are such that

(i) $\|\varphi(x+a) + f(x)\| = \|\varphi\|$,

(ii) $\int_{-\infty}^{+\infty} \varphi'(x+a) f(x)\,dx = 0$.

Then there exist positive constants A_1 and A_2 which are independent of f and a, and are such that

$$\int_{-\infty}^{+\infty} (\mathscr{L}_a f(x)) f(x)\,dx \geq A_1 \|f\|_m^2 - A_2\left(\|f\|_m^3 + \|f\|_m^4\right).$$

Proof. By translating f, it may be assumed that $a = 0$, so that $\mathscr{L}_a = \mathscr{L}_0 = \mathscr{L}$, say.

To begin, we have the easy estimate

$$\begin{aligned}(\mathscr{L}f, f) &= \int_{-\infty}^{+\infty} \widehat{Lf}(k)\overline{\hat{f}(k)}\,dk + \int_{-\infty}^{+\infty} (C - \varphi^p) f^2\,dx \\ &\geq A \int_{-\infty}^{+\infty} |k|^{2m} |\hat{f}(k)|^2\,dk - A_1\|f\|^2 \\ &\geq A\|f\|_m^2 - A_2\|f\|^2. \end{aligned} \tag{3.9}$$

Now let ψ_α denote an eigenfunction of $\mathscr{L}$ with eigenvalue α. Take ψ_α to be normalized so that $\|\psi_\alpha\| = 1$, and write

$$f(x) = p\psi_\alpha + f_1, \qquad \varphi(x) = q\psi_\alpha + \varphi_1, \tag{3.10}$$

where $p = (f, \psi_\alpha)$ and $q = (\varphi, \psi_\alpha)$ and, as before, $(\cdot, \cdot)$ denotes the L_2 inner product. Of course, q and φ_1 are independent of f and so $|q|$ and $\|\varphi_1\|$ are constants which are independent of f. Assumption (i) implies that

$$2(f, \varphi) + \|f\|^2 = 0. \tag{3.11}$$

If the relations in (3.10) are substituted into (3.11), and the result expanded using the identities $(\psi_\alpha, \psi_\alpha) = 1$ and $(f_1, \psi_\alpha) = (\varphi_1, \psi_\alpha) = 0$, there appears

$$2pq + 2(f_1, \varphi_1) + \|f\|^2 = 0.$$

Since $q \neq 0$, in consequence of (P_3), it is inferred that

$$p = \frac{-(f_1, \varphi_1)}{q} - \frac{\|f\|^2}{2q}.$$

Squaring both sides of this equation and using Schwarz' inequality gives

$$\begin{aligned} p^2 &\leq \|f_1\|^2\|\varphi_1\|^2/q^2 + A\left(\|f\|^3 + \|f\|^4\right) \\ &= \|f_1\|^2\left(\|\varphi\|^2/q^2 - 1\right) + A\left(\|f\|^3 + \|f\|^4\right). \end{aligned} \tag{3.12}$$

Now the spectral theorem is applied, taking into account properties (P_1) and (P_2) of $\mathscr{L}$ and assumption (ii) of the lemma, to obtain

$$(\mathscr{L}f, f) \geq \alpha p^2 + \beta\|f_1\|^2. \tag{3.13}$$

By (P_3), there is an $\eta_1 > 0$ such that

$$\eta_2 = \beta - (|\alpha| + \eta_1)\left(\|\varphi\|^2/q^2 - 1\right) > 0.$$

Combining (3.12), (3.13) and the last relation leads to the inequality

$$(\mathscr{L}f, f) \geq \eta_1 p^2 - (|\alpha| + \eta_1) p^2 + \beta \|f_1\|^2$$
$$\geq \eta_1 p^2 - (|\alpha| + \eta_1)(\|\varphi\|^2/q^2 - 1)\|f_1\|^2 + \beta\|f_1\|^2 - A_3(\|f\|^3 + \|f\|^4)$$
$$= \eta_1 p^2 + \eta_2 \|f_1\|^2 - A_3(\|f\|^3 + \|f\|^4)$$
$$\geq A_2 \|f\|^2 - A_3(\|f\|^3 - \|f\|^4), \qquad (3.14)$$

where $A_2 = \min(\eta_1, \eta_2)$.

Now let θ be any number such that $0 < \theta < 1$, multiply (3.9) by θ and (3.14) by $1-\theta$, and add the resulting relations to come to

$$(\mathscr{L}f, f) \geq A\theta \|f\|_m^2 + [A_2(1-\theta) - A_1\theta]\|f\|^2 - A_3(1-\theta)(\|f\|^3 + \|f\|^4).$$

If θ is chosen small enough, the constant in the square brackets on the right-hand side will be positive. It is then concluded that

$$(\mathscr{L}f, f) \geq A\theta\|f\|_m^2 - A_3(1-\theta)(\|f\|^2 + \|f\|^3),$$

and since $\|f\| \leq \|f\|_m$, this completes the proof of the lemma. ■

Lemma 4. Let $T > 0$ and suppose that u is a solution of the initial-value problem (2.3) defined at least on $\mathbf{R} \times [0, T)$. Define h and $\tilde{h}$ as follows:

$$h(x,t) = u(x,t) - \varphi(x + a(t)),$$
$$\tilde{h}(x,t) = h(x,t) + b(t)\varphi(x + a(t)),$$

where $a(t)$ and $b(t)$ are real-valued functions also defined at least for t in the range $[0, T)$.

Suppose that the following two statements hold for all t in $[0, T)$:

(1) $\|\varphi(\cdot + a(t)) + \tilde{h}(\cdot, t)\| = \|\varphi\|$,

(2) $\int_{-\infty}^{+\infty} [\varphi'(x + a(t))\tilde{h}(x,t)]\,dx = 0.$

Then there exist positive constants A_1 and A_2 (which depend on φ but are independent of a, b, u, and ψ) such that

$$\Lambda \geq G(\|h\|_m) - \gamma(|b(t)|),$$

where

$$G(x) = A_1 x^2 - A_2(x^3 + x^{p+3}), \qquad \gamma(x) = A_2(x^2 + x^4). \qquad (3.15)$$

Proof. As before, we have

$$\Lambda = \int_{-\infty}^{+\infty} (L_a h) h \,dx + \int_{-\infty}^{+\infty} R(\varphi, h)\,dx.$$

The second integral on the right-hand side can be estimated using (3.8) and Sobolev's theorem:

$$\left| \int_{-\infty}^{+\infty} R(h)\,dx \right| \leq A \int_{-\infty}^{+\infty} (|h|^3 + |h|^{p+2})\,dx \leq A(\|h\|_m^3 + \|h\|_m^{p+2}). \qquad (3.16)$$

To estimate the first integral, write it in the form

$$\int_{-\infty}^{+\infty} h(\mathscr{L}_a h)\,dx = \int_{-\infty}^{+\infty} \tilde{h}(\mathscr{L}_a \tilde{h})\,dx - 2b(t)\int_{-\infty}^{+\infty} \tilde{h}(\mathscr{L}_a \varphi)\,dx + b^2(t)\int_{-\infty}^{+\infty} \varphi(\mathscr{L}_a \varphi)\,dx. \qquad (3.17)$$

Since $\tilde{h}$ satisfies conditions (1) and (2), lemma 3 may be applied, with $f(x) = \tilde{h}(x,t)$, to obtain

$$\int_{-\infty}^{+\infty} h(\mathscr{L}_a h)\,dx \geq A_1 \|\tilde{h}\|_m^2 - A_2(\|\tilde{h}\|_m^3 + \|\tilde{h}\|_m^4). \qquad (3.18)$$

For any t in $[0, T)$,

$$\|\tilde{h}\|_m = \|h + b\varphi\|_m \geq \|h\|_m - |b|\,\|\varphi\|_m \geq \|h\|_m - A|b|,$$

and so (3.17) and (3.18) imply that

$$\int_{-\infty}^{+\infty} h(\mathscr{L}_a h)\,\mathrm{d}x \ge A_1\|h\|_m^2 - A_2\left(\|h\|_m^3 + \|h\|_m^4 + |b|^3 + |b|^4\right) - A\left(|b|\,\|h\|_m + |b|^2\right).$$

After an application of Young's inequality, and taking account of the fact that p is a positive integer, this becomes

$$\int_{-\infty}^{+\infty} h(\mathscr{L}_a h)\,\mathrm{d}x \ge A_1\|h\|_m^2 - A_2\left(\|h\|_m^3 + \|h\|_m^{p+3} + |b|^2 + |b|^4\right). \tag{3.19}$$

This estimate combined with (3.16) proves the lemma. ■

The conclusion of the theorem will now be established. Suppose $\varepsilon > 0$ to be given. A search is initiated for a $\bar{\delta} > 0$ such that if $0 < \delta < \bar{\delta}$, $\psi \in H^\infty$, and $\|\psi - \varphi\|_m < \delta$, then $d_m(u, \varphi) < \varepsilon$, for all positive t. The functions G and γ appearing in the conclusion of lemma 4 will be considered fixed since they depend only on φ and φ is fixed. Let $\delta_1 > 0$ be such that the function G is strictly increasing on $[0, \delta_1]$. Now define

$$\delta_2 = \min\left\{\varepsilon, \delta_1, \|\varphi'\|^2/2\|\varphi''\|, \|\varphi\|/15\right\}.$$

As previously noted, V and M are continuous functionals on H^m. Therefore Λ tends to zero as $\|\psi - \varphi\|_m$ tends to zero. The continuous function γ defined in (3.15) also vanishes at zero. It follows that a number $\bar{\delta} > 0$ exists such that if $0 < \delta < \bar{\delta}$, then

(i) $\delta < \delta_2$,

(3.20)

(ii) if $\|\psi - \varphi\| < \delta$, then $\Lambda + \gamma(3\delta/\|\varphi\|) < G(\delta_2)$.

Suppose that $0 < \delta < \bar{\delta}$, $\psi \in H^\infty$, and $\|\psi - \varphi\|_m < \delta$. Our aim is to apply lemma 4, and so we seek to construct continuous functions $a(t)$ and $b(t)$ satisfying the conditions (1) and (2). First define the function F: $[0, \infty) \times \mathbf{R} \to \mathbf{R}$ by

$$F(t, y) = \int_{-\infty}^{+\infty} u(x, t)\varphi'(x + y)\,\mathrm{d}x,$$

and search for $a(t)$ as a solution of the equation

$$F(t, a(t)) = 0. \tag{3.21}$$

For $t = 0$ this equation possesses a solution as the following remarks demonstrate. Consider the continuously differentiable function

$$\rho(y) = \int_{-\infty}^{+\infty} [\psi(x) - \varphi(x + y)]^2\,\mathrm{d}x.$$

Quite generally, it appears that

$$\lim_{y \to \pm\infty} \rho(y) = \|\psi\|^2 + \|\varphi\|^2 > \|\varphi\|^2,$$

and because of (3.20) combined with the definition of δ_2,

$$\rho(0) = \|\psi - \varphi\|^2 \le \delta_2^2 < \|\varphi\|^2.$$

Hence the function ρ must take its minimum at some finite value of y, say at $y = a_0$. Then $\rho'(a_0) = 0$, and upon differentiating under the integral, there appears $F(0, a_0) = 0$.

Now define

$$K(t, y) = \frac{(\partial F/\partial t)(t, y)}{(\partial F/\partial y)(t, y)},$$

and consider the initial-value problem

$$\mathrm{d}a/\mathrm{d}t = K(t, a(t)), \qquad a(0) = a_0. \tag{3.22}$$

Clearly, if $a(t)$ solves (3.22) on some interval $[0, T)$, then $F(t, a(t)) = 0$ on that interval.

To determine that (3.22) is solvable, the regularity of K is investigated. It follows from the smoothness of φ and the assumption that (2.3) is well posed in the sense prescribed earlier that both the numerator and denominator appearing in the definition of K are continuously differentiable on $[0, \infty) \times \mathbf{R}$. Moreover, the denominator has a lower bound, namely

$$\left|\frac{\partial F}{\partial y}\right| = \left|\int_{-\infty}^{+\infty}[u(x,t)-\varphi(x+y)]\varphi''(x+y)\,dx - \int_{-\infty}^{+\infty}[\varphi'(x+y)]^2\,dx\right|$$
$$\geq \|\varphi'\|^2 - \left(\int_{-\infty}^{+\infty}[u(x,t) - \varphi(x+y)]^2\,dx\right)^{1/2}\|\varphi''\|.$$

Hence K is defined and continuously differentiable on the set

$$\Omega = \left\{(t,y): \|u(\cdot,t)-\varphi(\cdot+y)\| < \frac{\|\varphi'\|^2}{\|\varphi''\|}\right\}.$$

In particular, the definition of δ_2 insures that $(0, a_0) \in \Omega$, and so the standard existence theory for ordinary differential equations implies that (3.22) has a unique C^1-solution defined at least on some interval $[0, T)$, where $T > 0$.

Let $[0, T_0)$ denote the maximal interval to which this solution may be extended (of course T_0 may be $+\infty$). For $t \in [0, T_0)$, define

$$h(x,t) = u(x,t) - \varphi(x+a(t)),$$

as before. Then it is established that

$$0 = F(t, a(t)) = \int_{-\infty}^{+\infty} h(x,t)\varphi'(x+a(t))\,dx,$$

for all t in $[0, T_0)$. Because of the way $a(t)$ is determined, it is a continuously differentiable function of t in its domain of definition.

In order to define $b(t)$, solve the equation

$$\|u(\cdot,t) + b\varphi(\cdot+a(t))\|^2 = \|\varphi\|^2 \tag{3.23}$$

for b. The solutions are

$$b = b(t) = -p(t) \pm \left(p^2(t) - \frac{\|u\|^2 - \|\varphi\|^2}{\|\varphi\|^2}\right)^{1/2}, \tag{3.24}$$

where

$$p(t) = (u(\cdot,t), \varphi(\cdot+a(t)))/\|\varphi\|^2.$$

It needs to be ascertained that the function b is well defined and continuous. To this end, let $T_1 = \sup\{T\colon T \leq T_0 \text{ and } \|h\| < \|\varphi\|/2 \text{ on } [0,T)\}$. (The assumption that $\delta < \|\varphi\|/15$ implies that the parameter T_1 is positive.) Since $u(x,t) = \varphi(x + a(t)) + h(x,t)$, we have

$$p(t) = \frac{\|\varphi\|^2 + (h(\cdot,t), \varphi(\cdot+a(t)))}{\|\varphi\|^2} \geq 1 - \frac{\|h(\cdot,t)\|}{\|\varphi\|} \geq 1/2, \tag{3.25}$$

for t in $[0, T_1)$, whereas for all t,

$$\frac{\|u(\cdot,t)\|^2 - \|\varphi\|^2}{\|\varphi\|^2} = \frac{\|\psi\|^2 - \|\varphi\|^2}{\|\varphi\|^2} \leq \frac{|\|\psi\| - \|\varphi\||(\|\psi\| + \|\varphi\|)}{\|\varphi\|^2} \leq \frac{\delta(2\|\varphi\|+\delta)}{\|\varphi\|^2} \leq \frac{3\delta}{\|\varphi\|} < \frac{1}{5}. \tag{3.26}$$

The quantity under the radical in (3.24) is thus seen to be positive, and consequently, for t in $[0, T_1)$, $b(t)$ is a well-defined real number if the positive square root is understood throughout. From (3.25) and (3.26) it follows readily that

$$|b(t)| \leq \frac{\|u(\cdot,t)\|^2 - \|\varphi\|^2}{2p(t)\|\varphi\|} \leq \frac{3\delta}{\|\varphi\|}. \tag{3.27}$$

Moreover $b(t)$ is continuous since $a(t)$ is continuous.

With the continuous functions $a(t)$ and $b(t)$ in hand on $[0, T_1)$, the function $\tilde{h}$ is defined as in lemma 4. Then plainly, from the way $b(t)$ is determined,

$$\|\varphi(\cdot + a(t)) + \tilde{h}(\cdot,t)\| = \|\varphi\|.$$

In addition, for $0 < t < T_1$,

$$\int_{-\infty}^{+\infty} \tilde{h}(x,t)\varphi'(x+a(t))\,dx$$
$$= \int_{-\infty}^{+\infty} h(x,t)\varphi'(x+a(t))\,dx + b(t)\int_{-\infty}^{+\infty} \varphi(x)\varphi'(x)\,dx = 0.$$

It therefore follows from lemma 4 and (3.27) that

$$\Lambda \geq G(\|h\|_m) - \gamma(|b(t)|) \geq G(\|h\|_m) - \gamma(3\delta/\|\varphi\|),$$

and hence, by (3.20),

$$G(\|h\|_m) \leq G(\delta_2). \tag{3.28}$$

Since G is increasing on $[0, \delta_1]$, and $\|h\|_m$ is a continuous function of t whose value at $t=0$ is less than δ_1, (3.28) establishes that, for $0 < t < T_1$

$$\|h\|_m \leq \delta_2. \tag{3.29}$$

This immediately implies that $T_1 = T_0 = \infty$. For if $T_1 < \infty$ and $T_1 < T_0$, then $\|h(x, T_1)\|_m \leq \delta_2 \leq \|\varphi\|/15$ contradicts the definition of T_1. On the other hand, if $T_1 = T_0 < \infty$, then from (3.29) it follows that the closure of $\{(t, a(t)): 0 \leq t < T_0\}$ is contained in Ω. Hence (3.22) is solvable on $[0, T_0 + \eta)$ for some $\eta > 0$, and the definition of T_0 is contradicted. Therefore the only possibility is $T_1 = T_0 = \infty$.

Thus (3.29) is seen to hold for all $t > 0$. Since $\delta_2 < \varepsilon$ and $d_m(u, \varphi) \leq \|h\|_m$, the proof of the theorem is now completed for the case $\psi \in H^\infty$.

It remains to treat the case of general $\psi \in H^{s_0}$. For a given $\varepsilon > 0$, define $\bar{\delta}$ as above, and suppose $\psi \in H^{s_0}$ satisfies $\|\psi - \varphi\|_m < \bar{\delta}/2$. Let $\{\psi_n\}$ be a sequence of functions in H^∞ such that $\|\psi_n - \varphi\|_{s_0} \to 0$ as $n \to \infty$, and $\|\psi_n - \varphi\|_m < \bar{\delta}$ for all n. Our well-posedness assumptions for problem (2.3) imply that the solution $u(x,t)$ of (2.3), with initial data $\psi(x)$, exists and satisfies

$$\lim_{n\to\infty} \|u_n(\cdot,t) - u(\cdot,t)\|_{s_0} = 0$$

for any fixed value of t. It follows that $d_m(u, \varphi) < \varepsilon$, and the proof of the theorem is thus complete. ■

In the applications to be considered in the next section, conditions (P_1), (P_2), and (P_3) will be verified directly for the particular equations under consideration. However, it is of interest to search for general classes of equations for which these conditions may reasonably be expected to hold. The following two propositions summarize some results in this direction.

Proposition 1. The spectrum of $\mathscr{L}_\varphi$ consists of the interval $[C, \infty)$ together with a finite number of eigenvalues.

Proof. This proposition follows easily from the spectral theory of closed operators as presented in ref. [30]. First, note that the essential spectrum of the operator $L + C$ is the interval $[C, \infty)$, while the operator $\mathscr{L}_\varphi$ is a perturbation of $L + C$ by a relatively compact operator (the verification of these facts is similar to that given in ref. [30], section IV.5.3, for the case where L is the Laplacian). Therefore, by theorem IV.5.35 of ref [30], the essential spectrum of $\mathscr{L}_\varphi$ is also $[C, \infty)$. It then follows from theorem IV.5.17 of ref. [30] that the dimensions of the nullspace and deficiency of $\mathscr{L}_\varphi - \lambda I$ are independent of λ if $\lambda \notin [C, \infty)$, with the possible exception of a set of isolated points $\{\lambda_n\}$. Moreover, since $\mathscr{L}_\varphi$ is self-adjoint, theorem V.3.16 of ref. [30] implies that $\mathrm{null}(\mathscr{L}_\varphi - \lambda I) = \mathrm{def}(\mathscr{L}_\varphi - \lambda I) = 0$ for $\lambda \notin \{\lambda_n\} \cup [C, \infty)$. This shows that the λ_n are isolated eigenvalues of $\mathscr{L}_\varphi$.

To show that the set of all λ_n is finite, it suffices to show that the spectrum of $\mathscr{L}_\varphi$ is bounded below. In fact, it will now be shown that if $K = (|\varphi|_\infty)^p + C$, then $\mathrm{spec}(\mathscr{L}_\varphi)$ does not intersect the interval $(-\infty, -K)$. To see this, let $\lambda < -K$, and consider $\mathscr{L}_\varphi - \lambda I = (L - \lambda I) + (C - \varphi^p)$. Since $L - \lambda I$ has symbol $(\alpha(k) - \lambda)$, and $\lambda < 0$, it appears that $L - \lambda I$ is invertible, as an operator on L_2, and that

$$\|(L - \lambda I)^{-1}\|_{2,2} = \sup_{k \in \mathbf{R}} \left| \frac{1}{\alpha(k) - \lambda} \right| = \frac{1}{|\lambda|}.$$

As mentioned in section 3, the symbol $\|\cdot\|_{2,2}$ denotes the operator norm in the algebra of bounded linear operators on L_2. On the other hand, one has that

$$\|C-\varphi^p\|_{2,2} \le K < |\lambda| = \frac{1}{\|(L-\lambda I)^{-1}\|_{2,2}}.$$

Hence, the Neumann series for the inverse of $(L-\lambda I)+(C-\varphi^p)$ converges, so that $(\mathscr{L}_\varphi-\lambda I)^{-1}$ exists and is bounded. This shows that $\lambda \notin \operatorname{spec}(\mathscr{L}_\varphi)$, and completes the proof of the proposition. ■

For any $\mu>0$, define $R_\mu=(L+\mu)^{-1}$. Notice that R_μ is a Fourier multiplier operator with symbol $1/(\mu+\alpha(k))$. From (3.2) it follows that $1/(\mu+\alpha(k)) \in L_2$, so that there is a well-defined function $K_\mu(x) \in L_2$ satisfying $(\widehat{K_\mu})(k)=1/(\mu+\alpha(k))$. It may be easily verified that, for any $f \in L_2$,

$$R_\mu f(x)=\int_{-\infty}^{+\infty} K_\mu(x-y) f(y)\,\mathrm{d}y.$$

(Notice that the existence of the integral is guaranteed by Hölder's Inequality.)

Proposition 2. Suppose that for $\mu>0$ sufficiently large, one has $K_\mu>0$ for all $x\in\mathbf{R}$. Then the least eigenvalue α of $\mathscr{L}_\varphi$ is a simple eigenvalue, and any eigenfunction $\psi(x)$ corresponding to α satisfies either $\psi(x)>0$ a.e. or $\psi(x)<0$ a.e. (Hence part of (P_2) is satisfied in this case, and the quantity on the left-hand side of the inequality in (P_3) is at least non-zero.)

Proof. We first establish the following lemma.

Lemma 5. There exists $\mu_1>0$ such that the operator $T=(\mathscr{L}_\varphi+\mu_1)^{-1}$ exists, is bounded on L_2, and has the property that if $f(x)\in L^2$ and $f(x)\ge 0$ for all $x\in\mathbf{R}$, and f is not zero a.e., then $Tf(x)>0$ for a.e. $x\in\mathbf{R}$.

Proof of lemma 5. Let $\nu>0$ be chosen such that $\nu+\varphi^p-C>0$. Then the equation

$$f=(\mathscr{L}_\varphi+\mu)g \tag{3.30}$$

may be rewritten in the form

$$(I-M)g=Kf,$$

where $Mg=R_{(\mu+\nu)}(\nu+\varphi^p-C)g$ and $Kf=R_{(\mu+\nu)}f$. If $\mu=\mu_1$ is chosen sufficiently large, then $\|M\|_{2,2}<1$, and so the solution $g=Tf$ of (3.30) exists and is given by

$$Tf=\sum_{n=0}^{\infty} M^n Kf,$$

where the series converges in L_2 norm. From the definitions of M and K, one sees that for any n, the operator M^nK is an integral operator with strictly positive kernel. Therefore, if $f(x)\ge 0$, each term of the series must be strictly positive at every $x\in\mathbf{R}$. Since convergence of a sequence of functions in L_2 implies pointwise convergence of some subsequence almost everywhere, this establishes the lemma. ■

Proof of proposition 2. Let μ_1 and T be as defined in lemma 5. Then $\operatorname{spec}(T)$ is the image of $\operatorname{spec}(\mathscr{L}_\varphi)$ under the transformation $\lambda\to(\lambda+\mu_1)^{-1}$. Denote the greatest eigenvalue of T by λ_0, and let ψ be an eigenfunction corresponding to λ_0. From the spectral theorem for bounded operators on L_2, one has, for all $\rho\in L_2$ such that $\|\rho\|=\|\psi\|$,

$$(T\psi,\psi)\ge(T\rho,\rho).$$

But, from lemma 5, one sees that $(T(|\psi|),|\psi|)\ge(T\psi,\psi)$, and it therefore follows that

$$(T(|\psi|),|\psi|)=(T\psi,\psi). \tag{3.31}$$

Now let ψ^+ and ψ^- be the positive and negative parts of ψ, so that $|\psi|=\psi^++\psi^-$ and $\psi=\psi^+-\psi^-$. Then from (3.31) one obtains

$$(T\psi^+,\psi^-)=0. \tag{3.32}$$

We claim that this implies that one of ψ^+ or ψ^- must vanish almost everywhere. Suppose the contrary. Then $\psi^-(x)>0$ for all x in a set of positive measure E, while $T\psi^+(x)>0$ for a.e. $x\in\mathbf{R}$, by lemma 5. In particular, there exists $\varepsilon>0$ such that $T\psi^+(x)>\varepsilon$ for all x in a set of positive measure in E. But this contradicts (3.32), and the claim is thus established. It follows that any eigenfunction ψ of T corresponding to λ_0 satisfies either $\psi(x)>0$ a.e. or $\psi(x)<0$ a.e. But these eigenfunctions are exactly the eigenfunctions of $\mathscr{L}_\varphi$ corresponding to the least eigenvalue α.

It remains to show that α is simple, and for this purpose it suffices to show that if ψ_1 and ψ_2 are any two eigenfunctions corresponding to α, then $(\psi_1,\psi_2)\neq 0$. Indeed, the result of the preceding paragraph shows that

$$(\psi_1,\psi_2)=\int_{-\infty}^{+\infty}\psi_1\psi_2\,\mathrm{d}x = \pm\int_{-\infty}^{+\infty}|\psi_1(x)\psi_2(x)|\,\mathrm{d}x,$$

whereas $|\psi_1(x)\psi_2(x)|>0$ for a.e. $x\in\mathbf{R}$. Therefore, $(\psi_1,\psi_2)\neq 0$. ■

In the sequel, it will often be convenient to note that the condition (P_3) in theorem 1 may be replaced by another condition, which is slightly stronger, but which has the advantage of being simpler to verify. For easy reference, this fact is stated below in the form of a separate theorem.

Theorem 2. In theorem 1, the condition (P_3) may be replaced by the following condition:

$$(P_3')\quad \int_{-\infty}^{+\infty}[\varphi(x)]^{p+2}\,\mathrm{d}x>\frac{p+1}{p}(|\alpha|-\beta)\times\int_{-\infty}^{+\infty}[\varphi(x)]^2\,\mathrm{d}x.$$

Proof. It is enough to show that the conditions (P_1), (P_2), and (P_3') together imply (P_3). To see this, note first that an integration of (3.3) yields

$$L\varphi+\left(C-\frac{\varphi^p}{p+1}\right)\varphi=0.$$

Hence

$$(\mathscr{L}_\varphi\varphi,\varphi)=(L\varphi+(C-\varphi^p)\varphi,\varphi) =\left(\frac{-p}{p+1}\varphi^{p+1},\varphi\right) =\frac{-p}{p+1}\int_{-\infty}^{+\infty}[\varphi(x)]^{p+2}\,\mathrm{d}x.$$

It therefore follows from (P_3') that

$$(\mathscr{L}_\varphi\varphi,\varphi)<(\beta-|\alpha|)\int_{-\infty}^{+\infty}[\varphi(x)]^2\,\mathrm{d}x,$$

and hence that

$$\frac{\beta\|\varphi\|^2-(\mathscr{L}_\varphi\varphi,\varphi)}{\beta+|\alpha|}>\frac{\|\varphi\|^2|\alpha|}{\beta+|\alpha|}. \tag{3.33}$$

On the other hand, properties (P_1) and (P_2), together with the spectral theorem, imply that

$$(\mathscr{L}_\varphi(\varphi-j\psi_\alpha),(\varphi-j\psi_\alpha)) >\beta((\varphi-j\psi_\alpha),(\varphi-j\psi_\alpha)), \tag{3.34}$$

where ψ_α denotes an eigenfunction for the eigenvalue α (with $\|\psi_\alpha\|=1$), and j denotes (φ,ψ_α). Expanding the inner products in (3.34) and simplifying yields

$$(\mathscr{L}_\varphi\varphi,\varphi)-j^2\alpha>\beta(\|\varphi\|^2-j^2),$$

and hence

$$j^2>\frac{\beta\|\varphi\|^2-(\mathscr{L}_\varphi\varphi,\varphi)}{\beta+|\alpha|}. \tag{3.35}$$

From (3.33) and (3.35) it follows that

$$(\varphi,\psi_\alpha)^2=j^2>\frac{\|\varphi\|^2|\alpha|}{\beta+|\alpha|}=\frac{\|\varphi\|^2\|\psi_\alpha\|^2}{1+(\beta/|\alpha|)},$$

and this is (P_3). ■

Whereas the theory detailed above for eq. (1.1) depended essentially on the existence of the invariant functionals V and M, a similar theory may be constructed for eqs. (1.2) using the invariant functionals V_1 and M.

Suppose that $D > 0$ and $\tilde{\varphi} = \tilde{\varphi}_D$ is a solitary-wave solution of (1.2) with speed $1 + D$. It follows that

$$(1+D)L(\tilde{\varphi})' + \left[D - (\tilde{\varphi})^p\right](\tilde{\varphi})' = 0,$$

and therefore the operator $\tilde{\mathscr{L}}_{\tilde{\varphi}}$ defined by

$$\tilde{\mathscr{L}}_{\tilde{\varphi}} f(x) = (1+D)Lf(x) + \left[D - (\tilde{\varphi})^p(x)\right] f(x)$$

has 0 as an eigenvalue, with eigenfunction $\tilde{\varphi}'$.

Theorem 3. Suppose that conditions (P_1), (P_2), and (P_3) of theorem 1 hold with φ replaced by $\tilde{\varphi}$ and $\mathscr{L}_\varphi$ by $\tilde{\mathscr{L}}_{\tilde{\varphi}}$. Then $\tilde{\varphi}$ is a stable solution of (1.2) (in the sense of theorem 1). The same conclusion holds if (P_3) is replaced by (P_3').

Proof. To prove theorem 3, one notes that if

$$\Delta M = M(u(\cdot, t)) - M(\tilde{\varphi}),$$

$$\Delta V_1 = V_1(u(\cdot, t)) - V(\tilde{\varphi}),$$

then

$$\Delta M + D\Delta V_1 = \int_{-\infty}^{+\infty} \tilde{\mathscr{L}}_{\tilde{\varphi}} h(x) \cdot h(x)\,dx + \int_{-\infty}^{+\infty} R(h, \tilde{\varphi})\,dx, \qquad (3.36)$$

where the expression denoted by $R(h, \tilde{\varphi})$ may be estimated as in (3.8). From (3.36), the proof then proceeds exactly as does the proof of theorem 1. ■

In fact, any application of theorem 1 to prove the stability of a solitary-wave solution of (1.1) automatically results in a proof of stability for a solitary-wave solution of (1.2). It is easily seen that the solitary-wave solutions $\tilde{\varphi}_D$ of (1.2) are related to the solitary-wave solutions φ_C of (1.1) by the identity

$$\tilde{\varphi}_{C/(1-C)}(\xi) = \left(\frac{1}{1-C}\right)^{1/p} \varphi_C(\xi), \qquad \xi \in \mathbf{R}.$$

Moreover, for $D = C/(1 - C)$, one has

$$\tilde{\mathscr{L}}_{\tilde{\varphi}_D}(f) = (1+D)\mathscr{L}_{\varphi_C}(f).$$

Since properties (P_1)–(P_3) and (P_3') are preserved if the operator in question is transformed by scalar multiplication, it follows that these properties hold for $\tilde{\mathscr{L}}_{\tilde{\varphi}_D}$ if and only if they hold for $\mathscr{L}_{\varphi_C}$.

4. Stability results for some model equations

The theory of the preceding sections will now be applied to various model equations for long-wave phenomena. These include, firstly, the (generalized) Korteweg–de Vries (KdV) equation

$$u_t + u_x + u^p u_x + u_{xxx} = 0, \qquad p \geq 1, \qquad (4.1)$$

whose solitary-wave solutions are given by $u(x, t) = \varphi^{p,C}(x - (1 + C)t)$, where

$$\varphi^{p,C}(y) = \left[\tfrac{1}{2}(p+1)(p+2)C\,\mathrm{sech}^2\left(\tfrac{1}{2}p\sqrt{C}\,y\right)\right]^{1/p},$$

and C is any positive real number. Also treated will be the Benjamin–Ono (BO) equation

$$u_t + u_x + uu_x - (\mathscr{H}u)_{xx} = 0, \qquad (4.2)$$

where $\mathscr{H}$ denotes the Hilbert transform, defined as a singular integral operator by

$$\mathscr{H}f(x) = \frac{1}{\pi}\,\text{p.v.} \int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\,dy,$$

or as a Fourier multiplier operator by $\widehat{(\mathscr{H}f)}(k) = \mathrm{i}(\mathrm{sign}\,k)\hat{f}(k)$. The solitary-wave solutions (discovered by Benjamin [2]) of this equation are given by $\varphi_\infty^C(x - (1 + C)t)$, where C is a positive constant and

$$\varphi_\infty^C(y) = \frac{4C}{C^2y^2 + 1}.$$

Finally, we will consider the Intermediate Long

Wave (ILW) equation

$$u_t + u_x + uu_x - (L_H u)_x = 0, \tag{4.3}$$

where H denotes a positive parameter, and L_H is the Fourier multiplier operator defined by

$$\widehat{(L_H u)}(k) = m_H(k)\hat{u}(k),$$
$$m_H(k) = k\coth(kH) - \frac{1}{H}.$$

Joseph [31] found that (4.3) has the solitary-wave solutions $u(x,t) = \varphi_H^C(x - (1+\tilde{C})t)$, where the functional form of φ_H^C is

$$\varphi_H^C(y) = \frac{C}{\cosh^2(ay) + \dfrac{C^2\sinh^2(ay)}{16a^2}},$$

and the wave speed $1+\tilde{C}$ and the parameter a are given as functions of C and H by

$$a\tan(aH) = C/4,$$
$$\tilde{C} = (1/H) - (4a^2/C) + (C/4). \tag{4.4}$$

To emphasize the similarities between the above equations, and to simplify notation, the quantities $-u_{xx}$ and $(\mathscr{H}u)_x$ will be denoted below by $L_0 u$ and $L_\infty u$, respectively. Therefore the KdV equation (in the case $p=1$) may be written as

$$u_t + u_x + uu_x - (L_0 u)_x = 0, \tag{4.5}$$

where $\widehat{(L_0 u)}(k) = k^2\hat{u}(k)$, and the BO equation becomes

$$u_t + u_x + uu_x - (L_\infty u)_x = 0, \tag{4.6}$$

where $\widehat{(L_\infty u)}(k) = |k|\hat{u}(k)$. Also, the notation φ_0^C will be used for the solitary-wave solutions of (4.5), as a substitute for the notation $\varphi^{1,C}$ defined earlier. Thus $\varphi_0^C(y) = 3C\,\mathrm{sech}^2[\frac{1}{2}\sqrt{C}\,y]$.

The operators associated by theorem 1 with the solitary waves φ_0^C, φ_H^C and φ_∞^C $(0<C<\infty, 0<H<\infty)$ will be denoted by $\mathscr{L}_0^C$, $\mathscr{L}_H^C$, and $\mathscr{L}_\infty^C$ respectively, where

$$\mathscr{L}_0^C f = L_0 f + (C - \varphi_0^C)f,$$
$$\mathscr{L}_H^C f = L_H f + (\tilde{C} - \varphi_H^C)f,$$
$$\mathscr{L}_\infty^C f = L_\infty f + (C - \varphi_\infty^C)f.$$

It will be convenient in what follows to take note of various similarity relations between the solitary waves under discussion and their associated operators. For any real number $\theta \neq 0$, define the dilation operator T_θ by $(T_\theta f)(x) = f(\theta x)$. Then the following identities are easily verified (for any C_0, $C_1 > 0$):

$$\begin{aligned}
\mathscr{L}_0^{C_0} &= \theta\left(T_{\sqrt{\theta}}\mathscr{L}_0^{C_1}T_{\sqrt{\theta}}^{-1}\right),\\
\varphi_0^{C_0} &= \theta\left(T_{\sqrt{\theta}}\varphi_0^{C_1}\right),\\
\mathscr{L}_\infty^{C_0} &= \theta\left(T_\theta\mathscr{L}_\infty^{C_1}T_\theta^{-1}\right),\\
\varphi_\infty^{C_0} &= \theta\left(T_\theta\varphi_\infty^{C_1}\right),
\end{aligned} \tag{4.7}$$

where $\theta = C_0/C_1$. Moreover, if $C_0, C_1, H_0, H_1 > 0$ and $C_0H_0 = C_1H_1$, then, defining $\theta = C_0/C_1 = H_1/H_0$, one obtains the formulas

$$\begin{aligned}
\mathscr{L}_{H_0}^{C_0} &= \theta\left(T_0\mathscr{L}_{H_1}^{C_1}T_0^{-1}\right),\\
\varphi_{H_0}^{C_0} &= \theta\left(T_\theta\varphi_{H_1}^{C_1}\right).
\end{aligned} \tag{4.8}$$

Lemma 6. Suppose $C_0, C_1 > 0$. Then any of the properties (P_1), (P_2), (P_3), or (P_3') holds for $\mathscr{L}_0^{C_0}$ (resp. $\mathscr{L}_\infty^{C_0}$) if and only if it holds for $\mathscr{L}_0^{C_1}$ (resp. $\mathscr{L}_\infty^{C_1}$). Also, if $H_0, H_1 > 0$ and $C_0H_0 = C_1H_1$, then any of the above properties holds for $\mathscr{L}_{H_0}^{C_0}$ if and only if it holds for $\mathscr{L}_{H_1}^{C_1}$.

Proof. It follows from (4.7) that $\mathrm{spec}(\mathscr{L}_0^{C_0}) = \{\theta\lambda\colon \lambda \in \mathrm{spec}(\mathscr{L}_0^{C_1})\}$; and also that ψ is an eigenfunction of $\mathscr{L}_0^{C_1}$ with eigenvalue λ, if and only if $T_{\sqrt{\theta}}\psi$ is an eigenfunction of $\mathscr{L}_0^{C_0}$ with eigenvalue $\theta\lambda$. Clearly properties (P_1), (P_2), (P_3), and (P_3') are preserved if the spectrum and eigenfunctions are transformed in this way. The same argument applies to the operators $\mathscr{L}_\infty^C$ and $\mathscr{L}_H^C$. ∎

The stability of the solitary-wave solutions of the KdV and BO equations has already been proved by Benjamin [18], Bona [19], and Bennett et al. [21]. Their proofs were essentially based on spectral analyses of the operators $\mathscr{L}_0^C$ and $\mathscr{L}_\infty^C$. The objective of this section is, firstly, to recast these results in the framework of the theory of the preceding section, and, secondly, to use these results, together with a perturbation argument, to establish the stability of certain of the solitary waves of the ILW equation.

Theorem 4. Let $C>0$ be given.

(a) Given any $\varepsilon>0$, there exists a $\delta>0$ such that if $\psi\in H^2$ and $\|\psi-\varphi_0^C\|_1<\delta$, then the solution $u(x,t)$ of (4.5) with initial value $u(x,0)=\psi(x)$ satisfies $d_1(u,\varphi_0^C)<\varepsilon$ for all $t\ge 0$.

(b) Given any $\varepsilon>0$, there exists a $\delta>0$ such that if $\psi\in H^s$, $s\ge 2$, and $\|\psi-\varphi_\infty^C\|_{1/2}<\delta$, then the solution $u(x,t)$ of (4.6) with initial data $u(x,0)=\psi(x)$ satisfies $d_{1/2}(u,\varphi_\infty^C)<\varepsilon$ for all $t\ge 0$.

Proof. The well-posedness of the initial-value problems for (4.5) and (4.6) in H^s, $s\ge 2$, may be found in refs. [28, 32]. By theorem 3 and lemma 6, it is enough to show that $\mathscr{L}_0^C$ and $\mathscr{L}_\infty^C$ each satisfy (P_1), (P_2), and (P_3') for some particular value of $C>0$. For convenience, consider the particular operators

$$\mathscr{L}_0^4 = L_0 + \left(4 - \frac{12}{\cosh^2(x)}\right),$$

$$\mathscr{L}_\infty^1 = L_\infty + \left(1 - \frac{4}{1+x^2}\right).$$

The eigenvalue problem for the ordinary differential operator $\mathscr{L}_0^4$ is classical (see, e.g., ref. [33]). The spectrum of $\mathscr{L}_0^4$ consists of the continuous part $[4,\infty)$, together with simple eigenvalues at $\lambda=-5$, $\lambda=0$, and $\lambda=3$. Hence properties (P_1) and (P_2) are satisfied, and (P_3') takes the form

$$\int_{-\infty}^{+\infty}[\varphi_0^4(x)]^3\,dx \ge 4\int_{-\infty}^{+\infty}[\varphi_0^4(x)]^2\,dx.$$

But elementary integrations show that

$$\int_{-\infty}^{+\infty}[\varphi_0^4(x)]^3\,dx = 1728\cdot\tfrac{16}{15},$$

$$\int_{-\infty}^{+\infty}[\varphi_0^4(x)]^2\,dx = 144\cdot\tfrac{4}{3},$$

so that the inequality is valid.

A complete spectral analysis of the operator $\mathscr{L}_\infty^1$ was carried out in ref. [21]. There, it was proved that $\operatorname{spec}(\mathscr{L}_\infty^1)$ consists of the continuous part $[1,\infty)$ together with simple eigenvalues at $\lambda=\frac{1}{2}(-\sqrt{5}-1)$, $\lambda=0$, and $\lambda=\frac{1}{2}(\sqrt{5}-1)$. Hence (P_1) and (P_2) are satisfied, and (P_3') becomes

$$\int_{-\infty}^{+\infty}[\varphi_\infty^1(x)]^3\,dx \ge 2\int_{-\infty}^{+\infty}[\varphi_\infty^1(x)]^2\,dx.$$

Again, elementary computations show that

$$\int_{-\infty}^{+\infty}[\varphi_\infty^1(x)]^3\,dx = 24\pi,$$

$$\int_{-\infty}^{+\infty}[\varphi_\infty^1(x)]^2\,dx = 8\pi,$$

and thus (P_3') is verified. ■

Remark. When combined with theorem 3, the proof of theorem 4 shows that the solitary-wave solutions of the equations

$$u_t + u_x + uu_x - u_{xxt} = 0$$

and

$$u_t + u_x + uu_x - (\mathscr{H}u)_{xt} = 0$$

are stable.

An examination of the symbols of the operators L_0, L_H, and L_∞ reveals a close relationship between the ILW equation and the two equations discussed in the preceding theorem. When H is near 0, the symbol $m_H(k)$ of L_H approximates that of L_0 (at least for small values of k) while $m_H(k)$ tends pointwise to the symbol of L_∞ as $H\to\infty$. This indicates that the ILW equation interpolates between the KdV and BO equations on the interval $0 < H < \infty$, approximating one or the other equation near the endpoints of the interval. (This relationship is also suggested by the nature of the physical phenomena which the equations are intended to model, see ref. [4].) Hence, one might expect that an analogue of theorem 4 holds for the ILW equation.

Such a result would follow from theorem 2 if the properties (P_1), (P_2), and (P_3') were verified for the operators $\mathscr{L}_H^C$ $(0 < C < \infty, 0 < H < \infty)$. Unfortunately, a spectral analysis of these operators is not available, and at present, the question of whether these properties are satisified for all values of $C > 0$ and $H > 0$ remains open. In theorem 5, a partial result will be obtained, namely, that $\mathscr{L}_H^C$ satisfies (P_1), (P_2), and (P_3') when CH is sufficiently near 0 (resp. ∞), by virtue of being a small perturbation of the operator $\mathscr{L}_0^{C/3}$ (resp. $\mathscr{L}_\infty^{C/4}$).

Before proving this result, we state some of the facts which we will need from perturbation theory (see ref. [30] for details).

Give $L_2 \times L_2$ the Hilbert space norm defined by $\|(f, g)\| = (\|f\|^2 + \|g\|^2)^{1/2}$, and for any closed operator T on L^2 with domain $D(T)$, define $G(T) = \{(f, g) \in L_2 \times L_2 \colon f \in D(T) \text{ and } T(f) = g\}$. Then a metric $\hat{\delta}$ on $C(L_2)$, the space of closed operators on L_2, may be defined as follows: for any $S, T \in C(L_2)$,

$$\hat{\delta}(S, T) = \| P_S - P_T \|_{B(L_2 \times L_2)},$$

where P_S and P_T are the orthogonal projections on $G(S)$ and $G(T)$, and $\| \quad \|_{B(L_2 \times L_2)}$ denotes the operator norm on the space of bounded operators on $L_2 \times L_2$.

Proposition 3. Let $S, T \in C(L_2)$, and suppose A is a bounded operator on L_2 with operator norm $\|A\|_{2,2}$. Then

$$\hat{\delta}(T + A, T) \le \|A\|_{2,2},$$

$$\hat{\delta}(S + A, T + A) \le 2\left(1 + \|A\|_{2,2}^2\right)\hat{\delta}(S, T).$$

If, in addition, S and T are invertible, then

$$\hat{\delta}(S, T) = \hat{\delta}(S^{-1}, T^{-1}).$$

Proposition 4. Let $T \in C(L_2)$ and let U denote an open subset of the complex plane whose boundary is a smooth contour Γ. Suppose that $(\text{spec}(T)) \cap \Gamma = \emptyset$ and $(\text{spec}(T)) \cap U$ consists of a finite number of eigenvalues of T, each with finite (algebraic) multiplicity. Then there exists $\delta > 0$ such that if $S \in C(L_2)$ and $\hat{\delta}(S, T) < \delta$, then $(\text{Spec}(S)) \cap \overline{U}$ consists of a finite number of eigenvalues of finite multiplicity, the sum of their multiplicities being equal to the sum of the multiplicities of the eigenvalues of T in U.

In particular, suppose $(\text{spec}(T)) \cap U$ consists of a single, simple eigenvalue μ. If $\{S_n\}$ is a sequence in $C(L_2)$ such that $\hat{\delta}(S_n, T) \to 0$ as $n \to \infty$, then for n large, $(\text{spec}(S_n)) \cap U$ consists of a single simple eigenvalue μ_n, and $\mu_n \to \mu$ as $n \to \infty$.

The well-posedness properties of eq. (4.3), which are necessary for the proof of the next theorem, have been established in ref. [28].

Theorem 5. There exist constants $\eta_1 > 0$ and $\eta_2 > 0$ such that if $0 < CH < \eta_1$ or $\eta_2 < CH < \infty$, then φ_H^C is a stable solution of (4.3). That is, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\psi \in H^{s_0}$ and $\|\psi - \varphi_H^C\|_{1/2} < \delta$, and $u(x, t)$ is the solution of (4.3) with initial data $u(x, 0) = \psi(x)$, then $d_{1/2}(u, \varphi_H^C) < \varepsilon$ for all $t \ge 0$.

Proof. By theorem 3 and lemma 6, it suffices to show that if H is sufficiently small or sufficiently large, then the operator $\mathscr{L}_H^C$ satisfies (P_1), (P_2), and (P_3') in the case $C = 4$. In the remainder of this proof, the value of C will be fixed at 4, and the superscripts will be dropped from $\mathscr{L}_H^C$ and φ_H^C, so that $\mathscr{L}_H$ and φ_H are understood to mean $\mathscr{L}_H^4$ and φ_H^4. Also, $\mathscr{L}_0$ and φ_0 will be used to denote $\mathscr{L}_0^{4/3}$ and $\varphi_0^{4/3}$, while $\mathscr{L}_\infty$ and φ_∞ will denote $\mathscr{L}_\infty^1$ and φ_∞^1.

Lemma 7. For $H > 0$, define $\gamma = \gamma(H) = \sqrt{H/3}$.

Then

(a) $\lim_{H\to\infty} \varphi_H(y) = \varphi_\infty(y)$, uniformly for $y \in \mathbf{R}$,

(b) $\lim_{H\to\infty} \hat{\delta}(\mathscr{L}_H, \mathscr{L}_\infty) = 0$,

(c) $\lim_{H\to 0} T_\gamma \varphi_H(y) = \varphi_0(y)$, uniformly for $y \in \mathbf{R}$,

(d) $\lim_{H\to 0} \hat{\delta}\left(\mathscr{L}_H, T_\gamma^{-1}\mathscr{L}_0 T_\gamma\right) = 0.$

Proof of lemma 7. For $y \in [0, \infty)$, define $g(y)$ to be the unique number in $[0, \pi/2)$ such that

$$g(y)\tan(g(y)) = y.$$

Then, in the present case where $C = 4$, the definitions of a and $\tilde{C}$ in (4.4) may be rewritten as

$$a = a(H) = \frac{g(H)}{H},$$

$$\tilde{C} = \tilde{C}(H) = \frac{1}{H} - a^2 + 1.$$

From L'Hôpital's rule one obtains

$$\lim_{y\to 0} \frac{g(y)}{\sqrt{y}} = 1, \qquad \lim_{y\to 0} \frac{1}{y}\left(1 - \frac{g^2(y)}{y}\right) = \frac{1}{3},$$

from which it follows that

$$\lim_{H\to 0} a(H) = \infty, \qquad \lim_{H\to 0} \tilde{C}(H) = \tfrac{4}{3}.$$

Also, it is clear that

$$\lim_{H\to\infty} a(H) = 0, \qquad \lim_{H\to\infty} \tilde{C}(H) = 1.$$

Since

$$\varphi_H(y) = \frac{4}{\cosh^2(ay) + \dfrac{\sinh^2(ay)}{a^2}},$$

$$\varphi_\infty(y) = \frac{4}{1 + y^2},$$

part (a) of the lemma follows immediately.

On the other hand, since

$$T_\gamma \varphi_H(y) = \frac{4}{\cosh^2(a\gamma y) + \dfrac{\sinh^2(a\gamma y)}{a^2}},$$

$$\varphi_0(y) = \frac{4}{\cosh^2(y/\sqrt{3})},$$

part (c) of the Lemma follows from the fact that

$$\lim_{H\to 0} a\gamma = \lim_{H\to 0} \frac{g(H)}{\sqrt{H}\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

For the proof of part (b), first write $\mathscr{L}_H = L_H + M_H$ and $\mathscr{L}_\infty = L_\infty + M_\infty$, where $M_H f(x) = [\tilde{C} - \varphi_H(x)]f(x)$ and $M_\infty f(x) = [1 - \varphi_\infty(x)]f(x)$. Next, note that $L_H - L_\infty$ and $M_H - M_\infty$ are bounded operators on L_2, with operator norms tending to 0 as $H \to \infty$. In fact, one has, for $f \in L_2$,

$$\left[\widehat{(L_H - L_\infty)f}\right](k) = [m_H(k) - |k|]\hat{f}(k),$$

so that

$$\|L_H - L_\infty\|_{2,2} = |m_H(k) - |k||_\infty$$
$$= \left|\frac{1}{H}(|kH| - (kH)\coth(kH) + 1)\right|_\infty$$
$$= \frac{1}{H}|p|_\infty,$$

where $p(k) = |k| - k\coth(k) + 1$. Therefore

$$\lim_{H\to\infty} \|L_H - L_\infty\|_{2,2} = \lim_{H\to\infty} (|p|_\infty)/H = 0.$$

Also, since

$$\|M_H - M_\infty\|_{2,2} \le |\tilde{C} - 1| + |\varphi_\infty - \varphi_H|_\infty,$$

it follows from part (a) that

$$\lim_{H\to\infty} \|M_H - M_\infty\|_{2,2} = 0.$$

Finally, $\hat{\delta}(\mathscr{L}_H, \mathscr{L}_\infty)$ may be estimated by means of proposition 4 as follows:

$$\begin{aligned}\hat{\delta}(\mathscr{L}_H, \mathscr{L}_\infty) &= \hat{\delta}(L_H + M_H, L_\infty + M_\infty)\\ &\le C(1 + \|M_H\|_{2,2})\\ &\quad \times \hat{\delta}(L_H, L_\infty + (M_\infty - M_H))\\ &\le C(1 + \|M_H\|_{2,2})\\ &\quad \times [\hat{\delta}(L_H, L_\infty) + \hat{\delta}(L_\infty, L_\infty + (M_\infty - M_H))]\\ &\le C(1 + \|M_H\|_{2,2})\\ &\quad \times [\|L_H - L_\infty\|_{2,2} + \|M_\infty - M_H\|_{2,2}].\end{aligned}$$

Since $\|M_H\|_{2,2}$ remains bounded as $H \to \infty$, this completes the proof of part (b).

To prove part (d), write $\mathscr{L}_H = L_H + M_H$ as above, and let $T_\gamma^{-1}\mathscr{L}_0 T_\gamma = L_\gamma + M_\gamma$, where $L_\gamma = T_\gamma^{-1} L_0 T_\gamma = \gamma^2 L_0$ and

$$\begin{aligned}(M_\gamma f)(x) &= \left(\tfrac{4}{3} - T_\gamma^{-1}\varphi_0 T_\gamma\right) f(x)\\ &= \tfrac{4}{3} f(x) - \left(T_\gamma^{-1}\varphi_0\right)(x) \cdot f(x).\end{aligned}$$

Next, it will be shown that

$$\lim_{H \to 0} \|M_H - M_\gamma\|_{2,2} = 0 \tag{4.9}$$

and

$$\lim_{H \to 0} \|(I + L_H)^{-1} - (I + L_\gamma)^{-1}\|_{2,2} = 0, \tag{4.10}$$

where I is the identity operator on L_2. The assertion (4.9) follows immediately from part (c) and the estimate

$$\|M_H - M_\gamma\|_{2,2} \le |\tilde{C}(H) - \tfrac{4}{3}| + |T_\gamma^{-1}\varphi_0 - \varphi_H|_\infty.$$

To prove (4.10), notice that $(I + L_H)^{-1} - (I + L_\gamma)^{-1}$ is a Fourier multiplier operator with symbol

$$\sigma(k) = \frac{1}{1 + \frac{1}{3}Hk^2} - \frac{1}{1 + k\coth kH - H^{-1}}.$$

Introducing the notation $y = Hk$ and $\beta(y) = (y\coth(y) - 1)/y^2$, one has

$$\begin{aligned}\|(I + L_H)^{-1} - (I + L_\gamma)^{-1}\|_{2,2} &= \sup_{k \in \mathbf{R}} |\sigma(k)|\\ &= \sup_{y \in \mathbf{R}} \left| \frac{y^2}{3H + y^2} \frac{3\beta(y) - 1}{1 + H^{-1}y^2\beta(y)} \right|\\ &\le \sup_{y \in \mathbf{R}} \frac{|3\beta(y) - 1|}{1 + H^{-1}y^2\beta(y)}.\end{aligned} \tag{4.11}$$

Now, from the definition of $\beta(y)$, it is easily seen that $\lim_{y \to 0}(3\beta(y) - 1) = 0$, that $y^2\beta(y)$ is an increasing function of y, and that there exists $\mu < \infty$ such that $0 \le \beta(y) < \mu$ for all real numbers y. Suppose that $\varepsilon > 0$ is given, and choose $\delta = \delta(\varepsilon) > 0$ so that $|3\beta(y) - 1| < \varepsilon$ for $|y| \le \delta$. Then for arbitrary $H > 0$, one has

$$\sup_{|y| \le \delta} \frac{|3\beta(y) - 1|}{1 + H^{-1}y^2\beta(y)} < \varepsilon. \tag{4.12}$$

Choose H_0 so small that

$$\frac{3\mu + 1}{1 + H_0^{-1}y^2\beta(y)} < \varepsilon \quad \text{for all } |y| \ge \delta.$$

Then, for $0 < H < H_0(\varepsilon)$,

$$\sup_{|y| \ge \delta} \frac{|3\beta(y) - 1|}{1 + H^{-1}y^2\beta(y)} < \varepsilon.$$

Combined with (4.12), this shows that the quantities in (4.11) are less than ε when $H < H_0(\varepsilon)$, thus establishing the validity of (4.10).

Finally, using proposition 3, the following estimate is obtained:

$$\begin{aligned}&\hat{\delta}\left(\mathscr{L}_H, T_\gamma^{-1}\mathscr{L}_0 T_\gamma\right)\\ &\quad = \hat{\delta}\left(L_H + M_H, L_\gamma + M_\gamma\right)\\ &\quad \le C(1 + \|M_H\|_{2,2})\\ &\qquad \times \hat{\delta}\left(L_H, L_\gamma + (M_\gamma - M_H)\right)\\ &\quad \le C(1 + \|M_H\|_{2,2})\\ &\qquad \times \left[\hat{\delta}\left(L_H, L_\gamma\right) + \hat{\delta}\left(L_\gamma, L_\gamma + (M_\gamma - M_H)\right)\right]\end{aligned}$$

$$\leq C(1+\|M_H\|_{2,2}) \times\left[4\hat{\delta}(L_H+I, L_\gamma+I)+\|M_\gamma-M_H\|_{2,2}\right] = C(1+\|M_H\|_{2,2})\left[4\hat{\delta}\left((L_H+I)^{-1},(L_\gamma+I)^{-1}\right) +\|M_\gamma-M_H\|_{2,2}\right].$$

Combined with (4.9) and (4.10), this completes the proof of (d), and lemma 7 is now established. ■

As was seen in the proof of theorem 4, the operators $\mathscr{L}_\infty$ and $\mathscr{L}_0$ satisfy properties (P_1), (P_2), and (P_3'). Denote the negative eigenvalue of $\mathscr{L}_0$ (resp. $\mathscr{L}_\infty$) by α_0 (resp. α_∞), and let the real number defined in (P_3) for $\mathscr{L}_0$ (resp. $\mathscr{L}_\infty$) be denoted by β_0 (resp. β_∞). Also, choose $K<\infty$ such that

$$K>\max\left(|\alpha_0|,|\alpha_\infty|\right),$$

$$K>\sup_{0<H<\infty}\left(|\varphi_H|_\infty+\tilde{C}(H)\right)$$

(the right-hand side of the last inequality is already known to be finite from the proof of lemma 7). As was seen in the proof of proposition 1 of the preceding section, it is a consequence of this choice of K that

$$(\operatorname{spec}(\mathscr{L}_H))\cap(-\infty,-K]=\emptyset, \quad \text{for all } H>0. \tag{4.13}$$

For any ε such that $0<\varepsilon<\beta_\infty/2$, define $\Gamma_{(\varepsilon)}$ to be the circle in the complex plane which is centered on the real axis and which intersects the real axis at $z=\beta_\infty-\varepsilon$ and $z=-K$. If $U_{(\varepsilon)}$ is the open disc bounded by $\Gamma_{(\varepsilon)}$, then $(\operatorname{spec}(\mathscr{L}_\infty))\cap\overline{U}_{(\varepsilon)}=\{\alpha_\infty,0\}$. Also choose circular contours Γ_1 and Γ_2, contained in $U_{(\varepsilon)}$, such that if U_1 and U_2 are the open discs bounded by Γ_1 and Γ_2, then $(\operatorname{spec}(\mathscr{L}_\infty))\cap\overline{U}_1=\{\alpha_\infty\}$, and $(\operatorname{spec}(\mathscr{L}_\infty))\cap\overline{U}_2=\{0\}$.

From lemma 7 and proposition 4, it follows that there exists $H_0>0$ such that if $H>H_0$, then $(\operatorname{spec}(\mathscr{L}_H))\cap\overline{U}_1$ and $(\operatorname{spec}(\mathscr{L}_H))\cap\overline{U}_2$ each consist of a single, simple eigenvalue. Since 0 is an eigenvalue of $\mathscr{L}_H$ (with eigenfunction φ_H'), we must then have $(\operatorname{spec}(\mathscr{L}_H))\cap\overline{U}_2=\{0\}$, which shows that 0 is a simple eigenvalue of $\mathscr{L}_H$, and hence (P_1) is satisfied. Similarly, an application of lemma 7 and proposition 4 to $\Gamma_{(\varepsilon)}$ shows that there exists $H_\varepsilon>0$ such that if $H>H_\varepsilon$, then $(\operatorname{spec}(\mathscr{L}_H))\cap\overline{U}_{(\varepsilon)}$ consists of a finite set of eigenvalues of total multiplicity 2. Therefore, if $H>\max(H_0,H_\varepsilon)$ it may be concluded that

$$(\operatorname{spec}(\mathscr{L}_H))\cap\overline{U}_\varepsilon=\{\alpha_H,0\},$$

where α_H is the simple eigenvalue in $(\operatorname{spec}(\mathscr{L}_H))\cap\overline{U}_1$. Taken together with (4.13), this shows that (P_2) is satisfied for large H.

Next, it follows from proposition 4 that $\lim_{H\to\infty}\alpha_H=\alpha_\infty$. Also, since ε may be taken arbitrarily small in the argument of the preceding paragraph, it has actually been shown that if

$$\beta_H=\inf\{\lambda\in\operatorname{spec}(\mathscr{L}_H):\ \lambda\neq\alpha_H \text{ and } \lambda\neq 0\},$$

then $\lim_{H\to\infty}\beta_H=\beta_\infty$. Finally, from the dominated convergence theorem and lemma 7 it follows that, as $H\to\infty$,

$$\int_{-\infty}^{+\infty}[\varphi_H(x)]^r\,\mathrm{d}x\to\int_{-\infty}^{+\infty}[\varphi_\infty(x)]^r\,\mathrm{d}x, \quad \text{for any } r>1.$$

Therefore, it appears that the inequality

$$\int_{-\infty}^{+\infty}[\varphi_H(x)]^3\,\mathrm{d}x>2(|\alpha_H|-\beta_H)\times\int_{-\infty}^{+\infty}[\varphi_H(x)]^3\,\mathrm{d}x$$

is true in the limit as $H\to\infty$. Since the inequality is strict, it must also hold for values of H which are sufficiently near to ∞. Thus (P_3') is established for large values of H.

The verification that (P_1), (P_2), and (P_3') hold for small values of H is similar. Since $\operatorname{spec}(T_\gamma^{-1}\mathscr{L}_0T_\gamma)=\operatorname{spec}(\mathscr{L}_0)$ for all $\gamma>0$, proposition 4 and lemma 7 may be used to show that when H is sufficiently small, $\mathscr{L}_H$ has simple eigenvalues $\gamma = \alpha_H < 0$ and $\lambda = 0$, and that if β_H is as defined above, then

$$\lim_{H\to 0} \alpha_H = \alpha_0 \quad \text{and} \quad \lim_{H\to 0} \beta_H = \beta_0.$$

Then from lemma 7(c) it follows that

$$\int_{-\infty}^{+\infty} [T_\gamma \varphi_H(x)]^3 \, dx > 2(|\alpha_H| - \beta_H) \times \int_{-\infty}^{+\infty} [T_\gamma \varphi_H(x)]^2 \, dx \tag{4.14}$$

for sufficiently small H. However, for any $r \ge 1$, one has

$$\int_{-\infty}^{+\infty} [T_\gamma \varphi_H(x)]^r \, dx = \frac{1}{\gamma} \int_{-\infty}^{+\infty} [\varphi_H(x)]^r \, dx,$$

and so (P_3') follows immediately from (4.14). This completes the proof of the theorem. ■

Remarks. The three equations studied in this section all happen to fall into the category of equations to which proposition 2 of the preceding section may be applied. In the case of KdV, where $\alpha(k) = k^2$, one has

$$K_\mu(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{ikx}}{\mu + k^2} \, dk = \frac{e^{-\sqrt{\mu}|x|}}{2\sqrt{\mu}} > 0$$

for all $x \in \mathbf{R}$. In the case of BO, where $\alpha(k) = |k|$, one has

$$K_\mu(x) = \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} \frac{e^{ikx}}{\mu + k} \, dk.$$

For $x > 0$, a change of the contour of integration (justified by Jordan's lemma) then shows that

$$K_\mu(x) = \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} \frac{i e^{-x\tau}}{\mu + i\tau} \, d\tau = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-x\tau}}{\mu^2 + \tau^2} \, d\tau > 0.$$

(For $x < 0$, the evenness of $\alpha(k)$ implies $K(x) = K(-x) > 0$.) Finally for ILW, $K_\mu(x)$ may be computed (in the case $\mu H > 1$) by means of a residue calculation as follows:

$$K_\mu(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{ikx}}{\mu - (1/H) + k \coth kH} \, dk = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{ikx/H}}{\mu H - 1 + k \coth k} \, dk = \sum_{\substack{\theta > 0 \\ \theta \cot \theta + H\mu - 1 = 0}} e^{-\theta x/H} \frac{2 \sin^2 \theta}{2\theta - \sin(2\theta)} > 0,$$

where it is assumed that $x > 0$. Again, for $x < 0$ one has $K_\mu(x) = K_\mu(-x) > 0$.

This section concludes with a result on the stability of solitary-wave solutions of eq. (4.1). (This result may also be found in ref. [22].)

Theorem 6. Suppose $p = 1$, 2, or 3. Then for all $C > 0$, $\varphi^{p,C}$ is a stable solution of (4.1). That is, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\psi \in H^2$ and $\|\psi - \varphi^{p,C}\|_1 < \delta$, then the solution $u(x,t)$ of (4.1) with initial data $u(x,0) = \psi(x)$ satisfies $d_1(u, \varphi^{p,C}) < \varepsilon$ for all $t \ge 0$.

Proof. Given a solitary wave $\varphi^{p,C}$ ($p \ge 1, C > 0$), define the operator $\mathscr{L}^{p,C}$ by

$$\mathscr{L}^{p,C} f = L_0 f + \left(C - (\varphi^{p,C})^p\right) f.$$

It is enough to show that $\mathscr{L}^{p,C}$ satisfies the conditions (P_1), (P_2), and (P_3') for all $C > 0$, if $p = 1$, 2, or 3.

As in (4.7) and (4.8), there exists an identity relating the various $\mathscr{L}^{p,C}$, namely (for p fixed),

$$\mathscr{L}^{p,C_0} = \theta\left(T_{\sqrt{\theta}} \mathscr{L}^{p,C_1} T_{\sqrt{\theta}}^{-1}\right),$$

where $\theta = C_0/C_1$. Hence, it suffices to prove that (P_1), (P_2), and (P_3') hold for a particular choice of C. If we fix $C = C_0 = 4/p^2$, then

$$\mathscr{L}^{p,C_0} f = -f''(x) + \left(\frac{4}{p^2} - \frac{2(p+1)(p+2)}{p^2} \operatorname{sech}^2(x)\right) f(x).$$

The spectral properties of this operator are well-known for all values of $p > 0$ (see, e.g., ref. [33], p. 768). The case $p = 1$ has already been considered in the proof of theorem 4. For $p \geq 2$, $\operatorname{spec}(\mathscr{L}^{p,C_0})$ consists of the continuous part $[4/p^2, \infty)$ together with simple eigenvalues at $\lambda = -(p+4)/p$ and $\lambda = 0$. Hence, for all $p \geq 2$, $\mathscr{L}^{p,C_0}$ satisfies (P_1) and (P_2).

To verify (P_3'), make use of the identity

$$\int_{-\infty}^{\infty} (\operatorname{sech} x^r)\,\mathrm{d}x = \frac{\Gamma(1/2)\Gamma(r/2)}{\Gamma((r+1)/2)}$$

to reduce (P_3') to the form

$$w^{(p+2)/p} \frac{\Gamma(1/2)\Gamma((p+2)/p)}{\Gamma((p+2)/p+1/2)} > \frac{p+1}{p}\left(\frac{4+p}{p} - \frac{4}{p^2}\right) w^{2/p} \frac{\Gamma(1/2)\Gamma(2/p)}{\Gamma(2/p+1/2)}, \tag{4.15}$$

where $w = 2(p+1)(p+2)/p^2$. By further simplifying (4.15), using the identity $\Gamma(s+1) = s\Gamma(s)$, one arrives at the inequality

$$p^3 - 4p - 16 < 0,$$

which is satisfied for $p = 2$ and $p = 3$. This completes the proof of the theorem. ■

5. Conclusion

It is apparent from the results exposed in the last section that the abstract criteria formulated in section 3 are effective in establishing the stability of travelling-wave solutions of equations appertaining to real physical situations. Moreover, the calculations leading to the presented conclusions of stability are relatively straightforward, thanks especially to the secondary criterion (P_3').

One should remark, however, that our criteria are probably not sharp in general. Indeed, consider the equation

$$u_t + u_x(1 + |u|^p) + u_{xxx} = 0, \tag{5.1}$$

where p is any positive real number. While this equation may have no particular significance as a model of physical phenomena, it is certainly useful as an example. As in (4.2), the solitary-wave solutions of (5.1) are given by the formula

$$u(x,t) = \varphi^{p,C}(x - (1+C)t),$$

where

$$\varphi^{p,C}(y) = \left[\tfrac{1}{2}(p+1)(p+2)C \times \operatorname{sech}^2\left(\tfrac{1}{2}pC^{1/2}y\right)\right]^{1/p}. \tag{5.2}$$

Under these circumstances, the results developed in the proof of theorem 6 apply for arbitrary $p \geq 2$, though certain of the calculations require a little more care. It transpires that both (P_3) and (P_3') fail for values of p strictly less than four. However, several different clues point in the same direction, and lead one to confidently expect stability of the solutions (5.2) of eq. (5.1) for all p less than four. Thus a sharper criterion for stability than that developed herein should be available.

Another issue of considerable interest that has been left open in the present analysis is the issue of stability or not of the entire class of intermediate depth equation solitary waves, and not just those bordering upon the Korteweg–de Vries equation or the Benjamin–Ono equation. According to our methods, proving stability of this class of waves will require a more direct spectral analysis of the associated linear operator $\mathscr{L}_\varphi$ than is provided by perturbation theory.

Perhaps more interesting still is the possibility of bringing the general scheme presented herein to bear upon the issue of stability of solitary-wave solutions of the two-dimensional Euler equations.

References

[1] D.J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a

new type of long stationary waves, Phil. Mag. (5) 39 (1895) 422.
[2] T.B. Benjamin, Internal waves of permanent form in fluids of great depth, J. Fluid Mech. 29 (1967) 559.
[3] H. Ono, Algebraic solitary waves in stratified fluids, J. Phys. Soc. Japan 39 (1975) 1082.
[4] T. Kubota, D. Ko and L. Dobbs, Weakly-nonlinear long internal gravity waves in stratified fluids of finite depth, J. Hydrodynamics 12 (1978) 157.
[5] T.B. Benjamin, J.L. Bona and J.J. Mahony, Model equations for long waves in nonlinear dispersive systems, Phil. Trans. R. Soc. London A 272 (1972) 47.
[6] T.B. Benjamin, Lectures on nonlinear wave motion, in: Lectures in Applied Mathematics, Vol. 15, A. Newell, ed. (American Mathematical Society, Providence, RI, 1974), pp. 3–47.
[7] T.B. Benjamin, J.L. Bona and D.K. Bose, Solitary-wave solutions for some model equations for waves in nonlinear dispersive media, in: Proc. Joint IUTAM/IMU Symp. on Applications of Methods of Functional Analysis for Problems in Mechanics, Marseille (1975), Springer Lecture Notes in Mathematics, No. 503 (Springer, Berlin, 1976), p. 207.
[8] T.B. Benjamin, J.L. Bona and D.K. Bose, Fixed-point theorems in Fréchet spaces and the existence of solitary waves (1986), in preparation.
[9] J.L. Bona, On solitary waves and their role in the evolution of long waves, in: Applications of Nonlinear Analysis in the Physical Sciences, H. Amann, N. Basley and K. Kirchgässner, eds. (Pitman, London, 1981), pp. 183–207.
[10] M. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform (Society for Industrial and Applied Mathematics, Philadelphia, 1981).
[11] K.O. Abdulloev, I.L. Bogolubsky and V.G. Makhankov, One more example of inelastic soliton interaction, Phys. Lett. A 56 (1976) 427.
[12] J.L. Bona, W.G. Pritchard and L.R. Scott, Solitary wave interaction, Phys. Fluids 23 (1980) 438.
[13] J.L. Bona, W.G. Pritchard and L.R. Scott, A comparison of solutions of two model equations for long waves, in: Lectures in Applied Mathematics, Vol. 20, N. Lebovitz, ed. (American Mathematical Society, Providence, 1983), pp. 235–267.
[14] R.E. Davis and A. Acrivos, Solitary internal waves in deep water, J. Fluid Mech. 29 (1967) 593.
[15] J.L. Hammack, A note on tsunamis: their generation and propagation in an ocean of uniform depth, J. Fluid Mech. 60 (1973) 769.
[16] J.L. Hammack and H. Segur, The Korteweg–de Vries equation and water waves. 2. Comparison with experiment, J. Fluid Mech. 65 (1974) 289.
[17] P.D. Weidman and T. Maxworthy, Experiments on strong interactions between solitary waves, J. Fluid Mech. 85 (1978) 417.
[18] T.B. Benjamin, The stability of solitary waves, Proc. R. Soc. London A 328 (1972), 153.
[19] J.L. Bona, On the stability theory of solitary waves, Proc. R. Soc. London A 344 (1975) 363.
[20] G. Scharf and W.F. Wreszinski, Stability for the Korteweg–de Vries equation by inverse scattering theory, Ann. Phys. 134 (1981) 56.
[21] D.P. Bennett, J.L. Bona, R.W. Brown, S.E. Stansfield and J.D. Stroughair, The stability of internal solitary waves, Math. Proc. Camb. Phil. Soc. 94 (1983) 351.
[22] M. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Commun. Pure Appl. Math. 39 (1986) 51.
[23] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, No. 30 (Princeton University Press, Princeton, NJ, 1970).
[24] J.-L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications, Vol. 1 (Dunod, Paris, 1968).
[25] F. Treves, Topological Vector Spaces, Distributions and Kernels (Academic Press, New York, 1967).
[26] T. Kato, On the Cauchy problem for the (generalized) Korteweg–de Vries equation, Studies in Applied Mathematics, Advances in Mathematics Supplementary Studies, Vol. 8 (1983), p. 93.
[27] J.-C. Saut, Sur quelques généralisations de l'equation de Korteweg–de Vries, J. Math. Pures Appl. 58 (1975) 21.
[28] L. Abdelhouhab, J.L. Bona, M. Felland and J.-C. Saut, The Cauchy problem for the Benjamin–Ono and the intermediate-depth equations (1986), to appear.
[29] J. Boussinesq, Essai sur la théorie des eaux courantes, Mém. prés. div. Sar. Acad. Sci. Inst. 23 (1877) 1.
[30] T. Kato, Perturbation Theory for Linear Operators, 2nd ed. (Springer, Berlin, 1976).
[31] R.I. Joseph, Solitary waves in a finite depth fluid, J. Phys. A 10 (1977) L225.
[32] J.L. Bona and R. Smith, The initial-value problem for the Korteweg–de Vries equation, Phil. Trans. R. Soc. London A 278 (1975) 555.
[33] P. Morse and H. Feshbach, Methods of Theoretical Physics, Vol. 1 (McGraw-Hill, New York, 1953).