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*A CRITERION FOR THE FORMATION OF SINGULARITIES FOR THE GENERALIZED KORTEWEG-de VRIES EQUATION**

J.P. Albert [1], J.L. Bona [2§] and M. Felland [3]

[1] Department of Mathematics
University of Oklahoma
Norman, OK 73019, USA

[2] Department of Mathematics and Applied Research Laboratory
The Pennsylvania State Univesity
University Park, PA 16802, USA

[3] Department of Mathematics and Computer Science
Clarkson University
Potsdam, NY 13676, USA

ABSTRACT: *It is shown that solutions u of the initial-value problem for the generalized Korteweg-de Vries equation*

$$u_t + f(u)_x + u_{xxx} = 0$$

form singularities in finite time in and only if the L_∞*-norm of u becomes infinite.*

KEY WORDS: Blow-up of solutions ● Korteweg-de Vries equation ● nonlinear dispersive waves ● singularities of solution of wave equations

RESUMO: *UM CRITÉRIO PARA A FORMAÇÃO DE SINGULARIDADES PARA A EQUAÇÃO GENERALIZADA DE KORTEWEG-de VRIES — As soluções u do problema de valor inicial para a equação generalizada de Korteweg-de Vries*

$$u_t + f(u)_x + u_{xxx} = 0$$

formam singularidades em tempo finito se e somente se a norma L_∞ *de u torna-se infinita.*

PALAVRAS-CHAVE: Soluções "explosivas" ● equação de Korteweg-de Vries ● ondas dispersivas não lineares ● singularidades de soluções de equações de onda

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1. INTRODUCTION

In this note a simple criterion is derived that relates to whether or not solutions of the initial-value problem for the Generalized Korteweg-de Vries equation

$$\begin{aligned} &u_t + f(u)_x + u_{xxx} = 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1}$$

form singularities in finite time. Here, f is a smooth, real-valued function of a real variable which, without loss of generality, may be assumed to have the property that $f(0) = f'(0) = 0$, and $u = u(x,t)$ is a real-valued function of the two independent variables x and t. The family of equations depicted in (1) arises in problems connected with modelling uni-directional wave propagation in nonlinear, dispersive media, and in such situations x is often proportional to distance in the direction of propagation of the wave motion and t is proportional to elapsed time (see [5], [14] and the references contained therein). The pure initial-value problem (1) comes about in presuming that the waveform u is known completely at some fixed instant of time, and then inquiring as to its subsequent development according to the model. It transpires that if u is the solution of (1) corresponding to smooth initial data u_0 that decays to zero at infinity appropriately, then $u(\cdot,t)$ loses regularity as a function of x at a finite value $t = t_0 > 0$ if and only if u becomes unbounded as t approaches t_0.

While the result obtained here is not definitive as regards the existence of singularities, it does show what to look for analytically, and it is especially useful information to keep in mind whilst searching numerically for evidence of the formation of singularities (cf. [6]).

The difference between the Generalized Korteweg-de Vries equations (GKdV equations henceforth) and the simple conservation laws

$$\begin{aligned} &u_t + f(u)_x = 0 \\ &u(x,0) = u_0(x), \end{aligned} \tag{2}$$

deserves remark. If, for example, $f(z) = z^{p+1}$, with p a positive integer, then it is well understood that solutions of (2) corresponding to positive, smooth initial data defined on all of the real line $\mathbb{R}$, and which tend to zero at infinity, remain bounded, but the derivative becomes unbounded in finite time. If the nonlinearity f is not too strong, by which we mean that p is not too large in the special case $f(z) = z^{p+1}$, then the dispersive term u_{xxx} overcomes the effect of nonlinear steepening observed in (2), essentially by converting the large gradients into a train of solitary waves. However, if the nonlinearity is too strong, then apparently the dispersive term does not necessarily overcome the nonlinear steepening, and the result may be the formation of singularities in the solution (see again, [6]).

The particular result described above concerning formation of singularities is new, insofar as we are aware, but the technical apparatus used to establish it already exists in the literature. We shall rely especially on the theory for quasi-linear evolution equations due to Kato [8,9] which, when applied to the GKdV equations yields telling results which form the backbone of the observation reported here. The earlier work of Schechter [12] also deserves mention, for the crucial differential inequalities are already set forth as examples in the exposition of his theory for nonlinear evolution equations.

The theoretical development is given in Section 2, where the relevant aspects of Kato's results are reviewed. Section 3 contains a few comments concerning the foregoing results and a considerable list of open questions in the general domain to which the paper is devoted.

2. A CRITERION FOR BLOW-UP

The main mathematical result announced in the introduction is now stated, as Theorem 2, and proved.

The terminology and notation used throughout is that of the modern theory of partial differential equations (see, for example, Lions' text [10]). In general, the norm of a function g in a Banach space X will be denoted $\|g\|_X$. However, if g lies in $X = H^s(\mathbb{R})$, the Sobolev class of functions in $L_2(\mathbb{R})$ whose derivatives up to order s also lie in $L_2(\mathbb{R})$, then the norm of g in $H^s(\mathbb{R})$ is denoted by $\|g\|_s$. Similarly, if g lies in $X = L_p(\mathbb{R})$, the norm of g is denoted $|g|_p$.

Here is Kato's result concerning the local well-posedness of the initial-value problem (1).

<u>Theorem 1</u>. Let $s_0 > \frac{3}{2}$. For any $A \geq 0$ there exists $T = T(A) > 0$ depending only on A such that if $u_0 \in H^s(\mathbb{R})$ with $s \geq s_0$ and $\|u_0\|_0 \leq A$, then there is a unique solution u of (1) in the space $C(0,T;H^s(\mathbb{R}))$. The function u has the property $\partial_t^k u \in C(0,T;H^{s-3k}(\mathbb{R}))$ so long as $s - 3k \geq -2$. Moreover, for any $s \geq s_0$, the correspondence $u_0 \to u$ defines a continuous map from $H^s(\mathbb{R})$ to $\bigcap_{s-3k\geq -2} C^k(0,T;\, H^{s-3k}(\mathbb{R}))$.

In addition, if u corresponds to $u_0 \in H^s(\mathbb{R})$ with $s \geq s_0$, as above, then for all $t \in [0,T]$,

$$\int_{-\infty}^{\infty} u^2(x,t)dx = \int_{-\infty}^{\infty} u_0^2(x)dx \tag{3}$$

and

$$\int_{-\infty}^{\infty}\left\{u_X^2(x,t) - 2f_1(u(x,t))\right\}dx = \int_{-\infty}^{\infty}\left\{u_{0X}^2(x) - 2f_1(u_0(x))\right\}dx, \tag{4}$$

where $f_1' = f$ and $f_1(0) = 0$. If $s \geq 2$, then for all $t \in [0,T]$,

$$\int_{-\infty}^{\infty}\left\{u_{XX}^2(x,t) - \frac{5}{3} f'(u(x,t))u_X^2(x,t)\right\}dx =$$

$$\int_{-\infty}^{\infty}\left\{u_{0XX}^2(x) - \frac{5}{3} f'(u_0(x))u_{0X}^2(x)\right\}dx + \tag{5}$$

$$\int_0^t\int_{-\infty}^{\infty}\left\{\frac{1}{12} f''''(u(x,s))u_X^5(x,s) + \frac{5}{3} f'(u(x,s))f''(u(x,s))u_X^3(x,s)\right\}dxds.$$

Remarks. The invariants (3) and (4) are standard aspects of equations of the form (1), or indeed of the more general equations (11) (cf. [1], [4], [5] and [11]). The relation (5) was first derived by Schechter [12] for the case $f(u) = u^{p+1}$, and later by Kato in the form reported here. Both authors used it in combination with a result of local existence to establish sufficient conditions for the existence of globally defined, smooth solutions of (1).

Theorem 2. Suppose $s \geq 2$ and let $u_0 \in H^s(\mathbb{R})$. Let T^* be the maximum value such that, for all $T \in (0,T^*)$, the solution u of (1) with initial data u_0 lies in $C(0,T;H^s(\mathbb{R}))$. Then either $T^* = +\infty$ and the solution u is global, or else

$$\sup_{0\leq t<T^*} |u(\cdot,t)|_\infty = +\infty . \tag{6}$$

Proof. Suppose there exists a finite constant B such that

$$|u(\cdot,t)|_\infty \leq B \quad \text{for} \quad 0 \leq t < T^*.$$

This assumption and the relations (3) and (4) yield a uniform bound on $\| u(\cdot,t)\|_1$ as follows. First, by (3),

$$\int_{-\infty}^{\infty} u^2(x,t)dx = \int_{-\infty}^{\infty} u_0^2(x)dx.$$

Hence, from (4), we derive that

$$\int_{-\infty}^{\infty} u_x^2(x,t)dx \le 2\int_{-\infty}^{\infty} |f_1(u(x,t))|dx + \int_{-\infty}^{\infty} u_{0x}^2(x)dx + 2\int_{-\infty}^{\infty} |f_1(u_0(x))|dx$$

$$\le 2K\left(\int_{-\infty}^{\infty} u^2(x,t)dx + \int_{-\infty}^{\infty} u_0^2(x)dx\right) + \int_{-\infty}^{\infty} u_{0x}^2(x)dx,$$

where

$$K = \sup_{|z|\le B} \frac{|f_1(z)|}{z^2} < +\infty .$$

It follows that

$$\| u(\cdot,t)\|_1^2 \le 4K\| u_0\|_0^2 + \| u_0^{\cdot}\|_1^2 \tag{7}$$

for all $t \in [0,T^*)$.

The relationship (7) may be combined with the equation (5) to obtain bounds on the $H^2(\mathbb{R})$-seminorm of the solution u that is in question. For, by (5),

$$\| u_{xx}(\cdot,t)\|_0^2 \le \frac{5}{3}\int_{-\infty}^{\infty} |f'(u(x,t))|u_x^2(x,t)dx + \| u_{0xx}\|_0^2$$

$$+ \frac{5}{3}\int_{-\infty}^{\infty} |f'(u_0(x))|u_{0x}^2(x)dx + \tag{8}$$

$$+ \int_0^t\int_{-\infty}^{\infty}\left\{\frac{1}{12}|f''''(u(x,s))|\,|u_x^5(x,s)| + |f'(u(x,s))f''(u(x,s))|\,|u_x^3(x,s)|\right\}dxds.$$

But since $|g|_\infty^2 \le \| g\|_0 \| g'\|_0$ for any $g \in H^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} |u_x(x,s)|^r dx \le |u_x(\cdot,s)|_\infty^{r-2}\| u_x(\cdot,s)\|_0^2 \le \| u_x(\cdot,s)\|_0^{\frac{r+2}{2}} \| u_{xx}(\cdot,0)\|_0^{\frac{r-2}{2}} .$$

Using this relation with r = 3 and r = 5 in (8) yields

$$\| u_{xx}(\cdot,t)\|_0^2 \le M\left[\| u_x(\cdot,t)\|_0^2 + \| u_{0x}\|_0^2 + \| u_{0xx}\|_0^2\right]$$

$$+M\int_0^t\left\{\| u_x(\cdot,s)\|_0^{7/2}\| u_{xx}(\cdot,s)\|_0^{3/2} + \| u_x(\cdot,s)\|_0^{5/2}\| u_{xx}(\cdot,s)\|_0^{1/2}\right\}ds,$$

where M depends only on the quantity

$$\sup_{|z|\le B} \{|f'(z)| + |f'''(z)| + |f'(z)f''(z)|\}.$$

An $H^2(\mathbb{R})$-bound on u may now be adduced from (9) and (7). From these inequalities, one obtains the relation

$$\| u_{xx}(\cdot,t)\|_0^2 \le \beta + \gamma\int_0^t \| u_{xx}(\cdot,s)\|_0^2 ds,$$

with the positive constants β and γ dependent only upon M, K, and $\| u_0\|_2$. Thus β and γ are independent of t so long as the original, boundedness assumption holds. A standard Gronwall-type argument gives

$$\| u_{xx}(\cdot,t)\|_0^2 \le \beta e^{\gamma t}, \tag{10}$$

for all $t \in [0,T^*)$.

It follows that $T^* = +\infty$. For if not, (10) would imply the existence of a finite constant A such that $\| u(\cdot,t)\|_2 \le A$ on $[0,T^*)$. Applying Theorem 1 with $s = 2$ and u_0 replaced by $u(\cdot,T^* - \frac{1}{2}T)$, where $T = T(A)$, would yield a solution of (1), in $C(0,\tilde{T};H^s(\mathbb{R}))$ and $\tilde{T}$ could be taken to be at least $T^* + \frac{1}{2}T$. This contradicts the maximality of T^* and establishes the assertion of Theorem 2.

Remarks. The boundedness assumption was only used in conjunction with the invariants (3) and (4) to show that $\| u_x(\cdot,t)\|_0$ was bounded on $[0,T^*)$. Consequently, it would also suffice in deriving our conclusions to assume that

$$\sup_{0\le t<T^*} \int_{-\infty}^{\infty} |f_1(u(\cdot,t))| dx$$

is finite. In case $f(u) = u^{p+1}$, this amounts to assuming that the $L_{p+2}(\mathbb{R})$-norm of u is bounded on $[0,T^*)$. However, as a practical criterion, particularly as related to numerical simulations, the boundedness of the $L_\infty(\mathbb{R})$-norm appears preferable even though it is a weaker condition.

Exactly the same considerations as presented above apply to the periodic initial-value problem where u_0 is taken to be a periodic function and the solution u of (1) maintains the same spatial periodicity. As numerical calculations have been performed on the periodic initial-value problem, it is perhaps worth formalizing this last remark.

Theorem 3. Let u_0 be a periodic function of period P which lies in $H^s([0,P])$ where $s \ge 2$. Then there exists a $T > 0$ depending only upon $\| u_0\|_s$ such that (1) possesses a unique, P-periodic solution $u \in C(0,T;H^s([0,P]))$. This solution also has $\partial_t^k u \in C(0,T;H^{s-3k}([0,P]))$ so long as $s - 3k \geq -2$, and the correspondence $u_0 \to u$ is continuous from $H^s([0,P])$ to $C(0,T;H^s([0,P]))$.

If T^* is the maximal time for the existence of the solution u, in the sense that $u \in C(0,T;H^s([0,P]))$ for all $0 < T < T^*$, then either $T^* = +\infty$ or else

$$\sup_{0 \leq t < T^*} |u(\cdot,t)|_{L_\infty([0,P])} = +\infty .$$

3. DISCUSSION

The principal issue to which this investigation was directed remains open, namely to decide whether or not solutions u of (1) actually form singularities. This point is important to clarify, and its resolution would aid our understanding of the interaction between nonlinearity and dispersion. The numerically obtained evidence reported in [6] and the instability result in [7] indicate that singularity formation may indeed occur in the context of (1) provided that f is sufficiently strong. In case $f(u) = u^{p+1}$, "sufficiently strong" means simply that $p \geq 4$, a critical exponent. (It should be pointed out that for $f(u) = u^{p+1}$, where $p > 4$, the results of Strauss [13] imply that if the initial data u_0 is sufficiently small, then the solution of (1) guaranteed by Theorem 1 corresponding to u_0 does in fact exist for all time. In fact, $|u(\cdot,t)|_\infty$ tends to zero as t approaches infinity. Similar results have been obtained by Albert [2,3] for model equations of the form (13), mentioned presently, in case $L = -\partial_x^2$). Whilst the case $f(u) = u^{p+1}$ with $p \geq 4$ in (1) does not arise naturally in models of real phenomena, similar issues crop up for more general equations to be discussed below where the critical exponent for the possibility of singularity formation in a pure power nonlinearity is one that occurs naturally.

The more general class of equations of GKdV type to which reference was made above has the form

$$u_t + f(u)_x - L(u)_x = 0. \tag{11}$$

Here, f and u are as before and L is a Fourier-multiplier operator

$$\widehat{L(v)}(\xi) = a(\xi)\hat{v}(\xi), \tag{12}$$

which reflects more general dispersion relations than that characterized by $-\partial_x^2$ in (1). The circumflexes above connote Fourier transforms. A discussion of linearized dispersion relations and the class of model equations (11) may be found in [1], [4], [5], and [11]. A local existence theory for this class of equations along the lines of that presented in Theorem 1 has been formulated recently in [1], building upon the earlier work of Saut [11]. A criterion for the existence of global solutions of (11), similar to that enunciated in Theorem 2 would be of considerable interest, but has thus far proved elusive.

It deserves remark in the context of more general dispersion relations that the phenomenon of singularity formation does not occur, no matter how strong the nonlinearity may be, for models with reasonable dispersion relations as in (12) which are written in the alternative form

$$u_t + f(u)_x + L(u)_t = 0. \qquad (13)$$

The model equations (13) were advocated in [5] as having certain advantages over the models (11) as far as the treatment of short-wave component was concerned.

Finally, even in the context of (1), it seems likely that in place of (6), it would suffice to assume only that there is a sequence $\{t_n\}_{n\geq 1}$ which converges to T^* from below for which $|u(\cdot,t_n)|_\infty$ is uniformly bounded in order that the conclusion that $T^* = +\infty$ hold. This result, if valid, also seems to be somewhat difficult.

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