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ON THE STRUCTURE OF THE EQUILIBRIUM PRICE SET OF OVERLAPPING-GENERATIONS ECONOMIES*

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This paper is concerned with generic properties of the set of price equilibria of overlapping-generations economies that include money. It is shown (in a C^1 topology) that if generations live for m periods and if there are n goods available in each period, then most economies will feature equilibrium price sets of dimension at most $(m-1)n$. It is likewise shown that for every k that ranges between 0 and $(m-1)n$ there are open sets of economies which possess locally k-dimensional, C^1 manifolds of equilibria. In the process of establishing these facts, a transversality theorem is proved which applies to maps, not necessarily Fredholm, between open subsets of non-separable, infinite-dimensional spaces.

1. Introduction

The study of equilibria of exchange economies with a finite number of agents and commodities has a substantial history, as is amply documented in the monographs of Arrow and Hahn (1971, ch. 9), Dierker (1982) and Mas-Colell (1985). One of the principal conclusions to emerge from this line of inquiry appears to be the basic proposition that equilibria of finite economies are, in general, locally unique. This fact is usually derived as a straightforward consequence of the inverse function theorem whenever the matrix of partial derivatives of the excess-demand function is surjective at a given equilibrium point. That the matrix of partial derivatives of the excess-demand function is typically surjective follows via arguments from transvers-

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ality theory whose application is now standard. Local uniqueness is a qualitative fact with considerable economic import, especially as regards comparative statics predictions.

Overlapping-generations models for economic activity were introduced by Samuelson (1958), and have been the object of considerable study recently. Their structure embodies an infinite number of agents and commodities, and consequently it is not obvious that the relatively simple analysis just enunciated continues to bear fruit. Indeed, the two mathematical assertions that are fundamental to the argument are not generally valid in infinite-dimensional settings. Firstly, if a linear mapping of an infinite-dimensional space is onto, it does not necessarily mean that it is one to one; indeed, the kernel may itself be of infinite dimension. In an economic frame this sort of situation has recently been observed by Kehoe et al. (1986b) in some examples of infinite-dimensional economies where at given equilibrium points the derivative of the excess-demand function is surjective with kernel of arbitrarily large dimension. Secondly, infinite-dimensional transversality theory is based upon Smale's extension of the Sard theorem [cf. Smale (1965)]. Smale's theorem assumes the maps to be Fredholm (see section 5) and the manifolds to be second countable (separable). These hypotheses appear to be somewhat restrictive with regard to several, natural economic frameworks.

Our purpose here is to explore properties of the set of equilibrium sequences of prices for a generic member of a general class composed of overlapping-generations economies which are not necessarily stationary and which feature general forms of the agents' utility functions. In the special case wherein each generation survives two periods, our main result states roughly that a typical model economy has an equilibrium price set of dimension no greater than n, where n is the number of commodities available in each period (assumed to be constant in time). For models with identical generations that live exactly two periods, Kehoe and Levine (1985) have shown that the equilibrium price set near certain steady states in particular subclasses of economies may have any dimension up to and including n. Thus in this special case, our results imply those of Kehoe and Levine to be sharp.

It appears that the first study of the structure of equilibria of overlapping-generations economies for models with many goods available in each period was undertaken by Balasko and Shell (1981). They focused on an economy where each generation contains a single Cobb–Douglas agent, and find that if the model features money, then there is a one-dimensional branch of equilibria, whereas the equilibrium is unique in the non-monetary case. Geanakoplos and Polemarchakis (1984) have emphasized that the crucial features leading to this early result are gross substitutability of excess demands and intertemporal separability of utilities. The implications of the gross-substitutability assumption have been explored further by Kehoe et al.

(1986a), who generalize considerably the previous results. All three studies are concerned with global properties of the equilibrium set. As mentioned already, the local behavior of equilibria lying near to a particular steady-state solution has been examined by Kehoe and Levine (1984, 1985). In a nice paper written at about the same time as the present script, Geanakoplos and Brown (to appear, and see also an earlier, as yet unpublished script) extend the results of Kehoe and Levine to general, non-stationary economies, obtaining results whose conclusions overlap with those of the present paper. Their theory is based on the multiplicative ergodic theorem and its application to dynamical systems. Central to their analysis is a notion of regularity and non-degeneracy of an equilibrium based upon existence and type of Lyapunov exponents for the economy in question. Whilst these properties can be shown to be robust in a sense, the topology used is rather strong and the probability measure that appears does not seem to arise naturally from economic considerations. Moreover, they impose a somewhat stringent rank condition on the partial derivatives of the excess-demand function which is crucial to their analysis, and higher order differentiability requirements, which are not needed using our approach. In any case, there seem to be insights gained from considering both approaches.

A natural mathematical setting for the analysis of overlapping-generations models is the space $\boldsymbol{R}^\infty$ of all real-valued sequences. While this space is suitable for questions of existence, its topology appears to be too coarse for a detailed study of the structure of the set of price equilibria. Consequently, our analysis will be conducted in a Banach-space setting to be introduced in section 2, along with a description of the model, the underlying economic assumptions, and a few other preliminary considerations.

In section 3, the elements of our analysis are used to study the important special case of stationary economies wherein all generations are the same. This is useful in motivating the later analysis, and is of interest in its own right. The assumption of homogeneity has the drawback that the derivative map of the aggregate excess-demand function displays a certain singularity at every stationary equilibrium. This technical inconvenience is overcome by a redefinition of the price space within each generation's lifetime. At a steady state, the local structure of equilibria will then be that of a C^1 manifold whose dimension ranges between 0 and n. This is in accordance with the findings of Kehoe and Levine, although we further demonstrate that this property is robust to general, non-stationary perturbations of the economy. Indeed, it is shown in section 4 as the main contribution of this paper that typical, overlapping-generations economies do not contain equilibrium price sets of higher dimension than their special, stationary counterparts. The upper bound on the dimensionality of the sets of equilibria is obtained first for the linearization of the excess-demand function of a truncated version of the economy in question. The result for the linearization of the entire, infinite-horizon economy then follows from the results pertaining to truncated economies after application of standard arguments from transversality theory.

As indicated above, the available infinite-dimensional transversality theory is based upon Smale's extension of the Sard theorem. Section 5 is devoted to a generalization of these ideas, which can be applied to the mathematical framework for overlapping-generations economies considered herein. Our main theorem in this regard places weaker requirements on the maps and dispenses with the second countability of the price domain. Although these facts represent a considerable relaxation of standard assumptions, the result may still appear restrictive in the context of overlapping-generations models. Obtaining similar conclusions without the restrictions implied by the use of standard transversality is an interesting open issue which warrants further study. Finally, section 6 presents an example of a simple economy whose analysis illustrates some of the methodological issues discussed in the earlier sections.

2. Notation and preliminary considerations

We begin with a standard formulation of a pure exchange, overlapping-generations economy. All economic transactions are assumed to take place at discrete times $t=0,1,2,\dots$. In each temporal period there are n different types of perishable goods available for consumption, the prices of which, in period t, are collected together in the price vector $p_t=(p_{1t},\dots,p_{nt})$. In the first instance consumers are presumed to survive exactly two periods, though this restriction will be relaxed subsequently. For $t>0$ the tth generation is made up of a finite number of agents with perfect foresight that consume in periods t and $t+1$. Their optimal consumption strategies are aggregated and represented by excess-demand functions y^t and z^t corresponding to their first and second period of life, respectively, and these are taken to be determined solely by the prevailing prices p_t and p_{t+1}. The following further assumptions about the excess-demand functions will be in effect throughout.

S (smoothness). For $t>0$ the functions $y^t(p_t,p_{t+1})$ and $z^t(p_t,p_{t+1})$ are C^2 mappings of $\mathbf{R}^{2n}_{++}$ into $\mathbf{R}^n$. Moreover, while these functions are allowed to depend on t, it is assumed that their partial derivatives are uniformly bounded on compact subsets of $\mathbf{R}^{2n}_{++}$, independently of t.

H (homogeneity). For all $t>0$, both y^t and z^t are homogeneous of degree zero.

WL (Walras' law). If $t>0$, then for every p_t,p_{t+1} in $\mathbf{R}^n_{++}$, the relation $p_t y^t(p_t,p_{t+1})+p_{t+1}z^t(p_t,p_{t+1})=0$ holds good. (The products are vector-space inner products.)

BD (boundedness). For each t the functions y^t and z^t are both bounded below on $\mathbf{R}^{2n}_{++}$.

Here, and subsequently, the usual notation $\mathbf{R}^k_{++}$ has been employed to indicate the interior of the non-negative orthant in Euclidean k-space, $\mathbf{R}^k$.

The foregoing assumptions are standard. Each has an economic basis that is well understood, and which will not be elaborated here. The article of Kehoe and Levine (1984) provides a good guide to the literature. The degree of freedom entailed in the restriction H allows for the possibility of normalizing prices and so considering y^t and z^t to be defined on a slightly simpler domain. For reasons of expositional convenience, we shall defer to the next section the reformulation of the price space which seems most appropriate for our analytical framework.

There is also postulated a 0th generation that lives only in the first period, and which is endowed with an aggregate of M units of fiat money. (One may also consider the situation where $M<0$, corresponding to an economy that initially has external debt. We generally restrict to the case $M>0$, though this is relaxed briefly in section 3.) The associated excess-demand function z^0 will be assumed to depend on both p_1 and M, and so to define a mapping of $\mathbf{R}^{n+1}_{++}$ into $\mathbf{R}^n$. This function is supposed to satisfy S, BD, and H, where the homogeneity encompasses all $n+1$ variables. In lieu of the budget constraint WL we assume

WL′ (modified Walras' law). For every p_1 in $\mathbf{R}^n_{++}$ and $M>0$, we have $p_1 z^0(p_1, M) = M$.

This situation corresponds to supposing that only the initial generation is endowed with money, the unit price of which has been incorporated into the specification of M. The constant M will therefore represent the total money supply of the economy in all subsequent generations.

A (monetary) *equilibrium* is a sequence $\mathbf{p} = \{p_t\}_{t=1,\ldots}$ of price vectors $p_t, t=1,2,\ldots$, for which the market clears at each positive period. That is, the sequence $\mathbf{p}$ has the properties that

$$z^0(p_1, M) + y^1(p_1, p_2) = 0,$$

and, for $t>1$, (2.1)

$$z^{t-1}(p_{t-1}, p_t) + y^t(p_t, p_{t+1}) = 0.$$

In our study it will be convenient to work with the aggregate excess-demand function F for the entire, infinite-horizon economy. The function F is a mapping of $\Pi_1^\infty \mathbf{R}^n_{++}$ to $\Pi_1^\infty \mathbf{R}^n$ whose tth component is simply the tth expression on the left-hand side of eq. (2.1). In this notation, the economy is represented by F and an equilibrium is a sequence $\boldsymbol{p}$ of price vectors such that $F(\boldsymbol{p})=0$. The set of all equilibrium price sequences corresponding to F will be denoted by $W(F)$.

The domain (and range) of F will be restricted to the Banach space ℓ^n_∞, the space of all sequences $\boldsymbol{p}=\{p_t\}_{t=1,\dots}$ with finite norm $\|\boldsymbol{p}\|=\sup_{t\geq 1}\{|p_t|\}$, where $|p_t|$ denotes the maximum norm on $\boldsymbol{R}^n$, for example. More precisely, F will be considered as a mapping of the interior E of the set $\Pi_1^\infty \boldsymbol{R}^n_{++} \cap \ell^n_\infty$. The tools of standard infinite-dimensional calculus [cf. Lang (1972)] are available in a setting such as that considered here wherein F is defined on an open subset of a Banach space.

A few remarks are in order regarding our choice of state space and its topology. As indicated in the introduction, $\Pi_1^\infty \boldsymbol{R}^n$ has a topology that is too weak for a detailed analysis of the structure of the set of price equilibria $W(F)$: indeed, two price sequences are close in the usual product topology on $\Pi_1^\infty \boldsymbol{R}^n$ if a large enough number of the leading components are close, even if the remaining (infinite number of) components are unrelated to each other. The example in section 6 shows that in the product topology, steady states and equilibrium price sequences that are unbounded may all belong to the same topological component. Although the restriction of F to E precludes consideration of equilibrium price sequences that grow unboundedly, a change of the independent variables may extend the range to which our theory applies. For instance, in the case with one good in each period where every consumer lives for two periods, one might set $x_1=p_1$, and for $t>1$, $x_t=p_t/p_{t-1}$. An analysis based on bounded sequences $\{x_t\}_{t\geq 1}$ would then include all the equilibrium price sequences which correspond to uniformly bounded rates of growth (inflation), a collection with a broad range of economic interest. Because of the homogeneity assumption H, the above change of variables leads to an easily comprehended modification of the existing problem.[1] Likewise, no restriction to generality is implied in considering the range of F as contained in ℓ^n_∞ since the analysis will only focus on the zeroes of F.[2]

An important technical result that is needed in the further development is contained in the following lemma.

Lemma 2.1. *The function F is a C^1 mapping on E.*

Proof. Let $F_{i,t}$ denote the aggregate excess-demand function of good i at time t. Thus $F_{i,t}$ is simply component number $n(t-1)+i$ of the function F.

[1]Although the above transformation of variables is not a diffeomorphism, it does map $\boldsymbol{R}^\infty_{++}$ homeomorphically onto itself.

[2]It can be shown using the mean-value theorem that under Assumption S the image of F is uniformly bounded for any closed ball in ℓ^n_∞ containing equilibrium points and lying in E.

The domain of $F_{i,t}$ consists of triples of price vectors (p_{t-1}, p_t, p_{t+1}) in $\boldsymbol{R}^{3n}_{++}$ which we temporarily write as x_t. Let $\boldsymbol{p}=(\ldots, p_t, p_{t+1}, \ldots)$ be an element of E and denote by $DF_{i,t}(\boldsymbol{p})$ the $3n$-vector of partial derivatives of $F_{i,t}$, evaluated at x_t. Then there is an $r>0$ such that $\{\boldsymbol{q}:\|\boldsymbol{q}-\boldsymbol{p}\|\leqq 2r\}$ lies entirely in E. For any $t\geqq 1$ and $1\leqq i\leqq n$, the mean-value theorem ensures that for $\tilde{\boldsymbol{p}}$ in $\{\boldsymbol{q}:\|\boldsymbol{q}-\boldsymbol{p}\|\leqq r\}$, there is an s in $[0,1]$ such that

$$|F_{i,t}(\boldsymbol{p})-F_{i,t}(\tilde{\boldsymbol{p}})|\leqq \sup_{0\leqq s\leqq 1}\{|DF_{i,t}(s\boldsymbol{p}+(1-s)\tilde{\boldsymbol{p}})|\}\,|x_t-\tilde{x}_t|,$$

where $\tilde{x}_t$ corresponds to $\tilde{\boldsymbol{p}}$ in the same way that x_t does to $\boldsymbol{p}$. For any s in $[0,1]$, the tth component of $s\boldsymbol{p}+(1-s)\tilde{\boldsymbol{p}}$ lies in the set

$$\{v: r\leqq v_i\leqq \|\boldsymbol{p}\|+r,\ 1\leqq i\leqq n\}\quad \text{in}\quad \boldsymbol{R}^n.$$

It follows that

$$\|F(\boldsymbol{p})-F(\tilde{\boldsymbol{p}})\|\leqq 3\,\mathrm{mn}\,\|\boldsymbol{p}-\tilde{\boldsymbol{p}}\|,$$

where m is a bound whose existence is guaranteed by assumption S on the first partial derivatives of the excess-demand functions. Hence, F is continuous at $\boldsymbol{p}$, and $\boldsymbol{p}$ was an arbitrary point of E. Applying the definition of the norm of an operator in ℓ^n_∞ [cf. Taylor (1958, p. 220)], and making use of the assumption on the second partial derivatives of the excess-demand functions, a similar argument shows that $DF=(\ldots, DF_{1,t}, \ldots, DF_{n,t}, \ldots)$ is continuous, and so F is confirmed to be C^1 on E. The lemma is thus established.

The collection of all economies F satisfying the assumptions put forth above will be denoted by ε. Since our aim here is to investigate local properties of $W(F)$, it suffices to fix attention on a bounded, open ball B centered about a point in E. As already remarked in the proof of Lemma 2.1, B may be chosen in such a way that there is an $r>0$ with the property that $p_{it}\geqq r$ for any $t\geqq 0$ and $1\leqq i\leqq n$. A metric d_B is introduced on ε as follows. Let F and G designate two elements of ε and let $\{y^t, z^t\}$, $\{\tilde{y}^t, \tilde{z}^t\}$ stand for the component functions of F and G, respectively. Define $d_B(F,G)$ by

$$d_B(F,G)=\min\{\Gamma, 1\},$$

where

$$\Gamma=\sup\{\gamma(t,\boldsymbol{p}): 0\leqq t, \boldsymbol{p}\in B\},$$

and

$$\gamma(t,\boldsymbol{p})=|y^t(p_t,p_{t+1})-\tilde{y}^t(p_t,p_{t+1})|+|Dy^t(p_t,p_{t+1})-D\tilde{y}^t(p_t,p_{t+1})|$$

$$+\left|z^t(p_t,p_{t+1})-\tilde{z}^t(p_t,p_{t+1})\right|+\left|Dz^t(p_t,p_{t+1})-D\tilde{z}^t(p_t,p_{t+1})\right|.$$

In this definition $Dy^t(p_t,p_{t+1})$ is the matrix of partial derivatives of y^t at (p_t,p_{t+1}), $|Dy^t(p_t,p_{t+1})|$ is the norm of $Dy^t(p_t,p_{t+1})$, and similarly for the remaining variables. The function d_B provides a complete, metric topology on ε. Note that because the partial derivatives of each generation's excess-demand functions are homogeneous of degree minus one and obey WL, the metric d_E would yield the discrete topology on ε. Hence the sort of localization proposed above is necessary in order to obtain our principal results using the present functional-analytic setting.

The following lemma is a straightforward consequence of a generalized version of the implicit function theorem [see Schwartz (1967, p. 278)]. Recall that a linear map $T:X\to Y$ is termed a *splitting surjection* if T is onto and $\ker(T)$ has a closed complement in X.

Lemma 2.2. Let F be an element of ε, and $\boldsymbol{p}$ in $W(F)$. Assume that $DF(\boldsymbol{p})$ is a splitting surjection, let $K=\ker DF(\boldsymbol{p})$, and let L be a closed complement of K in ℓ^n_∞. Suppose that $\boldsymbol{p}=\boldsymbol{p}_K+\boldsymbol{p}_L$ where $\boldsymbol{p}_K\in K$ and $\boldsymbol{p}_L\in L$.

Then there exist open neighborhoods $U_1\subset K$ of $\boldsymbol{p}_K$, $U_2\subset L$ of $\boldsymbol{p}_L$, and $V\subset\varepsilon$ of F, and a continuous function $g:U_1\times V\to U_2$ such that for every $(\tilde{\boldsymbol{p}}_K,\tilde{\boldsymbol{p}}_L,G)\in U_1\times U_2\times V$, $G(\tilde{\boldsymbol{p}}_K+\tilde{\boldsymbol{p}}_L)=0$ if and only if $\tilde{\boldsymbol{p}}_L=g(\tilde{\boldsymbol{p}}_K,G)$.

The standard formulation of the implicit function theorem implies that if $DF(\boldsymbol{p})$ is surjective, then the equilibrium set is formed locally of C^1 manifolds, although at this stage they may be modeled on infinite-dimensional spaces.

3. Stationary economies

The elements of our analysis will first be applied to the important, special case wherein the economic aspects of the various generations do not change over time. Such an economy is called *stationary* and evidently possesses the property that there is a pair (y,z) of excess-demand functions for which $(y^t,z^t)\equiv(y,z)$ for every $t\geqq 1$ and such that hypotheses S, H, WL and BD hold. A stationary economy is thus fully specified by the triple (z^0,y,z) of excess-demand functions.

An equilibrium $(p_1,p_2,\ldots)$ of a stationary economy given by the excess-demand functions (z^0,y,z) is called a *steady state* if $p_t=p$ for every $t>0$, where p is a fixed element of $\boldsymbol{R}^n_{++}$. Thus $\boldsymbol{p}=(p,p,\ldots)$ is a steady state if

$$z^0(p,M)+y(p,p)=0, \tag{3.1}$$

and

$$z(p,p)+y(p,p)=0. \tag{3.2}$$

The existence of steady states in this general context has been studied by Kehoe and Levine (1984). Their results ensure that for every pair of excess-demand functions (y, z), which satisfy S, H, WL, BD and a standard boundary condition, there is $p \in \boldsymbol{R}^n_{++}$ such that $y(p,p)+z(p,p)=0$. Let us assume that for such a p there is an excess-demand function z^0 and an M such that $z^0(p,M)+y(p,p)=0$. In our analysis concern will be particularly with situations where $M \neq 0$. In the case where $M<0$, it is assumed that WL′ holds locally around the stationary point.

The following lemma is an extension of Lemma 4.4 in Araujo and Scheinkman (1977). Let $\boldsymbol{p}=(p,p,\ldots)$ be a steady state of a stationary economy, let F be the aggregate excess-demand function defined earlier for this economy, and let $DF(\boldsymbol{p})$ denote the derivative map of F evaluated at $\boldsymbol{p}$. Denote by $D_1 y$ (respectively, $D_2 y$) the $n \times n$ matrix of partial derivatives of y with respect to the first (respectively, last) n coordinates of $\boldsymbol{R}^{2n}$, and similarly for $D_1 z$ and $D_2 z$.

Lemma 3.1. *Let $\boldsymbol{p}=(p,p,\ldots)$ be a steady state of a stationary economy given by the triple of excess-demand functions (z^0, y, z). Suppose that the characteristic equation*

$$\det\{\lambda^2 D_2 y(p,p)+\lambda[D_1 y(p,p)+D_2 z(p,p)]+D_1 z(p,p)\}=0$$

has no roots equal to one in absolute value, and let r be the number of roots less than one in absolute value. Suppose $D_2 y(p,p)$ to be an isomorphism. If $r \geqq n$, then $DF(\boldsymbol{p})$ is surjective and $\dim\ker[DF(\boldsymbol{p})]=r-n$, and if $r<n$, then $DF(\boldsymbol{p})$ is not surjective.

Remark. Araujo and Scheinkman assume that $r=n$ in their discussion, however, their proof is straightforwardly extended to substantiate the broader statement given in Lemma 3.1. It is required in the proof (see their footnote 11) that at the steady state the tangent space of the initial conditions manifold [the kernel of the linearization of (3.1)] be transversal to the tangent space of the stable manifold of the system (3.2).

Kehoe and Levine (1984) demonstrate that at any given steady state, $2n-1$ roots of the characteristic equation associated with the linearization of (3.2) are a priori unrestricted, but that assumption H entails that one of the roots must be equal to one. The following result is a consequence.

Theorem 3.1. *Assume that $\boldsymbol{p}$ is a steady state of (z^0, y, z). Then $DF(\boldsymbol{p})$ is not a splitting surjection.*

Proof. Let r be the number of roots of the characteristic equation associated with the linearization of (3.2) evaluated at the steady state which are less than 1 in absolute value. Suppose that $DF(\boldsymbol{p})$ is surjective, so that $r \geqq n$.

Since, by assumption H, 1 itself is certainly a root, we find from Lemma 3.1 that one can construct operators arbitrarily close in the norm topology to F whose derivatives at $\boldsymbol{p}$ are surjective with kernels of dimension $r-n$. Similarly, one can construct operators arbitrarily close to F whose derivatives at $\boldsymbol{p}$ are surjective with kernels of dimension $r-n+1$. Since in the norm topology the set of splitting surjections whose kernels have a given dimension is open [see Abraham and Robbin (1967, p. 42)], $DF(\boldsymbol{p})$ cannot be surjective. This contradiction leaves only the stated conclusion as a valid possibility.

We proceed to make a redefinition of the price space and use this renormalized domain in showing that for every k between 0 and n, there are stationary economies where the derivative map of the aggregate excess-demand function, evaluated at a particular steady state, is surjective with a k-dimensional kernel.

Let $x_1=p_1$, and $x_t=p_t/p_{1t-1}$, for all t, where $p_t=(p_{1t},p_{2t},\dots)$. Then by the homogeneity assumption H, (3.1) and (3.2) can be rewritten as

$$z^0(x_1,M)+y\left(\left(1,\frac{x_1^1}{x_{11}}\right),x_2\right)=0, \tag{3.3}$$

and, for $t>1$,

$$z\left(\left(1,\frac{x_{t-1}^1}{x_{1t-1}}\right),x_t\right)+y\left(\left(1,\frac{x_t^1}{x_{1t}}\right),x_{t+1}\right)=0, \tag{3.4}$$

where for each $t>0$, $x_t=(x_{1t},x_{2t},\dots)$ and $(1,x_t^1/x_{1t})$ is a vector in $\boldsymbol{R}^n$ the first coordinate of which equals one and the last $n-1$ components, denoted x_t^1/x_{1t}, are the last $n-1$ coordinates of the n-vector x_t/x_{1t}.

Suppose that $\boldsymbol{p}=(p,p,\dots)$ is a steady state of (z^0,y,z). If $D_2y(p,p)$ is an isomorphism, then by the implicit function theorem there exists a C^1 function g defined on a neighborhood of (p,p) such that

$$g(p_{t-1},p_t)=(p_t,p_{t+1}) \quad \text{if and only if} \quad z(p_{t-1},p_t)+y(p_t,p_{t+1})=0.$$

Let $(x,x)=(p,p)/p_{11}$ where p_{11} is as above. Then if $D_2y(p,p)$ is an isomorphism, there is a C^1 function $\hat{g}$ defined on a neighborhood of (x,x) such that

$$\hat{g}(x_{t-1},x_t)=(x_t,x_{t+1}) \quad \text{if and only if}$$

$$z((1,x_{t-1}^1/x_{1t-1}),x_t)+y((1,x_t^1/x_{1t}),x_{t+1})=0.$$

Denote by G the matrix of partial derivatives of g at (p,p), and by $\hat{G}$ the matrix of partial derivatives of $\hat{g}$ at (x, x). Then G has a unit eigenvalue whose eigenvector is (p, p), and $\hat{G}$ has a zero eigenvalue whose eigenvector is $(x, 0)$. Our purpose now is to demonstrate that the remaining $2n-1$ eigenvalues of G are also eigenvalues of $\hat{G}$.

In what follows it will be assumed that none of the eigenvalues of G equals zero, and that $\hat{G}$ has only one zero eigenvalue. Both properties hold generically. The main result of the present section is based upon the following simple facts.

Fact 1. Let $Q_M \subset \boldsymbol{R}^{2n}$ be the set of (p_t, p_{t+1}) such that $p_t y(p_t, p_{t+1}) = -M$ where M is non-zero. Then

(i) Q_M defines a $(2n-1)$-dimensional, C^1 manifold,
(ii) G maps the tangent space of Q_M at (p, p) onto itself, and
(iii) the tangent space of Q_M at (p, p) coincides with the generalized eigenspace of G that excludes the eigenvector (p, p).

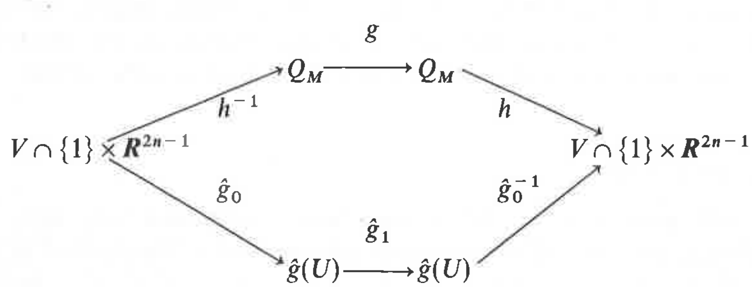
Proof. See Kehoe and Levine (1984, p. 85).

Fact 2. For some neighborhood $U \subset \boldsymbol{R}^{2n}$ of (x, x), $\hat{g}(U)$ is a C^1 manifold of dimension $2n-1$.

Proof. Let $\{0\} \times \boldsymbol{R}^{2n-1}$ be the subspace of vectors in $\boldsymbol{R}^{2n}$ whose first coordinate equals zero, and $\{1\} \times \boldsymbol{R}^{2n-1}$ the set of all vectors in $\boldsymbol{R}^{2n}$ whose first coordinate equals 1. Then $\hat{G}$ is one-to-one on $\{0\} \times \boldsymbol{R}^{2n-1}$, so it follows from a standard argument [cf. Lang (1972, p. 16)] that there is a neighborhood $V \subset \boldsymbol{R}^{2n}$ of (x, x) such that $\hat{g}(V \cap \{1\} \times \boldsymbol{R}^{2n-1})$ is a C^1 manifold of dimension $2n-1$. But $\hat{g}(V \cap \{1\} \times \boldsymbol{R}^{2n-1}) = \hat{g}(U)$ for some neighborhood U in $\boldsymbol{R}^{2n}$.

Let U and V be as defined in the discussion above relating to Fact 2. Define $h: Q_M \to \{1\} \times \boldsymbol{R}^{2n-1}$ by $h(p_t, p_{t+1}) = (p_t, p_{t+1})/p_{1t}$. Then h is generically a diffeomorphism on a neighborhood of (p, p). Denote by $\hat{g}_0$ the restriction of $\hat{g}$ to $V \cap \{1\} \times \boldsymbol{R}^{2n-1}$, and by $\hat{g}_1$ the restriction of $\hat{g}$ to $\hat{g}(U)$.

Fact 3. The following diagram commutes, and therefore $\hat{g}_1$ is a local diffeomorphism at (x, x).



Proof. Obvious from the definitions.

From Facts 1, 2, and 3, the following result obtains.

Lemma 3.2. Let $h: Q_M \to \{1\} \times \boldsymbol{R}^{2n-1}$ *be defined by* $h(p_t, p_{t+1}) = (p_t, p_{t+1})/p_{1t}$, *and assume that h is a diffeomorphism on a neighborhood of* (p,p). *Suppose that none of the eigenvalues of G equals zero and that* $\hat{G}$ *has only one zero eigenvalue. Then G has a unit eigenvalue with eigenvector* (p,p), *and the remaining* $2n-1$ *eigenvalues of G are also eigenvalues of* $\hat{G}$.

Let $DF(\boldsymbol{x})$ be the derivative map at $\boldsymbol{x} = (x, x, \ldots)$ of the left-hand side of the system of equations in (3.3) and (3.4), and let x be as defined above. Then from Lemmas 3.1 and 3.2 the following result is readily deduced.

Theorem 3.2. Let the conditions of Lemma 3.2 be satisfied. Let r be the number of eigenvalues of G less than one in absolute value, and assume that G has only one eigenvalue equal to one in absolute value. If $r \geqq n-1$ *then* $DF(\boldsymbol{x})$ *is surjective with kernel of dimension* $r-n+1$.

Remarks. 1 (comparative statics). Since r ranges between 0 and $2n-1$, it follows from Theorem 3.2 and the implicit function theorem that if one considers the metric topology on the space of aggregate excess-demand functions on the price domain given in terms of the x-variables, then for every k between 0 and n, there are open sets of economies in ε which possess locally k-dimensional, C^1 manifolds of equilibria. Observe that the mathematical setting allows for the possibility of introducing perturbations which result in every generation to be varied, whereas in earlier comparative statics analyses [cf. Kehoe and Levine (1985) and Geanakoplos and Brown (1985)] the perturbations allowed only resulted in changing a finite number of generations.

2 (more general life spans and non-monetary economies). After minor modifications in notation, the analysis can be shown to include modeling situations wherein generations live an arbitrary number of periods and where the economy does not feature money. An interesting result obtains if time is considered to extend from $-\infty$ to ∞. At a steady state $\boldsymbol{x}$ of the form $(\ldots, x, x, x, \ldots)$, $DF(\boldsymbol{x})$ is generally invertible. Hence, in the doubly open-ended version of the model, stationary equilibria are usually isolated.

4. The main result

In this section it is shown that most economies can only display equilibrium price sets of dimension no greater than n. The method of proof is and p, which belongs to B, is an equilibrium point of F. Thus for any τ, $DF^{\tau}(p^{\tau})$ can be viewed as a linear map from $\boldsymbol{R}^{n(\tau+1)}$ onto $\boldsymbol{R}^{n\tau}$ and therefore its kernel has to be of dimension n. But for every τ, the vectors $a^{1,\tau+1},\dots,a^{n+1,\tau+1}$ belong to $\ker[DF^{\tau}(p^{\tau})]$, and, as just ascertained, there is a τ for which they are linearly independent, a clear contradiction.

Remarks. *1* (redefinition of the price space). The price variables in Theorem 4.1 are spot prices, and the analysis thus applies to equilibrium sequences of spot-price vectors that belong to ℓ^n_{∞}. The same conclusions would obtain if the metric topology on the space of economies were related to the renormalized prices that were introduced in the preceding section.[3] Indeed, under this latter specification of the price space, there is a strengthening of our results since, as mentioned earlier, it has been demonstrated that there are open sets of economies containing C^1 manifolds of equilibria having any dimension up to n.

Throughout our study we have restricted the price domain to a fixed, but arbitrary ball B in the price domain E. This local approach has been taken because consideration of the entire set E gives rise to the discrete topology on the space ε of economies considered here, because of the hypotheses H and WL. This is an area where the present theory could be considerably improved.

2 (regular economies). In attempting to extend the theory of finite-dimensional regular economies to our setting, one might say that an economy F is *regular* if $DF(x)$ is surjective for every equilibrium x.[4] The results of section 3 indicate that this concept may not be generic in the present type of modeling configurations. Another limitation of this concept of regularity owes to the fact that surjectivity places uniform restrictions on the derivatives of the excess-demand functions. An illustrative example may be obtained by considering the operator A whose matrix in the standard basis is given as

$$\begin{bmatrix} 1 & & & & \\ & 1/2 & & & \\ & & 1/4 & & \\ & & & 1/8 & \\ & & & & \ddots \end{bmatrix},$$

where every off-diagonal term is equal to zero and for $t=1,2,\dots,$ the

[3]The proof of this assertion parallels the one given in the text. Observe that the proofs of Theorem 4.1 and Proposition 4.1 do not depend intrinsically upon the domain considered, and the corresponding proof of Lemma 4.1 is facilitated by the fact that the formulated change in the x-variables defines a diffeomorphism between both truncated domains.

[4]The class of examples presented in section 6 illustrates that this concept is not vacuous if one uses an appropriate specification of the price domain.

diagonal term of the tth row is equal to 2^{1-t}. For any τ, the composite map $\text{proj}_\tau \circ A$ is certainly surjective. However, A is a compact operator on ℓ_∞ and the range of A is plainly infinite dimensional. Hence the range of A cannot be closed [see Rudin (1973, Theorem 4.18)] and so A is certainly not surjective. As A is linear, $DA(\mathbf{x})=A$, and so A is seen to lack the defining property of a regular economy. This observation also indicates that the specification of easily verified, sufficient conditions for the regularity of an economy may be difficult. In section 5 a related, but somewhat easier issue is treated, namely to specify conditions on a class of economies so that almost all members of the class are regular.

3 (more general lifespans and non-monetary economies). To infer an upper bound on the dimension of the kernel of the derivative map in Proposition 4.1, it is sufficient to assume that the dimensionality of each generation's commodity space is uniformly bounded. In particular, consider the special situation where each generation lives exactly m periods, and where there are n goods available in each period. For such economies, it is true that most will not possess equilibrium price sets of dimension greater than $(m-1)n$. Our results can also be stated to encompass any $M \in \mathbf{R}$. Observe, however, that if $M=0$ the upper bound $(m-1)n$ may be reduced by one dimension, as the price level is now indeterminate.

5. An infinite-dimensional transversality theorem

Some of the results obtained in the preceding section indicate that without additional restrictions, regular economies as defined here are not generic, even if the price domain is suitably respecified. Because the equilibria of regular economies are C^1 manifolds of bounded dimension, which is exactly the situation to which Theorem 4.1 applies, it is of special interest to set forth further restrictions on the set ε which will imply that almost every economy in this restricted subset is regular. The provision of such conditions will be the thrust of the present section, where an appropriate transversality theorem is proved which applies to the class of economies under study. This transversality result appears to prevail under somewhat weaker assumptions than those of the standard transversality theory in infinite-dimensional spaces, but of course the context is very special. We begin with a few preliminary definitions and remarks.

Let M and N be two Banach spaces, and let $B(M,N)$ be the space of bounded linear operators that map M into N. An operator $H \in B(M,N)$ is said to be *Fredholm* if it has closed range, finite-dimensional kernel, and finite-dimensional cokernel. If H is Fredholm, then its *index* is defined as $\dim \ker(H) - \dim \operatorname{coker}(H)$. The set $F(M,N)$ of Fredholm operators from M to N is an open subset of the space $B(M,N)$ if the latter space is equipped with its norm topology, and, in addition, the index is constant on components of $F(M,N)$ [see Schechter (1971)]. It is well known that if $H=J+K$

where J is an isomorphism of M onto N and $K \in B(M,N)$ is a compact operator, then H is a Fredholm operator of index zero.

A not necessarily linear C^1 function $f: M \to N$ is called a *Fredholm map* if its derivative $Df(m): M \to N$ is a Fredholm operator for all $m \in M$. Since f is a C^1 mapping, $Df(m)$ depends continuously on m and so the index of $Df(m)$ is therefore independent of m and, by definition, this number is the index of the mapping f. A point $m \in M$ is *regular* if the derivative $Df(m)$: $M \to N$ is a splitting surjection, otherwise it is *singular*. The images of the singular points under f are called the *singular values* or *critical values* and their complement in N the *regular values.*

Theorem 5.1 [Smale (1965)]. *Let M and N be C^q-Banach manifolds with M separable and $f: M \to N$ a Fredholm map which is C^q. If q is greater than the index of f, then the regular values form a residual set in N.*

Theorem 5.2. Let ϕ: $N \times M \to D$ be a C^q map where M, N and D are Banach spaces with M and N separable. If 0 *is a regular value of ϕ and if for every $n \in N, \phi_n = \phi(n, \cdot)$ is a Fredholm map whose index is less than q, then the set $\{n \in N : 0$ is a regular value of $\phi_n\}$ is residual in N.*

Remark. This sort of infinite-dimensional transversality theorem has been a useful tool for demonstrating generic properties of classes of functions, in particular classes of solutions of various partial differential equations [cf. Foias and Temam (1977) and Uhlenbeck (1976)].

Proof. The proof follows a standard line [see e.g. Quinn (1970)]. Since 0 is a regular value of ϕ, $P = \phi^{-1}(0)$ is a separable, C^q-Banach manifold. Let π be the C^q mapping which is the restriction to P of the projection of $N \times M$ onto N. If a point $(n,m) \in N \times M$ lies in P, then

$$\dim \ker [D\pi(n,m)] = \dim \{(n',m') \in T_{(n,m)}(P) : n' = 0\} = \dim \ker [D\phi_n(m)]$$

and

$$\begin{aligned} \dim \operatorname{coker} [D\pi(n,m)] &= \operatorname{codimension}(T_{(n,m)}(P) + \{0\} \times M) \\ &= \operatorname{codimension}(\ker [D\phi(n,m)] + \{0\} \times M) \\ &= \dim \operatorname{coker} [D\phi_n(m)], \end{aligned}$$

where $T_{(n,m)}(P)$ is the tangent space to P at the point (n,m). The assumption on ϕ_n thus guarantees that π is a Fredholm map of index less than q. By Theorem 5.1, the set $\{n \in N : n$ is a regular value of $\pi\}$ is residual. However, if n is a regular value for π, it follows from an examination of the definitions that 0 is a regular value for ϕ_n [cf. Abraham and Robbin (1967, Theorem 19.1)]. Thus the result is established.

Let ε, B and F be as in section 4 and let

$$\hat{\varepsilon} = \{F \in \varepsilon \colon \text{for every } \boldsymbol{x} \in B, \text{ range}\,[DF(\boldsymbol{x})] \text{ is closed}\}, \tag{5.1}$$

equipped with the relative topology induced by ε. The extra demand placed on ε in (5.1) suffices to establish a genericity theorem of the sort contemplated above. Notice that $\hat{\varepsilon}$ certainly may have a non-empty interior if the underlying domain B is small enough. For example, if $F \in \varepsilon$ is such that at some point $\boldsymbol{p}$, $DF(\boldsymbol{p})$ is a splitting surjection, then it follows that if $\boldsymbol{q}$ is near enough to $\boldsymbol{p}$ in E and G is near enough to F in ε, then $DG(\boldsymbol{q})$ is also surjective, so that DG certainly has closed range on some ball B.

Theorem 5.3. *There exists a residual set $\hat{\varepsilon}_1$ in $\hat{\varepsilon}$ such that every $F \in \hat{\varepsilon}_1$ is regular on B.*

The proof of this theorem is based upon the proof of Theorem 4.1 and the following two lemmas.

Lemma 5.1. *Let $F \in \varepsilon$, and $\boldsymbol{x}$ in E. Suppose that $DF(\boldsymbol{x})$ is represented by the matrix*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A is a $\tau n \times \tau n$ matrix of scalars. If range $[DF(\boldsymbol{x})]$ is closed, then for any $\tau n \times \tau n$ matrix A', if $DF'(\boldsymbol{x})$ is represented by the matrix

$$\begin{bmatrix} A' & B \\ C & D \end{bmatrix},$$

then the range of $DF'(\boldsymbol{x})$ will also be closed.

Proof. See Dunford and Schwartz (1958, p. 513, Exercise 17).

Lemma 5.2. *Let $F \in \hat{\varepsilon}$ and let $\boldsymbol{x} \in E$. Suppose that for every $\tau \geqq 1$ $DF^{\tau}(\boldsymbol{x})$ is surjective. Then $DF(\boldsymbol{x})$ is surjective.*

Proof. Since $DF^{\tau}(\boldsymbol{x})$ is surjective for every τ, it follows from Proposition 4.1 that $\dim K = \dim \ker[DF(\boldsymbol{x})] \leqq n$. Therefore K has a closed complement L (i.e.,

$\ell^n_\infty = K + L$) such that $DF(x)$ is one-to-one on L. Let T be the restriction of $DF(x)$ to L. As T has closed range, it follows from the closed graph theorem that T has a continuous inverse, which is to say that $T^{-1} \in B(\text{range}[DF(x)], L)$.

To prove that $DF(x)$ is surjective, consider a vector $y = (y_1, y_2, y_3, \dots) \in \ell^n_\infty$ and search for another vector $z \in \ell^n_\infty$ such that $DF(x)z = y$. Since $DF^\tau(x)$ is surjective for every τ, the elements

$$e_1 = (y_1, 0, 0, 0, 0, \dots)$$
$$e_2 = (y_1, y_2, 0, 0, 0, \dots)$$
$$e_3 = (y_1, y_2, y_3, 0, 0, \dots)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\qquad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdot$$

all belong to the range of $DF(x)$. As T^{-1} is continuous, the set $\{T^{-1}(e_n)\}_{n=1,\dots}$ is bounded. By a Cantor diagonalization process, there exists a subsequence of $\{T^{-1}(e_n)\}_{n=1,\dots}$ that converges in each coordinate to some $z \in \ell^n_\infty$. Since each row of $DF(x)$ contains only a finite number of non-zero scalars, it follows that $DF(x)z = y$.

Proof of Theorem 5.3. For any τ let $\hat{\varepsilon}^\tau$ be the set $\{F^\tau : F \in \hat{\varepsilon}\}$. Let $\hat{\varepsilon}^\tau$ be endowed with the topology induced by $\hat{\varepsilon}$. As Lemma 5.1 holds, we may argue exactly as in the proof of Lemma 4.1 that there is an open dense set $\hat{\varepsilon}^\tau_1$ in $\hat{\varepsilon}^\tau$ such that every $F^\tau \in \hat{\varepsilon}^\tau_1$ is transversal to 0 on B^τ. Hence, there exists an open dense set $\hat{\sigma}_\tau \subset \hat{\varepsilon}$ such that for every $F \in \hat{\sigma}_\tau$, F^τ is transversal to 0 on B^τ. Therefore, the set of all $F \in \hat{\varepsilon}$ such that F^τ is transversal to 0 on B^τ for every τ, is residual in $\hat{\varepsilon}$. Theorem 5.3 is thus a consequence of Lemma 5.2

6. Examples of regular economies

An example is presented of a class of regular economies where, after a redefinition of commodity prices, the set of price equilibria belongs to ℓ_∞. While this class of examples is somewhat special, it does capture the essence of certain, more general, stationary economies. Indeed, it can be demonstrated that if every agent has an upward-sloping offer curve, the properties possessed by the class defined below extend to the models examined by Gale (1973).

Consider the simple situation in which each generation consists of a single agent and assume that there exists but one consumption good available in each time period. For $t = 0, 1, 2, \dots$, let c^t_t, respectively c^t_{t+1}, denote the consumption of agent t in period t, respectively $t+1$. For all $t > 0$, let $u^t(c^t_t, c^t_{t+1}) = \log(c^t_t) + \alpha c^t_{t+1}$ be the utility function of the tth agent, where α is a constant greater than 1, and let $(w^t_t, w^t_{t+1}) = (1, 0)$ be his endowment of goods in periods t and $t+1$, respectively. Agent 0 is endowed with a positive quantity of M units of money. Given a price sequence $\boldsymbol{p}=(p_1,p_2,\dots)$, we presume as usual that agent 0 maximizes c_1^0 subject to WL′, whereas, for every $t>0$, agent t maximizes u^t subject to WL.

Consider any price sequence $\boldsymbol{p}$ and any $t>0$. Then the pair (c_t^t, c_{t+1}^t) solves the constrained utility-maximization problem of agent t if

$$\frac{1}{p_t c_t^t}=\frac{\alpha}{p_{t+1}}, \tag{6.1}$$

$$p_t c_t^t + p_{t+1} c_{t+1}^t = p_t. \tag{6.2}$$

Since $w_{t+1}^t=0$, then for the price sequence $\boldsymbol{p}$ to be in equilibrium, it must be the case that

$$p_{t+1} c_{t+1}^t = M \quad \text{for} \quad \text{all } t \geqq 0. \tag{6.3}$$

Supposing the trivial optimization problem of trader 0 to be solved, it follows from (6.1), (6.2), and (6.3) that a price sequence $\boldsymbol{p}$ is an equilibrium if, and only if, for all $t>0$

$$p_{t+1}=\alpha(p_t - M)>0. \tag{6.4}$$

But for every $p_1' \geqq M(1-1/\alpha)^{-1}$, there exists a price sequence with p_1' as its first coordinate which satisfies (6.4). Moreover, if

$$p_1' = \frac{M}{1-1/\alpha},$$

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$\ell^n_\infty = K + L$) such that $DF(x)$ is one-to-one on L. Let T be the restriction of $DF(x)$ to L. As T has closed range, it follows from the closed graph theorem that T has a continuous inverse, which is to say that $T^{-1} \in B(\text{range}[DF(x)], L)$.

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all belong to the range of $DF(x)$. As T^{-1} is continuous, the set $\{T^{-1}(e_n)\}_{n=1,\ldots}$ is bounded. By a Cantor diagonalization process, there exists a subsequence of $\{T^{-1}(e_n)\}_{n=1,\ldots}$ that converges in each coordinate to some $z \in \ell^n_\infty$. Since each row of $DF(x)$ contains only a finite number of non-zero scalars, it follows that $DF(x)z = y$.

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Consider the simple situation in which each generation consists of a single agent and assume that there exists but one consumption good available in each time period. For $t = 0, 1, 2, \ldots$, let c^t_t, respectively c^t_{t+1}, denote the consumption of agent t in period t, respectively $t+1$. For all $t > 0$, let $u^t(c^t_t, c^t_{t+1}) = \log(c^t_t) + \alpha c^t_{t+1}$ be the utility function of the tth agent, where α is a constant greater than 1, and let $(w^t_t, w^t_{t+1}) = (1, 0)$ be his endowment of goods in periods t and $t+1$, respectively. Agent 0 is endowed with a positive quantity of M units of money. Given a price sequence $\boldsymbol{p}=(p_1, p_2, \ldots)$, we presume as usual that agent 0 maximizes c_1^0 subject to WL′, whereas, for every $t>0$, agent t maximizes u^t subject to WL.

Consider any price sequence $\boldsymbol{p}$ and any $t>0$. Then the pair (c_t^t, c_{t+1}^t) solves the constrained utility-maximization problem of agent t if

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Since $w_{t+1}^t = 0$, then for the price sequence $\boldsymbol{p}$ to be in equilibrium, it must be the case that

$$p_{t+1} c_{t+1}^t = M \quad \text{for} \quad \text{all } t \geqq 0. \tag{6.3}$$

Supposing the trivial optimization problem of trader 0 to be solved, it follows from (6.1), (6.2), and (6.3) that a price sequence $\boldsymbol{p}$ is an equilibrium if, and only if, for all $t>0$

$$p_{t+1} = \alpha(p_t - M) > 0. \tag{6.4}$$

But for every $p'_1 \geqq M(1-1/\alpha)^{-1}$, there exists a price sequence with p'_1 as its first coordinate which satisfies (6.4). Moreover, if

$$p'_1 = \frac{M}{1-1/\alpha},$$

it follows from (6.4) that for $t>1$,

$$p'_t = \frac{M}{1-1/\alpha},$$

and if

$$p'_1 > \frac{M}{1-1/\alpha},$$

then by (6.4), $p'_t \to \infty$ as $t \to \infty$. Therefore, the set of equilibrium prices of this economy is not contained in ℓ_∞.

Let $x_1 = p_1$ and, for $t > 1$, $x_t = p_t/p_{t-1}$. Then for every price sequence $p = (p_1, p_2, \ldots)$ which satisfies (6.4), its associated vector $x = (x_1, x_2 \ldots)$ has the property that either $x_t = 1$ for all $t > 1$, or $x_t \to \alpha$ as $t \to \infty$; in either case, x is an element of ℓ_∞. Furthermore, after linearization of the aggregate excess-demand function at equilibrium points, the methods laid down in section 3 for stationary economies readily confirm that such an economy is regular when considered as a function on its redefined price space. Its equilibrium set consists of two components.[5] One component is the single vector

$$x = \left(\frac{1}{1-1/\alpha}, 1, 1, 1, \ldots\right),$$

where the derivative map of the aggregate excess-demand function is an isomorphism. The other component is a one-dimensional, C^1 manifold.

[5]Under quite general conditions, Dierker (1972) has shown that regular finite economies have an odd number of equilibria. Note, however, that the number of components depends on the topology chosen. For instance, in the product topology, the equilibrium set of this class of economies comprises a unique component.

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