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Dispersive Blowup of Solutions of Generalized Korteweg–de Vries Equations*

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The strong effect of dispersion on short-wavelength disturbances featured by the Korteweg–de Vries equation and some of its generalizations is exploited to provide solutions of these equations that correspond to infinitely smooth initial data, which exhibit a specified loss of spatial smoothness at particular times. The points in space-time at which smoothness is lost may even comprise an arbitrary discrete subset of the upper half-plane $\{(x, t): x \in \mathbb{R}, t \geqslant 0\}$. Our results are related to recent work on smoothing of solutions of such equations, some of which are sharpened here, and they show that in certain aspects these earlier results are not far from being optimal. The theory makes use of new results concerning well-posedness of such equations in weighted Sobolov spaces and some detailed analysis of the linear Korteweg–de Vries equation. © 1993 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to explore the singularities of solutions of certain one-dimensional wave equations that are caused by a focusing effect related to the dispersive properties of the equation. Whilst the phenomenon to be explained presently appears to be a property of quite a number of wave equations that feature an unbounded, linearized dispersion relation, it is examined here in the relatively narrow context of generalized Korteweg–de Vries equations of the form

$$u_t + u^p u_x + u_{xxx} = 0, \tag{1.1}$$

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where p is a non-negative integer and $u = u(x, t)$ is a real-valued function of the two real variables x and t which in applications are usually proportional to distance measured in the direction of the waves' propagation and elapsed time, respectively. The case $p = 1$ is the classical Korteweg–de Vries equation written in scaled variables and a travelling frame of reference, whilst $p = 0$ recovers the linearized Korteweg–de Vries equation. Though the equations are non-linear for positive p, it is noteworthy that the loss of smoothness suffered by some solutions is associated only with the linearized dispersion relation possessed in common by all the equations (1.1), and hence the term "dispersive blowup" as a descriptive label.

As far as we know, the gist of the idea that comes to the fore here first appeared in an extended remark in the paper of Benjamin *et al.* [4] in their discussion of the Korteweg–de Vries equation's potential as a model for small-amplitude, long-wavelength, surface water waves. The crux of their remark centers on a particular property of the linearized group and phase velocity for the Korteweg–de Vries equation (KdV equation henceforth). One aspect of this will become clear in Section 4 where the fundamental solution of the linearized, initial-value problem, a function related to the Airy function, is studied in some detail. Another way to see what is involved is to take $p = 0$ in (1.1) and consider a simple harmonic wavetrain of the form $\cos(kx - \omega t)$ where the wavenumber k is viewed as fixed but arbitrary. Demanding this wavetrain be a solution of (1.1) with $p = 0$ leads to the dispersion relation for the frequency ω as a function of wavenumber k, namely

$$\omega = \omega(k) = k(1 - k^2). \tag{1.2}$$

Especially important is the fact that the group velocity $c_g(k) = \omega'(k)$ and the phase velocity $c(k) = \omega(k)/k$ are both unbounded, assigning arbitrarily large negative values to short-wave components. Because of this property, it is possible using Fourier's principle to specify initial data arranged in such a way that infinitely many, widely spaced, short-wave components will coalesce at a single point at some given time and thereby result in loss of spatial smoothness of the solution at that time. Thus it was shown in the last cited paper that for (1.1) with $p = 0$, an infinitely differentiable, $L_2(\mathbb{R})$-solution can become unbounded at a single point in space-time. Benjamin *et al.* [4] suggested that similar effects would be found for the KdV equation itself, and went on to view this purported state of affairs as evidence in favour of a heuristic argument for a different model equation, the so-called regularized long-wave equation which has a linearized dispersion relation more closely allied to that of the full, two-dimensional Euler equations for surface wave motion, and which does not possess the property of dispersive blowup. The supposition of Benjamin *et al.* [4]

concerning solutions of the KdV equation follows as a corollary to the main results about dispersive blowup derived here.

Our work relates in general ways to papers of Cohen [10], Ginibre and Tsutsumi [12], Ginibre *et al.* [13], Kato [15], Kenig *et al.* [16, 17], Ponce [24–26], and Sachs [27] wherein it was shown that in certain function classes, the KdV equation is smoothing. That is, some solutions $u(x, t)$ of the KdV equation are, for almost every $t > 0$, smoother in various ways than they were at $t = 0$. From a smoothing result, it is adduced at once by running time backward that certain solutions form singularities in finite time. The methods employed here are quite different from those such as the inverse-scattering transform, energy-type estimates, and harmonic analysis techniques used in the above-mentioned papers, and they lead to more specific conclusions in some cases.

The manner in which our results are obtained is quite simple, and was outlined in our earlier note (Bona and Saut [5]). As explained in Section 4, the evolution equation in question is viewed as a linear equation forced by its non-linearity. Looked at this way, the equation may be solved by linear techniques, so resulting in an equivalent integral equation in which the solution of the linearized initial-value problem appears explicitly. Initial data is then specified along the lines suggested by Benjamin *et al.* [4] that features loss of regularity for the solution of the linearized equation. It is then shown that the non-linear term in the integral equation remains smooth, and so the full solution of the equation is inferred to form the same singularity that was present for the linear equation.

The main ingredient that is used to establish control of the non-linear term in the integral equation is an existence theory for the evolution equation in certain weighted Sobolev spaces. This topic has been the subject of considerable effort, as witnessed by the works of Kato [15] and Kruzhkov and Faminskii [19]. Unfortunately, the particular results needed here seem not to have been established elsewhere, and consequently they are derived in Section 3 of the present paper.

Section 2 contains some preliminary results that will be used in the later sections, whilst Section 5 records commentary on likely extensions of the present theory. Some of the results in Section 2 are new, whilst others are just restatements of theorems in the literature. The Appendices contain the proofs of a couple of the most technical facts described in the body of the paper.

2. Preliminary Results

The generalized KdV equations (GKdV equations henceforth)

$$(\text{GKdV}) \quad u_t + u^p u_x + u_{xxx} = 0 \tag{2.1a}$$

introduced earlier will be posed for $x \in \mathbb{R}$, the real line, and $t>0$, subject to the auxilliary initial condition that

$$u(x, 0)=\psi(x) \tag{2.1b}$$

for $x \in \mathbb{R}$, where ψ is some given function. The initial-value problem (2.1) is locally well posed in reasonable function classes, and if $p<4$ it is globally well posed as the following results on smooth solutions attest.

THEOREM 2.1. *Let $k \geqslant 2$ be an integer and let $\psi \in H^k(\mathbb{R})$. Then the following holds true regarding the initial-value problem* (2.1).

(i) *If $p<4$ there exists a unique solution u of GKdV corresponding to the initial value ψ which, for any $T>0$ and $R>0$, belongs to $C(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+1}(-R, R))$. Moreover, for any $T>0$ and $R>0$, the correspondence $\psi \mapsto u$ is continuous from $H^k(\mathbb{R})$ into $C(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+1}(-R, R))$.*

(ii) *If $p=4$, the same conclusions as those in* (i) *hold provided that $\|\psi\|_0$ is not too large.*

(iii) *If $p>4$ and if $\|\psi\|_1$ is not too large, then the same conclusions enunciated in* (i) *continue to be valid.*

(iv) *If $p \geqslant 4$, but ψ is unrestricted in size, then there exists a positive $T^*=T^*(\|\psi\|_1)$ such that the conclusions in* (i) *hold for all T in the interval* $(0, T^*)$.

(v) *For all $T>0$ such that for all $R>0$ the solution u of the GKdV equation lies in $C(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+1}(-R, R))$, it is also true that for all $R>0$, $\partial_t^j u$ lies in $C(0, T; H^{k-3j}(\mathbb{R})) \cap L_2(0, T; H^{k+1-3j}(-R, R))$ for any j such that $k-3j>-3$. Moreover, the correspondence $\psi \mapsto u$ is continuous from $H^k(\mathbb{R})$ into $\bigcap_{k-3j>-3} C^j(0, T; H^{k-3j}(\mathbb{R})) \cap H^j(0, T; H^{k+1-3j}(-R, R))$, for all $R>0$.*

Notation. In the above theorem, and throughout, the notation is that which is currently standard in the theory of partial differential equations. If X is any Banach space of functions, its usual norm will be denoted $\| \ \|_X$ except for a few abbreviations to be explained now. If $X=L_p(\mathbb{R})$ for some $p \in [1, \infty]$, then the norm of $f \in X$ will be written as $|f|_p$. If k is any non-negative integer and $X=H^k(\mathbb{R})$, the Sobolev space of $L_2(\mathbb{R})$-functions whose first k derivatives lie also in $L_2(\mathbb{R})$, the norm of a function f wil be abbreviated to $\|f\|_k$. Similar abbreviation will apply to the dual spaces $H^{-k}(\mathbb{R})$ of the $H^k(\mathbb{R})$, and indeed to the spaces $H^s(\mathbb{R})$ for any real s (cf. Lions and Magenes [21] for a discussion of these spaces). Thus the norm in $L_2(\mathbb{R})$ is denoted both by $| \ |_2$ and by $\| \ \|_0$. All the spaces $H^s(\mathbb{R})$ are Hilbert spaces, but the only inner product that will intervene in our analysis is that of $L_2(\mathbb{R})$, which is written simply as $(\ ,\ )$. In a couple of places in our development, appeal will also be made to the spaces $W_p^k(\mathbb{R})$ of $L_p(\mathbb{R})$-functions whose first k derivatives also lie in $L_p(\mathbb{R})$ and their dual spaces $W_q^{-k}(\mathbb{R})$ where $p^{-1}+q^{-1}=1$. The norm in these spaces wil not be abbreviated. It is well known that $f \in W_q^{-k}(\mathbb{R})$ if and only if f can be represented as a finite sum of distributional derivatives of order at most k of functions belonging to $L_q(\mathbb{R})$.

If X is any Banach space, $C(0, T; X)$ is the space of continuous functions from $[0, T]$ into X with the maximum norm

$$\|u\|_{C(0,\,T;\,X)} = \max_{0 \leqslant t \leqslant T} \|u(t)\|_X.$$

Similarly, $L_p(0, T; X)$ is the collection of Borel measurable, X-valued functions u defined on $[0, T]$ for which $\|u(t)\|_X$ lies in $L_p(0, T)$.

Remarks. The results (i) and (iv) may be found in Kato [15], except for the continuous dependence in the space $L_2(0, T; H^{k+1}(-R, R))$. This latter result follows from the approximation technique of Bona and Smith [7] coupled with Kato's proof of the local smoothing effect (Kato [15, formula 31]). A sketch of the proof appears in Appendix A to this paper. Conclusions (ii) and (iii) derive from (iv) and the a priori bounds that obtain if $\|\psi\|_0$ is not too big for $p=4$ and if $\|\psi\|_1$ is not too big for $p>4$ (cf. Schechter [30] and Strauss [32]). Part (v) follows from parts (i)–(iv) upon differentiating the equation j times with respect to t and arguing inductively on j for $j=1, ..., m$ where m is the largest integer less than $1+\frac{1}{3}k$.

It is worth comment that recent numerical simulations of the initial-value problem (2.1) indicate that solutions u corresponding to initial data $\psi \in H^k(\mathbb{R})$, $k \geqslant 1$, need not remain in the class $H^k(\mathbb{R})$ for all time if $p \geqslant 4$ (see Bona *et al.* [8, 9]). Indeed, it was shown in Albert *et al.* [2] that $\|u(\cdot, t)\|_k$ becomes infinite in finite time if and only if $|u(\cdot, t)|_\infty$ becomes infinite in finite time, and the numerical simulations indicate convincingly that the $L_\infty(\mathbb{R})$-norm of certain solutions becomes unbounded at a finite value of t. However, it appear that this sort of singularity formation subsists essentially on the non-linearity and is not directly related to the results obtained here. Consequently we term it "non-linear blowup" to distinguish it from the dispersive blowup that is the focus for the present. The only result here that bears upon non-linear blowup is Corollary 3.2 which strengthens slightly the just mentioned theory of Albert *et al.* [2]. Note also that the argument of Saut and Temam [29], when applied in the present context, shows that if $\psi \in H^s(\mathbb{R})$, $s \geqslant 2$, but $\psi \notin H^r(\mathbb{R})$ for some $r > s$, then the solution u of (2.1) emanating from ψ does not lie in $H^r(\mathbb{R})$ for any t in its interval of existence as an $H^s(\mathbb{R})$-valued solution of (2.1).

If $\psi \in H^s(\mathbb{R})$ where $s > \frac{3}{2}$ is not necessarily an integer, most of the above results are still valid (cf. Abdelouhab *et al.* [1], Bona and Scott [6], Kato, [14, 15], Saut [28], and Saut and Temam [29]).

In addition to the theory for (2.1) pertaining to data ψ in $H^k(\mathbb{R})$ where $k \geqslant 2$, there is also a theory of weak solutions that applies when $k=0$ or $k=1$. Stated now are results in this direction which will be used in Section 4 (see also the recent papers of Ginibre and Tsutsumi [12], Ginibre *et al.* [13], and Kenig *et al.* [16, 17], for other, very nice results concerning solutions corresponding to fairly rough initial data).

THEOREM 2.2. *Let $\psi \in H^k(\mathbb{R})$ be initial data for the problem* (2.1). *Then the following conclusions obtain.*

(i) *If $k=0$ and $p<4$, then there exists a solution u of* (2.1) *which, for any $T>0$ and $R>0$, lies in $L_\infty(\mathbb{R}^+; L_2(\mathbb{R})) \cap L_2(0, T; H^1(-R, R))$. A bound for the norm of u in this space can be given that depends only on T, R, and $|\psi|_2$.*

(ii) *If $k=1$ and $p<4$, then there exists a solution u of* (2.1) *which, for any $T>0$ and $R>0$, lies in $L_\infty(\mathbb{R}^+; H^1(\mathbb{R})) \cap L_2(0, T; H^2(-R, R))$. A bound for the norm of u in this space can be given that depends only on T, R, and $\|\psi\|_1$.* If $p \geqslant 4$ *and $\|\psi\|_1$ is sufficiently small, then the same result holds.*

Remarks. For $p=1$, the result (i) is in Kato [15]. By paying a little more attention to the details of Kato's argument, the improved range of p may be obtained as is shown in the sketch presented below. Part (ii) appears to be new, and consequently a complete proof is offered here. A partial result in this direction has recently been obtained by Ponce [24]. A similar result may also be deduced from the work of Kenig *et al.* [18].

Proof. (i) Suppose that $k=0$ but $1<p<4$. We follow closely the argument of Kato [15], especially his Theorem 6.2 and Theorem 7.1. Let $\{\psi_j\}_{j=1}^\infty$ denote a sequence of $H^\infty(\mathbb{R})$-functions such that $\psi_j \to \psi$ in $L_2(\mathbb{R})$ and, say, $\|\psi_j\|_0 \leqslant \|\psi\|_0$ for all j, and for each j let u_j be the associated smooth solution of the GKdV equation with initial data ψ_j as guaranteed by Theorem 2.1. By virtue of Kato's formula (33) in the proof of his Theorem 6.2,

$$\int_0^T \int_{-R}^R |\partial_x u_j(x, t)|^2 \, dx \, dt \leqslant K,$$

where K depends only on T, R, and $\|\psi_j\|_0$, and so if R and T are fixed, is independent of j. Moreover, for all $t \geqslant 0$,

$$\|u_j(\cdot, t)\|_0 = \|\psi_j\|_0 \leqslant \|\psi\|_0.$$

The idea now is to pass to the limit as $j \to \infty$, for which process Aubin's compactness theorem (Aubin [3], Lions and Peetre [22], Lions [20, Chap. 1, Sect. 5]) presents itself as a potentially useful tool. However, some sort of bound on $\partial_t u_j$ is needed to apply this theorem. Since $\{u_j\}_{j=1}^{\infty}$ is uniformly bounded in $L_\infty(0, T; L_2(\mathbb{R})) \cap L_2(0, T; H^1(-R, R))$ for each fixed T and R, it transpires that $\{\partial_x^3 u_j\}_{j=1}^{\infty}$ is uniformly bounded in $L_\infty(0, T; H^{-3}(\mathbb{R})) \cap L_2(0, T; H_{\text{loc}}^{-2}(\mathbb{R}))$ for each fixed T. Thus a j-independent bound on the non-linear term $\partial_x(u_j^{p+1})$ will yield a j-independent bound on $\partial_t u_j$ by virtue of the differential equation. Abbreviating u_j by u, we argue as follows. Let $\phi \in H_0^1(-R, R)$ with $\|\phi\|_{H_0^1(-R,R)} \leqslant 1$. Then for each $t \in [0, T]$,

$$\int_{-R}^{R} u^p u_x \phi \, dx = -\frac{1}{p+1} \int_{-R}^{R} u^{p+1} \phi_x \, dx,$$

and so by Hölder's inequality, a Sobolev imbedding theorem, and a classical interpolation inequality, we have

$$\begin{aligned}\left| \int_{-R}^{R} u^p u_x \phi \, dx \right| &\leqslant \frac{1}{p+1} \|u(\cdot, t)\|_{L_{2p+2}(-R,R)}^{p+1} \|\phi_x\|_{L_2(-R,R)} \\ &\leqslant c \|u(\cdot, t)\|_{H^{p/(2p+2)}(-R,R)}^{p+1} \\ &\leqslant c \|u(\cdot, t)\|_{L_2(-R,R)}^{(p+2)/2} \|u(\cdot, t)\|_{H^1(-R,R)}^{p/2} \\ &\leqslant C \|u(\cdot, t)\|_{H^1(-R,R)}^{p/2},\end{aligned}$$

because of the t- and j-independent bound on $\|u(\cdot, t)\|_0$. Since this inequality holds for all such ϕ, there is implied the relation

$$\|u^p u_x\|_{H^{-1}(-R,R)} \leqslant C \|u(\cdot, t)\|_{H^1(-R,R)}^{p/2}.$$

Integrating the $(\frac{4}{p})$-th power of this relation over $[0, T]$ leads to

$$\begin{aligned}\int_0^T \|u_j^p \partial_x u_j\|_{H^{-1}(-R,R)}^{4/p} \, dt &\leqslant C \int_0^T \|u_j(\cdot, t)\|_{H^1(-R,R)}^2 \, dt \\ &\leqslant CK,\end{aligned}$$

which holds independently of j. It follows that $\{\partial_t u_j\}_{j=1}^{\infty}$ is bounded in $L_{4/p}(0, T; H^{-2}(-R, R))$, independently of j. Thus in case $4/p > 1$, we are in a position to pass to the limit as $j \to \infty$ in the way that is by now classical (see Kato [15], Lions [20], Strauss [32], Temam [34]), obtaining a global solution u of the GKdV equation that, for each $T > 0$, lies in

$$L_\infty(0, T; L_2(\mathbb{R})) \cap L_2(0, T; H_{\text{loc}}^1(\mathbb{R})) \cap C(0, T; H_{\text{loc}}^{-1/2}(\mathbb{R}))$$

such that $u(\cdot, t) \to \psi$ as $t \to 0$ at least locally in $H^{-1/2}$.

(ii) Let $r: \mathbb{R} \to \mathbb{R}$ be a strictly increasing, bounded, smooth function, all of whose derivatives are bounded. We proceed as in Section 6 of Kato [15]. Let $\{\psi_j\}_{j=1}^{\infty}$ be a sequence of $H^{\infty}(\mathbb{R})$-functions such that $\psi_j \to \psi$ in $H^1(\mathbb{R})$. Let $v = u_j$ denote the smooth solution of (2.1) corresponding to the initial datum ψ_j, for $j = 1, 2, \ldots$. As is well known, if $p < 4$, $\|v(\cdot, t)\|_1$ is bounded, independently of t with a bound that depends only on $\|\psi_j\|_1$. Hence $\|v(\cdot, t)\|_1$ is bounded, independently of j and t. If $p \geq 4$, then the theory worked out by Strauss [32] or Schechter [30] contains the same conclusions in case $\|\psi_j\|_1$ is small enough, and the latter is true for large enough j provided $\|\psi\|_1$ is small enough.

A j-independent bound on v in the space $L_2(0, T; H^2_{\mathrm{loc}}(\mathbb{R}))$ is now derived. Differentiate a solution v of the GKdV equation with respect to x, multiply by rv_x, and integrate the resulting expression over $\mathbb{R}$. After several integrations by parts, there appears the relation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} rv_x^2\,dx &+ \frac{3}{2}\int_{-\infty}^{\infty} r_x v_{xx}^2\,dx - \frac{1}{2}\int_{-\infty}^{\infty} r_{xxx} v_x^2 \\ &+ \frac{1}{(p+1)(p+2)}\int_{-\infty}^{\infty} r_{xxx} v^{p+2}\,dx - 2\int_{-\infty}^{\infty} r_x v^p v_x^2\,dx \\ &= -\frac{1}{p+1}\int_{-\infty}^{\infty} rv^{p+1} v_{xxx}\,dx. \end{aligned} \tag{2.2}$$

Using the GKdV equation satisfied by v, the right-hand side of (2.2) may be expressed as

$$\begin{aligned}-\frac{1}{p+1}\int_{-\infty}^{\infty} rv^{p+1} v_{xxx}\,dx &= \frac{1}{(p+1)(p+2)}\frac{d}{dt}\int_{-\infty}^{\infty} rv^{p+2}\,dx \\ &\quad - \frac{1}{2(p+1)^2}\int_{-\infty}^{\infty} r_x v^{2p+2}\,dx. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) and rearranging terms leads to

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} r\left(\frac{1}{2}v_x^2 - \frac{1}{(p+1)(p+2)}v^{p+2}\right)dx &+ \frac{3}{2}\int_{-\infty}^{\infty} r_x v_{xx}^2\,dx \\ = \int_{-\infty}^{\infty} r_{xxx}&\left(\frac{1}{2}v_x^2 - \frac{1}{(p+1)(p+2)}v^{p+2}\right)dx \\ + \int_{-\infty}^{\infty} r_x v^p&\left(2v_x^2 - \frac{1}{2(p+1)^2}v^{p+2}\right)dx. \end{aligned} \tag{2.4}$$

Using the aformentioned properties of r and the fact that the $H^1(\mathbb{R})$-norm of v is bounded independently of j and t, it follows readily that the right-hand side of (2.4) is bounded, independently of j and t, say by a constant $c=c(\|\psi\|_1)$ which depends only on $\|\psi\|_1$ for j sufficiently large. Integrating (2.4) over the temporal interval $[0, t]$, where $0<t\leqslant T$, and estimating the right side by cT, we obtain that

$$\frac{3}{2}\int_0^t\int_{-\infty}^{\infty} r_x v_{xx}^2\,dx \leqslant cT - \int_{-\infty}^{\infty} r\left(\frac{1}{2}v_x^2 - \frac{1}{(p+1)(p+2)}v^{p+2}\right)dx\Bigg|_0^t.$$

Since r and $\|v\|_1$ are bounded, independently of t and of j, it may thus be concluded that

$$\int_0^T\int_{-\infty}^{\infty} r_x v_{xx}^2\,dx \leqslant c, \tag{2.5}$$

where the constant in (2.5) depends only upon T, r, and on $\|\psi\|_1$, at least for j sufficiently large. By choosing the increasing function r appropriately, it follows that

$$\int_0^T\int_{-R}^{R} v_{xx}^2\,dx \leqslant c, \tag{2.6}$$

where c depends only on T, R, and $\|\psi\|_1$. This is the desired a priori information.

Once it has been inferred that the sequence $\{u_j\}_{j=1}^{\infty}$ is bounded in $L_\infty(\mathbb{R}^+; H^1(\mathbb{R}))$ and, for each $T>0$ and $R>0$, in $L_2(0, T; H^2(-R, R))$, standard arguments show that the sequence $\{u_j\}_{j=1}^{\infty}$ converges to a function u which is a global distributional solution of (2.1) and which lies in $L_\infty(\mathbb{R}^+; H^1(\mathbb{R}))\cap L_2(0, T; H^2_{\text{loc}}(\mathbb{R}))$, for all $T>0$. It also follows that $u\in C(\mathbb{R}^+; H^{-1/2}(\mathbb{R}))$, $u\in C_w(\mathbb{R}^+; H^1(\mathbb{R}))$, and that $u_t\in C_w(\mathbb{R}^+; H^{-2}(\mathbb{R}))\cap L_2(0, T; H^{-1}_{\text{loc}}(\mathbb{R}))$, for all $T>0$. Here $C_w(I; X)$ denotes the functions $u: I\to X$ which are continuous when X is given the weak topology.

This concludes the proof of the theorem. ∎

While the GKdV equation does not necessarily possess global solutions for arbitrary smooth data when p is larger than 3, it is interesting to note that if $p=2q$ is even, and the sign of the non-linearity is changed, then no matter how large q is, global solutions obtain for any smooth data. Thus consider the initial-value problem

$$u_t - u^{2q}u_x + u_{xxx} = 0, \tag{2.7a}$$

$$u(x, 0) = \psi(x). \tag{2.7b}$$

Regarding (2.7), the following result applies for any $q\in\mathbb{N}$.

THEOREM 2.3. *Suppose that k is a non-negative integer and $\psi\in H^k(\mathbb{R})$.*

Then if $k = 0$ and $q \leqslant 2$, or $k = 1$ and q is unrestricted, there exists a solution u of (2.7) with initial data ψ such that, for any $T>0$ and $R>0$, $u \in L^\infty(0, T; H^k(\mathbb{R})) \cap L^2(0, T; H^{k+1}(-R, R))$. If $k=2$ and q is arbitrary, the solution is unique. For any $q \geqslant 0$, if $k \geqslant 2$, then $u \in C(0, T; H^k(\mathbb{R})) \cap L^2(0, T; H^{k+1}(-R, R))$, and $\partial_t^j u \in C(0, T; H^{k-3j}(\mathbb{R})) \cap L_2(0, T; H^{k+1-3j}(-R, R))$ for all j such that $k-3j>-3$. Moreover, the correspondence $\psi \mapsto u$ is continuous from $H^k(\mathbb{R})$ to $\bigcap_{k-3j>-3} C^j(0, T; H^{k-3j}(\mathbb{R})) \cap H^j(0, T; H^{k+1-3j}(-R, R))$ for all $T>0$ and $R>0$.

Remarks. Despite the more agreeable situation regarding the global well-posedness of the initial-value problem (2.7), it has exactly the same behaviour as does (2.1) regarding dispersive blowup, again emphasizing the scant role played by non-linearity in our theory. In particular, it is suggested that what we earlier called non-linear blowup is independent of dispersive blowup.

The proof of Theorem 2.3 follows by applying Kato's general theory (cf. Kato [15, Sects. 4, 5, 6, 7]). The central difference between the GKdV equation and (2.7a) is that the third invariant integral is positive definite for (2.7a), whilst this property is lacking for (2.1a). More exactly, if u is a sufficiently regular solution of GKdV, then

$$\int_{-\infty}^{\infty} u(x, t)\, dx, \qquad \int_{-\infty}^{\infty} u^2(x, t)\, dx,$$

$$\int_{-\infty}^{\infty} [\tfrac{1}{2}u_x^2(x, t) - c_p u^{p+2}(x, t)]\, dx$$

are all time invariant. Here $c_p = 1/(p+1)(p+2)$. For (2.7a), the corresponding invariants are

$$\int_{-\infty}^{\infty} u(x, t)\, dx, \qquad \int_{-\infty}^{\infty} u^2(x, t)\, dx,$$

$$\int_{-\infty}^{\infty} [\tfrac{1}{2}u_x^2(x, t) + c_{2q} u^{2q+2}(x, t)]\, dx.$$

It follows that the $H^1(\mathbb{R})$-norm of solutions of (2.7a) is always bounded, whereas this conclusion seems to apply to GKdV for initial data unrestricted in size only if $p<4$.

3. Existence Theorems in Weighted Spaces

In this section some technical theorems that are central to our main line of argument will be established, namely an existence theory for the initial-value problem (2.1) set in weighted, L_2-based Sobolev spaces. The results established here are similar to those proved by Kato [15] and Kruzhkov and Faminski [19], although the precise theorems obtained appear to be new in case $p > 1$.

The weights w of interest for the blow-up results in Section 4 have the following properties. The function $w = w_\sigma(x)$ is a non-decreasing, C^∞ function depending on a positive parameter σ for which

$$w(x) = w_\sigma(x) = \begin{cases} 1 & \text{for } x < 0, \quad \text{and} \\ (1+x^2)^\sigma & \text{for } x > 1. \end{cases} \tag{3.1}$$

The class $L_2(\mathbb{R}, w)$ is the class of measurable functions which are square integrable with respect to the measure $w^2(x)\,dx$. The class $H^k(\mathbb{R}, w)$ is the subspace of $L_2(\mathbb{R}, w)$ consisting of all those elements whose first k distributional derivatives also lie in $L_2(\mathbb{R}, w)$. The space $H^k(\mathbb{R}, w)$ is given the obvious Hilbert space structure.

THEOREM 3.1. *Let p and k be non-negative integers and let the parameter σ associated with the weight w be non-negative, but otherwise arbitrary. Suppose the initial data ψ in (2.1) to lie in $H^k(\mathbb{R}, w)$.*

(a) *If $k = 0$ or 1 and $p < 4$, then there exists a solution u of (2.1) corresponding to the initial data ψ such that, for any $T > 0$, u belongs to $L_\infty(0, T; H^k(\mathbb{R}, w)) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$.*

(b) *If $k \geqslant 2$ and $p < 4$, then there exists a unique solution u of (2.1) corresponding to the initial data ψ such that, for any $T > 0$, u belongs to $C(0, T; H^k(\mathbb{R}, w)) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$. Moreover, the mapping that associates u to ψ is continuous from $H^k(\mathbb{R}, w)$ into $C(0, T; H^k(\mathbb{R}, w)) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$.*

(c) *If $k \geqslant 1$ and $p \geqslant 4$, then corresponding to each ψ there is a $T^* = T^*(\|\psi\|_1)$ and a solution u of (2.1) corresponding to ψ such that for any $T \in (0, T^*)$, u lies in $L_\infty(0, T; H^k(\mathbb{R}, w)) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$. If $k \geqslant 2$, u lies in $C(0, T; H^k(\mathbb{R}, w)) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$ and is unique within its function class. Moreover, the mapping that associates u to ψ is continuous. If $\|\psi\|_1$ is small enough, then $T^* = +\infty$.*

Remark. In fact, the value T^* appearing in part (c) of the theorem is exactly the same as the value T^* that intervenes in part (iv) of Theorem 2.1. That is, if the initial data ψ lies in $H^k(\mathbb{R}; w_\sigma)$, then as long as the solution $u(\cdot, t)$ remains in $H^k(\mathbb{R})$, it follows that it lies in $H^k(\mathbb{R}; w_\sigma)$ also.

Proof. We follow the standard pattern of deriving bounds on solutions of (2.1) corresponding to smooth initial data and then passing to the limit as the smooth data converges to the given data ψ. To this end, let $\{\psi_j\}_{j=1}^{\infty}$ be a sequence of $C_0^\infty(\mathbb{R})$-functions that converges to ψ in $H^k(\mathbb{R}, w)$. Without loss of generality, it may be assumed that $\|\psi_j\|_{H^k(\mathbb{R}, w)} \leqslant \|\psi\|_{H^k(\mathbb{R}, w)}$ for all j. Let $u = u_j$ denote the unique solution of (2.1) corresponding to ψ_j. This solution will be local or global, depending on the value of p (see Theorem 2.1), but while it exists it will be a C^∞-function of x and t all of whose partial derivatives lie in $L_2(\mathbb{R})$. Let $v = wu$ so that v satisfies the equation

$$v_t + v_{xxx} + v\left(6\frac{w_x w_{xx}}{w^2} - 6\frac{w_x^3}{w^3} - \frac{w_{xxx}}{w}\right) + v_x\left(6\frac{w_x^2}{w^2} - 3\frac{w_{xx}}{w}\right)$$

$$-3\frac{w_x}{w}v_{xx} + \frac{1}{w^p}v^p v_x - \frac{w_x}{w^{p+1}}v^{p+1} = 0. \tag{3.2}$$

By assumption, $v(x, 0)$ lies in $H^\infty(\mathbb{R})$ and, in addition, $\|v(\cdot, 0)\|_k$ is bounded independently of j. Since

$$\frac{w_x}{w} \geqslant 0$$

and

$$6\frac{w_x w_{xx}}{w^2} - 6\frac{w_x^3}{w^3} - \frac{w_{xxx}}{w}, \qquad 6\frac{w_x^2}{w^2} - 3\frac{w_{xx}}{w},$$

$$\frac{1}{w^p}, \qquad \frac{w_x}{w^{p+1}}$$

are all smooth, bounded functions of x, Kato's general theory [14, 15] assures that at least locally in time, there is a unique C^∞-function $v(x, t)$ solving (3.2) with the given initial value which has its partial derivatives in $L_2(\mathbb{R})$. By uniqueness for H^∞-solutions of (2.1), it follows that $v/w = u$.

Consider first the case $k = 0$, because it is particularly important in Section 4 and because the proof in this case is illustrative of the argument in the general case. Multiply (3.2) by v, integrate over $\mathbb{R}$, and integrate by parts appropriately to reach the relation

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} v^2\,dx + 3\int_{-\infty}^{\infty} \frac{w_x}{w} v_x^2$$

$$= \int_{-\infty}^{\infty} \theta v^2\,dx + \frac{2}{p+2}\int_{-\infty}^{\infty} \frac{w_x}{w^{p+1}} v^{p+2}\,dx, \tag{3.3}$$

where

$$\theta = \frac{w_{xxx}}{w} + 6\frac{w_x^3}{w^3} - 6\frac{w_x w_{xx}}{w^2} - \frac{3}{2}\left(\frac{w_{xx}}{w}\right)_x + 3\left(\frac{w_x^2}{w^2}\right)_x + \frac{3}{2}\left(\frac{w_x}{w}\right)_{xx}.$$

The definition of w entails that θ is a bounded, smooth function. Thus Gronwall's lemma my be applied to (3.3) if we are able to estimate appropriately the second term on the right-hand side of (3.3). To this end, remember that $v/w = u$ and that $\|u\|_0$ is bounded, independently of j and of p.

Taking $p = 1$ in (3.3), we begin by observing that

$$\int_{-\infty}^{\infty} \frac{w_x}{w^2} v^3\,dx = \int_{-\infty}^{\infty} \frac{w_x}{w} uv^2 \leqslant \|u\|_0\,\|v\|_0 \left|\frac{w_x}{w} v\right|_\infty \leqslant c\,\|v\|_0 \left|\frac{w_x}{w} v\right|_\infty.$$

Furthermore, since w_x/w and $(w_x/w)_x$ are both bounded functions, it follows that

$$\begin{aligned}
\left|\frac{w_x}{w} v\right|_\infty &\leqslant \left\|\frac{w_x}{w} v\right\|_0^{1/2} \left\|\left(\frac{w_x}{w} v\right)_x\right\|_0^{1/2} \\
&\leqslant 2\left|\frac{w_x}{w}\right|_\infty \|v\|_0^{1/2} \left\{\left\|\left(\frac{w_x}{w}\right)_x v\right\|_0^{1/2} + \left\|\frac{w_x}{w} v_x\right\|_0^{1/2}\right\} \\
&\leqslant c\left\{\|v\|_0 + \|v\|_0^{1/2} \left\|\frac{w_x}{w} v_x\right\|_0^{1/2}\right\},
\end{aligned}$$

where here, and henceforth, c denotes various constants that depend upon the weight w and the norm of the initial data ψ, but which are independent of t and j. It also follows that

$$\left\|\frac{w_x}{w} v_x\right\|_0 \leqslant c\left\|\left(\frac{w_x}{w}\right)^{1/2} v_x\right\|_0.$$

Hence an application of Young's inequality leads to the estimate

$$\int_{-\infty}^{\infty} \frac{w_x}{w^2} v^3\,dx \leqslant c\,\|v\|_0^2 + \frac{3}{2}\int_{-\infty}^{\infty} \frac{w_x}{w} v_x^2\,dx.$$

Combining this estimate with (3.3) and the fact that θ is bounded gives

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} v^2\,dx + 2\int_{-\infty}^{\infty} \frac{w_x}{w} v_x^2\,dx \leqslant c\int_{-\infty}^{\infty} v^2\,dx,$$

from which it is readily adduced that $\|v(\cdot, t)\|_0$ is bounded, uniformly on bounded time intervals.

Suppose now that $2 \leqslant p \leqslant 4$. The crucial term on the right-hand side of (3.3) is then handled differently. First, for any p, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{w_x}{w^{p+1}} v^{p+2}\, dx &= \int_{-\infty}^{\infty} u^2 \frac{w_x}{w^{p-1}} v^p\, dx \leqslant c \left| \frac{w_x}{w^{p-1}} v^p \right|_\infty \leqslant c \left| \left(\frac{w_x}{w^{p-1}} v^p \right)_x \right|_1 \\
&\leqslant c \left| \left(\frac{w_x}{w^{p-1}} \right)_x v^p \right|_1 + c \left| \frac{w_x}{w^{p-1}} v^{p-1} v_x \right|_1 \\
&\leqslant c \left| \left(\frac{w_{xx}}{w^{p-1}} - (p-1) \frac{w_x^2}{w^p} \right) v^p \right|_1 \\
&\quad + c \left| \frac{w_x^{1/2}}{w^{1/2}} \frac{1}{w^{p-2}} v^{p-1} \right|_2 \left| \left(\frac{w_x}{w} \right)^{1/2} v_x \right|_2 .
\end{aligned} \tag{3.4}$$

If $p=2$, the properties of w allow us to continue this bound as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{w_x}{w^{p+1}} v^{p+2}\, dx &\leqslant c\, |v|_2^2 + c\, |v|_2 \left| \left(\frac{w_x}{w} \right)^{1/2} v_x \right|_2 \\
&\leqslant c\, |v|_2^2 + \int_{-\infty}^{\infty} \frac{w_x}{w} v_x^2\, dx.
\end{aligned}$$

One may then use this in (3.3) along with Gronwall's lemma to conclude the desired result. If $p > 2$, the estimate in (3.4) is extended in different ways. For $p = 3$, write

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{w_x}{w^4} v^5\, dx &\leqslant c \left| u \left(\frac{w_{xx}}{w} - 2 \frac{w_x^2}{w^2} \right) v^2 \right|_1 + c \left| u \left(\frac{w_x}{w} \right)^{1/2} v \right|_2 \left| \left(\frac{w_x}{w} \right)^{1/2} v_x \right|_2 \\
&\leqslant c \left| \left(\frac{w_{xx}}{w} - 2 \frac{w_x^2}{w^2} \right) v^2 \right|_2 + c \left| \left(\frac{w_x}{w} \right)^{1/2} v \right|_\infty \left| \left(\frac{w_x}{w} \right)^{1/2} v_x \right|_2 \\
&\leqslant c\, |v|_2 \left| \left(\frac{w_{xx}}{w} - 2 \frac{w_x^2}{w^2} \right) v \right|_\infty \\
&\quad + c \left| \left(\frac{w_x}{w} \right)^{1/2} v \right|_2^{1/2} \left| \left[\left(\frac{w_x}{w} \right)^{1/2} v \right]_x \right|_2^{1/2} \left| \left(\frac{w_x}{w} \right)^{1/2} v_x \right|_2 \\
&\leqslant c\, |v|_2 \left| \left(\frac{w_{xx}}{w} - 2 \frac{w_x^2}{w^2} \right) v \right|_2^{1/2} \left| \left[\left(\frac{w_{xx}}{w} - 2 \frac{w_x^2}{w^2} \right) v \right]_x \right|_2^{1/2} \\
&\quad + c |v|_2 \left| \left(\frac{w_{xx}}{w} \right)^{1/2} v_x \right|_2 + c |v|_2^{1/2} \left| \left(\frac{w_x}{w} \right)^{1/2} v_x \right|_2^{3/2}
\end{aligned}$$

$$\leqslant c\,|v|_2^2 + c\,|v|_2^{3/2}\left|\left(\frac{w_{xx}}{w_x} - 2\frac{w_x}{w}\right)\frac{w_x}{w}\,v_x\right|_2^{3/2}$$

$$+\frac{1}{2}\left|\left(\frac{w_x}{w}\right)^{1/2} v_x\right|_2^2$$

$$\leqslant c\,|v|_2^2 + \int_{-\infty}^{\infty} \frac{w_x}{w}\,v_x^2\,dx.$$

This estimate applied to (3.3) gives, as before, the advertised bound on $|v(\cdot, t)|_2$. Finally, consider the case $p=4$. Following the same lines of argument that appear in the cases $p=1, 2, 3$, we are only able to derive the upper bound

$$\int_{-\infty}^{\infty} \frac{w_x}{w^s}\,v^6\,dx \leqslant c_1\,|v|_2^2 + c_2(|u|_2)\left|\left(\frac{w_x}{w}\right)^{1/2} v_x\right|_2^2,$$

where $c_2 = c_2(|u|_2)$ is an absolute constant times the fourth power of $|u|_2$. This latter bound is effective in conjunction with (3.3) only for $|u|_2$ sufficiently small. In fact, without working hard, one determines that $|u|_2 < 9^{1/4}$ suffices, so the restriction on $|u|_2$ for which (3.5) becomes useful in (3.3) is not so stringent.

If $p>4$ or if $p=4$ and the initial data has relatively large $L_2(\mathbb{R})$-norm, the line of argument given above breaks down. However, the situation may be retrieved at the cost of extra hypotheses. First, assume that $\|\psi_j\|_1$ is bounded, independently of j, and that this bound is sufficiently small for the theory in unweighted spaces to imply that the solutions u_j corresponding to the initial data ψ_j, $j=1, 2, \ldots$, are globally defined and that their $H^1(\mathbb{R})$-norm is bounded, independently of t and j.

Again consider the last term in (3.3), but now proceed in a simpler manner to form the estimate

$$\left|\int_{-\infty}^{\infty} \frac{w_x}{w^{p+1}}\,v^{p+2}\,dx\right| = \left|\int_{-\infty}^{\infty} \frac{w_x}{w}\,u^p v^2\,dx\right| \leqslant c\,|u|_\infty^p\,|v|_2^2 \leqslant c\,|v|_2^2. \quad (3.5)$$

Here c depends upon $\sup_j \|\psi_j\|_1$ and w, but is independent of t and j. Thus Gronwall's lemma applies immediately and for any $T>0$, a j-independent bound on u_j in $L_\infty(0, T; L_2(\mathbb{R}, w))$ thereby results. Notice that this bound subsists only on $|u_j(\cdot, t)|_\infty$ having a j-independent bound on the interval $[0, T]$. Thus if $\|u_j(\cdot, t)\|_1$ is bounded on $[0, T]$, independently of j, it follows that u_j is bounded in $L_\infty(0, T; L_2(\mathbb{R}, w))$ with a finite bound not dependent upon j.

Attention is now turned to the case $k=1$ but p arbitrary, where it is remarked at the outset that, because of the first part of the proof, there is in hand a bound on $u=u_j$ for $0\leqslant t\leqslant T$ in $L_2(\mathbb{R}, w)$, or, what is the same, a bound on $|v|_2$ which depends only upon $\|\psi\|_1$ and T, though if $p\geqslant 4$ the values of T may be restricted. Now multiply (3.2) by $-v_{xx}-(p+1)^{-1}v^{p+1}/w^p$ and integrate the result over $\mathbb{R}$ to obtain the equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} v_x^2\,dx-\frac{1}{(p+1)(p+2)}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{v^{p+2}}{w^p}\,dx\\
&\quad+3\int_{-\infty}^{\infty}\frac{w_x}{w}v_{xx}^2\,dx+\frac{3}{p+1}\int_{-\infty}^{\infty}\frac{v^{p+1}}{w^{p+1}}v_{xx}\,dx\\
&\quad-\int_{-\infty}^{\infty}\theta_1 v v_{xx}\,dx-\frac{1}{p+1}\int_{-\infty}^{\infty}\theta_1\frac{v^{p+2}}{w^p}\,dx\\
&\quad-\int_{-\infty}^{\infty}\theta_2 v_x v_{xx}\,dx-\frac{1}{p+1}\int_{-\infty}^{\infty}\theta_2\frac{v^{p+1}}{w^p}v_x\,dx\\
&\quad+\frac{1}{p+1}\int_{-\infty}^{\infty}\frac{w_x}{w^{p+1}}v^{p+1}v_{xx}\,dx+\frac{1}{(p+1)^2}\int_{-\infty}^{\infty}\frac{w_x}{w^{2p+1}}v^{2p+2}\,dx=0, \qquad (3.6)
\end{aligned}$$

where

$$\theta_1(x)=-\frac{w_{xxx}}{w}-6\left(\frac{w_x}{w}\right)^2+6\frac{w_x w_{xx}}{w^2}$$

and

$$\theta_2(x)=-3\frac{w_{xx}}{w}+6\left(\frac{w_x}{w}\right)^2.$$

The detailed properties of w imply that both θ_1 and θ_2 are bounded functions, along with their derivatives to all orders. Since the initial data $\psi=\psi_j$ lies in $H^1(\mathbb{R}, w)$, it certainly lies in $H^1(\mathbb{R})$ and therefore the theory in unweighted Sobolev spaces implies that $u=v/w$ lies in $L_\infty(0, T; H^1(\mathbb{R}))$ with a bound in this space that depends only upon $\|\psi\|_1$ (and in case $p\geqslant 4$, either the bound entails a restriction on the size of $\|\psi\|_1$ or else the bound depends on T and holds only for $T<T^*=T^*(\|\psi\|_1)$). It follows that $|u|_\infty$ is bounded on $\mathbb{R}\times[0, T]$, since $L_\infty(0, T; H^1(\mathbb{R}))\subsetneq L_\infty(\mathbb{R}\times[0, T])$, and again the bound only depends on $\|\psi\|_1$ for $p<4$, but may depend on T if $p\geqslant 4$ and the size of $\|\psi\|_1$ is not suitably restricted. Using this information, it follows immediately that

$$\left|\int_{-\infty}^{\infty} \frac{v^{p+2}}{w^p}\theta_1\,dx\right| \leqslant c\int_{-\infty}^{\infty} v^2\,dx \leqslant c,$$

$$\left|\int_{-\infty}^{\infty} v_x v_{xx}\theta_2\,dx\right| \leqslant c\int_{-\infty}^{\infty} v_x^2\,dx,$$

$$\left|\int_{-\infty}^{\infty} vv_{xx}\theta_1\,dx\right| \leqslant c\left\{1+\int_{-\infty}^{\infty} v_x^2\,dx\right\},$$

$$\left|\int_{-\infty}^{\infty} \frac{v^{p+1}}{w^p}v_x\theta_2\,dx\right| \leqslant c\left\{1+\int_{-\infty}^{\infty} v_x^2\,dx\right\}.$$

On the other hand, partial integration shows

$$\int_{-\infty}^{\infty} \frac{v^{p+1}}{w^{p+1}}v_{xx}\,dx = -\int_{-\infty}^{\infty} (u^{p+1})_x v_x\,dx = -(p+1)\int_{-\infty}^{\infty} u^p u_x v_x\,dx,$$

so that

$$\left|\int_{-\infty}^{\infty} \frac{v^{p+1}}{w^{p+1}}v_{xx}\,dx\right| \leqslant \frac{p+1}{2}\{|u|_\infty^{2p}|u_x|_2^2+|v_x|_2^2\} \leqslant c\left\{1+\int_{-\infty}^{\infty} v_x^2\,dx\right\}$$

since $|u|_\infty$ is bounded on $\mathbb{R}\times[0,T]$ independently of j. The two terms not yet treated that appear on the right-hand side of (3.7) do not give significantly more trouble, as we see now. Consider first the penultimate term in (3.6) and proceed by writing

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{v^{p+1}v_{xx}w_x}{w^{p+1}}\,dx &= -(p+1)\int_{-\infty}^{\infty} u_x u^p v_x w_x\,dx - \int_{-\infty}^{\infty} u^{p+1}v_x w_{xx}\,dx\\ &= -(p+1)\int_{-\infty}^{\infty} v_x^2 u^p \frac{w_x}{w}\,dx\\ &\quad +(p+1)\int_{-\infty}^{\infty} v\frac{w_x^2}{w^2}u^p v_x\,dx - \int_{-\infty}^{\infty} vv_x u^p\frac{w_{xx}}{w}\,dx.\end{aligned}$$

Since w_x/w, w_{xx}/w, and u are bounded, it follows that

$$\left|\int_{-\infty}^{\infty} \frac{v^{p+1}v_{xx}w_x}{w^{p+1}}\,dx\right| \leqslant c\left\{\int_{-\infty}^{\infty} v^2\,dx + \int_{-\infty}^{\infty} v_x^2\,dx\right\}.$$

The final term in (3.6) is written simply as

$$\int_{-\infty}^{\infty} v^{2p+2}\frac{w_x}{w^{2p+1}}\,dx = \int_{-\infty}^{\infty} u^{2p}v^2\frac{w_x}{w}\,dx,$$

and consequently

$$\left| \int_{-\infty}^{\infty} v^{2p+2} \frac{w_x}{w^{2p+1}} \, dx \right| \leqslant c \int_{-\infty}^{\infty} v^2 \, dx.$$

Combining the last few inequalities with the relation (3.6), it is deduced that

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} v_x^2 \, dx + 3 \int_{-\infty}^{\infty} \frac{w_x}{w} v_{xx}^2$$
$$\leqslant c \left\{ 1 + \int_{-\infty}^{\infty} v_x^2 \, dx \right\} + \frac{1}{(p+1)(p+2)} \frac{d}{dt} \int_{-\infty}^{\infty} \frac{v^{p+2}}{w^p} \, dx.$$

Integrating this latter inequality over the interval $[0, t]$ and estimating in a straightforward way gives a relation to which Gronwall's lemma applies, namely

$$\frac{1}{2} |v_x(\cdot, t)|_2^2 + 3 \int_0^t \int_{-\infty}^{\infty} \frac{w_x}{w} v_{xx}^2 \, dx \, ds$$
$$\leqslant c(t, \|\psi\|_{H^1(\mathbb{R}, w)}, w) \left\{ 1 + \int_0^t \int_{-\infty}^{\infty} v_x^2 \, dx \, ds \right\}.$$

Thus it is concluded that $v_x = (wu_j)_x$ is bounded in $L_\infty(0, T; L_2(\mathbb{R}))$, independently of j. Because of the relationship

$$wu_{jx} = (wu_j)_x - \frac{w_x}{w} wu_j$$

and the fact that wu_j is already known to be bounded in $L_\infty(0, T; L_2(\mathbb{R}))$, independently of j, it transpires that u_j is bounded in $L_\infty(0, T; H^1(\mathbb{R}, w))$, independently of j.

Attention is now given to the cases wherein $k \geqslant 2$ and p is arbitrary. Here the argument is relatively straightforward, and is made inductively, assuming at stage k that u_j is known to be bounded in $L_\infty(0, T; H^{k-1}(\mathbb{R}, w))$, independently of j. Abbreviate $\partial^k/\partial x^k$ by ∂_x^k, apply ∂_x^k to (3.2), and take the $L_2(\mathbb{R})$-inner product of the result with $\partial_x^k v$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} |\partial_x^k v|^2 \, dx - 3 \int_{-\infty}^{\infty} \partial_x^k \left(\frac{w_x}{w} v_{xx} \right) \partial_x^k v \, dx + \int_{-\infty}^{\infty} \partial_x^k (\theta_1 v) \partial_x^k v \, dx$$
$$+ \int_{-\infty}^{\infty} \partial_x^k (\theta_2 v_x) \partial_x^k v \, dx + \int_{-\infty}^{\infty} \partial_x^k \left(\frac{v^p v_x}{w^p} - \frac{v^{p+1} w_x}{w^{p+1}} \right) \partial_x^k v \, dx = 0. \qquad (3.7)$$

The properties of θ_1 and θ_2, the induction hypothesis, plus an integration by parts allows the conclusion that

$$\left|\int_{-\infty}^{\infty} \partial_x^k(\theta_1 v)\partial_x^k v\, dx\right| + \left|\int_{-\infty}^{\infty} \partial_x^k(\theta_2 v_x)\partial_x^k v\, dx\right| \leqslant c\left\{1+\int_{-\infty}^{\infty} |\partial_x^k v|^2\, dx\right\}.$$

By using Leibniz' rule, performing additional integrations by parts, and recalling that w_x/w and its derivatives are bounded, one finds that

$$-3\int_{-\infty}^{\infty} \partial_x^k\left(\frac{w_x}{w} v_{xx}\right)\partial_x^k v\, dx = 3\int_{-\infty}^{\infty} \frac{w_x}{w}(\partial_x^{k+1} v)^2\, dx + \text{lower order terms}.$$

It is easily shown using the induction hypothesis that

$$\text{lower order terms} \leqslant c\left\{1+\int_{-\infty}^{\infty} (\partial_x^k v)^2\right\}.$$

Consider now the last integral in (3.7) and notice that

$$\int_{-\infty}^{\infty} \partial_x^k\left(\frac{v^p v_x}{w^p}\right)\partial_x^k v\, dx = \int_{-\infty}^{\infty} \partial_x^k(u^p v_x)\partial_x^k v\, dx,$$

so therefore

$$\begin{aligned}
&\int_{-\infty}^{\infty} \partial_x^k\left(\frac{v^p v_x}{w^p}\right)\partial_x^k v\, dx \\
&= \int_{-\infty}^{\infty} u^p\, \partial_x^{k+1} v\, \partial_x^k v\, dx + p\int_{-\infty}^{\infty} u^{p-1}u_x(\partial_x^k v)^2\, dx \\
&\quad + \sum_{j=2}^{k}\binom{k}{j}\int_{-\infty}^{\infty} \partial_x^j(u^p)\partial_x^{k-j+1} v\, \partial_x^k v\, dx \\
&= \frac{p}{2}\int_{-\infty}^{\infty} u^{p-1}u_x(\partial_x^k v)^2\, dx + \sum_{j=2}^{k}\binom{k}{j}\int_{-\infty}^{\infty} \partial_x^j(u^p)\partial_x^{k-j+1} v\, \partial_x^k v\, dx.
\end{aligned}$$

Our induction hypothesis includes the fact that $\|\psi_j\|_k$ is bounded, independently of j, and so the theory in unweighted spaces implies $\partial_x^j u$ to be bounded in $L_\infty(0, T; H^1(\mathbb{R}))$, and so in $L_\infty(\mathbb{R}\times[0, T])$, independently of j, for $0 \leqslant j \leqslant k-1$. Therefore, since $k \geqslant 2$, it is concluded that

$$\left|\int_{-\infty}^{\infty} \partial_x^k\left(\frac{v^p v_x}{w^p}\right)\partial_x^k v\, dx\right| \leqslant c\left\{1+\int_{-\infty}^{\infty} (\partial_x^k v)^2\, dx\right\} + \left|\int_{-\infty}^{\infty} \partial_x^k(u^p)v_x\, \partial_x^k v\, dx\right|.$$

When $\partial_x^k(u^p)$ is developed into products of derivatives of u by Leibniz' formula, the only term not known to be bounded is $pu^{p-1}\,\partial_x^k u$. Thus

$$\left|\int_{-\infty}^{\infty} \partial_x^k(u^p) v_x\, \partial_x^k v\, dx\right|$$
$$\leqslant c\left\{1+\int_{-\infty}^{\infty} (\partial_x^k v)^2\, dx\right\} + p\left|\int_{-\infty}^{\infty} u^{p-1}\,\partial_x^k u v_x\, \partial_x^k v\, dx\right|$$
$$\leqslant c\left\{1+\int_{-\infty}^{\infty} (\partial_x^k v)^2\, dx\right\} + p\,|u|_\infty^{p-1}\,|v_x|_\infty\,|\partial_x^k u|_2\,|\partial_x^k v|_2$$
$$\leqslant c\left\{1+\int_{-\infty}^{\infty} (\partial_x^k v)^2\, dx\right\} + c\,|v_x|_2^{1/2}\,|v_{xx}|_2^{1/2}\,|\partial_x^k v|_2.$$

Since $k\geqslant 2$, $|v_x|_2$ is known to be bounded on $[0, T]$, and consequently it is adduced that

$$\left|\int_{-\infty}^{\infty} \partial_x^k\left(\frac{v^p v_x}{w^p}\right)\partial_x^k v\, dx\right| \leqslant c\left\{1+\int_{-\infty}^{\infty} (\partial_x^k v)^2\, dx\right\}.$$

The second part of the last integral in (3.7) also presents no difficulty if one first writes

$$\int_{-\infty}^{\infty} \partial_x^k\left(\frac{v^{p+1}w_x}{w^{p+1}}\right)\partial_x^k v\, dx = \int_{-\infty}^{\infty} \partial_x^k\left(vu^p\,\frac{w_x}{w}\right)\partial_x^k v\, dx,$$

and then makes use of the property that w_x/w and all of its derivatives are bounded, the induction hypothesis, and Leibniz' rule to conclude that the latter integral is bounded above by

$$c\left\{1+\int_{-\infty}^{\infty} (\partial_x^k v)^2\, dx\right\}.$$

Putting together the information just derived, one discovers that

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} |\partial_x^k v|^2\, dx \leqslant c\left\{1+\int_{-\infty}^{\infty} |\partial_x^k v|^2\, dx\right\},$$

from which it follows by Gronwall's lemma again that $|\partial_x^k v|_2$ is bounded for $0\leqslant t\leqslant T$, independently of j. Thus $\partial_x^k v = \partial_x^k(wu)$ is bounded in $L_\infty(0, T; L_2(\mathbb{R}))$, independently of j. But, it follows from Leibniz' rule that

$$\begin{aligned}
\partial_x^k(wu) &= w\,\partial_x^k u + \sum_{r=1}^{k} \binom{k}{r} \partial_x^r w\, \partial_x^{k-r} u \\
&= w\,\partial_x^k u + \sum_{r=1}^{k} \binom{k}{r} \partial_x^r w\, \partial_x^{k-r}\left(\frac{v}{w}\right) \\
&= w\,\partial_x^k u + \sum_{r=1}^{k} \binom{k}{r} \partial_x^r w \sum_{i=0}^{k-r} \binom{k-r}{i} \partial_x^i\left(\frac{1}{w}\right) \partial_x^{k-r-i} v \\
&= w\,\partial_x^k u + \sum_{r=1}^{k} \sum_{i=0}^{k-r} \binom{k}{r}\binom{k-r}{i} \frac{\partial_x^r w}{w}\left[w\,\partial_x^i\left(\frac{1}{w}\right)\right] \partial_x^{k-r-i} v.
\end{aligned}$$

Both $\partial_x^r w/w$ and $w\,\partial_x^i(1/w)$ are bounded functions if $r, i \geqslant 0$. Hence the induction hypothesis implies all the terms in the double summation to be bounded in $L_\infty(0, T; L_2(\mathbb{R}))$, independently of j. It thus follows that $w\,\partial_x^k u_j$ is bounded in $L_\infty(0, T; L_2(\mathbb{R}))$, independently of j.

Summarizing the accomplishments thus far, the following two propositions have been demonstrated.

(i) If $k=0$ and $p=0, 1, 2$, or 3, then for any $T>0$, u_j is bounded in $L_\infty(0, T; L_2(\mathbb{R}, w)) \cap L_2(0, T; H^1_{\text{loc}}(\mathbb{R}))$ with a bound which is independent of j. The same conclusion holds for $p=4$ provided that $\|\psi\|_0$ is not too large. If $k \geqslant 1$ and p is arbitrary, u_j is bounded in $L_\infty(0, T; L_2(\mathbb{R}, w)) \cap L_2(0, T; H^1_{\text{loc}}(\mathbb{R}))$ with a bound that is independent of j for any $T>0$ such that $\|u_j(\cdot, t)\|_1$ is bounded on $[0, T]$ with a j-independent bound. Thus, if $\|\psi\|_1$ is small enough, it follows from Theorem 2.1 that T is unrestricted in the last assertion.

(ii) If $k \geqslant 1$ and $p=0, 1, 2$, or 3, then for any $T>0$ u_j is bounded in $L_\infty(0, T; H^k(\mathbb{R}, w)) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$ with a bound that is independent of j. The same conclusion holds for $p \geqslant 4$ provided $T>0$ is such that $\|u_j(\cdot, t)\|_1$ is bounded on $[0, T]$ independently of j. Thus if $\|\psi\|_1$ is not too large, T is unrestricted.

From the information summarized in (i) and (ii) above, one may pass to the limit as $j \to \infty$ and obtain solutions as advertised in the statement of the theorem. In case $k=0$ or $k=1$, the limiting procedure uses weak-star compactness arguments as in Theorem 2.2, while if $k \geqslant 2$ the limit can be inferred to be strong by use of Kato's general semigroup theory [14, 15] or the special regularization techniques of Bona and Smith [7], Saut and Temam [29], or Temam [34] (see the Appendix for example). The details are not brief, but they are straightforward in light of the a priori bounds in hand, and so are omitted.

This concludes the proof of Theorem 3.1. ∎

An interesting corollary bearing upon the possibility of non-linear blowup emerges from the proof exhibited above. This result amounts to a strengthening of the theorem recorded in Albert *et al.* [2]. As mentioned before, their result states that, as regards the initial-value problem (2.1) where $p \geqslant 4$ and $k \geqslant 2$ and with initial data $\psi \in H^k(\mathbb{R})$, either the corresponding solution u exists for all time as an $H^k(\mathbb{R})$-soution of (2.1) or else there is a finite value $T^* > 0$ such that

$$\lim_{t \uparrow T^*} |u(\cdot, t)|_\infty = +\infty.$$

COROLLARY 3.2. *Let* $\psi \in H^k(\mathbb{R}, w)$ *where* $k \geqslant 2$ *and suppose the parameter* σ *associated with* w *to be non-negative. Let* u *be the solution of* (2.1) *associated with* ψ, *where* p *is any non-negative integer. Let* T^* *be the maximum time for which, for all* T *with* $0 < T < T^*$, u *lies in* $C(0, T; H^k(\mathbb{R}, w))$. *Then either* $T^* = +\infty$ *or*

$$\lim_{t \uparrow T^*} |u(\cdot, t)|_\infty = +\infty.$$

Proof. This theorem may be deduced by using the known results in unweighted Sobolev spaces along with a careful examination of the steps given in the proof of Theorem 3.1. Of course the theorem has additional content only if $p \geqslant 4$.

Suppose that u is uniformly bounded on any finite time interval. It will be shown in consequence of this assumption that u is bounded in $H^k(\mathbb{R}, w)$ uniformly on bounded time intervals. From this it is straightforward to deduce the desired result by iterating the local existence theory for (2.1).

Notice first that the crucial ingredient in the proof of Theorem 3.1 for the derivation of the a priori bound on the solution u in $L_2(\mathbb{R}, w)$ for $p \geqslant 4$ and data which was not necessarily small was the knowledge that $|u|_\infty$ was bounded (see (3.5)). Once a bound is inferred to exist in $L_2(\mathbb{R}, w)$, the further derivation of an a priori bound in $H^1(\mathbb{R}, w)$ also only makes essential use of bounds on $|u|_\infty$ in conjunction with the already derived bound in $L_2(\mathbb{R}, w)$ (see (3.4) ff.). For bounds on u in $H^k(\mathbb{R}, w)$ where $k \geqslant 2$, proceed inductively using (3.7). The main point to observe is that in the estimation of the last term on the right-hand side of (3.7), the bound in (3.8) is still effective because $|u|_\infty^{p-1}$ is bounded and, because of the result of Albert *et al.* [2], so is $|\partial_x^k u|_2$.

The corollary is thus established. ∎

Remark. Corollary 3.2 has an interesting interpretation as regards the possibility of solutions of GKdV equations forming singularities in the $H^k(\mathbb{R})$ norm in finite time. As is now apparent, a singularity will form in $H^k(\mathbb{R})$ if and only if it forms in $H^k(\mathbb{R}, w_\sigma)$ for any $\sigma \geqslant 0$ such that the initial data ψ lies in $H^k(\mathbb{R}, w_\sigma)$. Thus one may expect that non-linear blowup, if it occurs, does not subsist upon the bad behaviour of the data at infinity, but rather on the non-linear term locally overpowering dispersive effects.

The calculations leading to a priori bounds actually show more than was claimed in Theorem 3.1. In fact, one may deduce from the relations (3.3), (3.6), and (3.7) that on any time interval $[0, T]$ of existence of an $H^k(\mathbb{R}; w_\sigma)$-valued solution u of (2.1), the integral

$$\int_0^T \int_{-\infty}^{\infty} \frac{w_x}{w} (\partial_x^{k+1} u)^2 w^2 \, dx \, dt$$

is finite. A simple argument based on the translation invariance of the differential equation thus shows that

$$\int_0^T \int_{x_0}^{\infty} (x - x_0) w^2(x) (\partial_x^{k+1} u(x, t))^2 \, dx \, dt < +\infty$$

for any finite value x_0, a result reminiscent of some of those in [19]. It follows that for almost every $t \in [0, T]$

$$\int_{x_0}^{\infty} (x - x_0) w^2(x) (\partial_x^{k+1} u(x, t))^2 \, dx < +\infty.$$

4. Dispersive Blowup

The results concerning dispersive blowup are now stated and proved. The proofs involve relatively careful calculations of integrals of products of the Airy function

$$\mathrm{Ai}(\xi) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{1}{3} \theta^3 + \theta \xi\right) d\theta, \tag{4.1}$$

along with the existence theory in weighted spaces established in Section 3. The type of result that these considerations will yield is that for particular choices of the parameters p and k associated with the strength of the non-linearity and the smoothness of the initial data, respectively, there are choices of infinitely smooth initial data for which the kth derivative of the solution u corresponding to this data becomes unbounded in finite time. In fact, we shall even be able to provide initial data in $H^k(\mathbb{R}) \cap C^\infty(\mathbb{R})$ for which

$$\lim_{\substack{x \to x_n \\ t \to t_n}} |\partial_x^k u(x, t)| = +\infty$$

for $n = 1, 2, 3, \ldots$, where $\{(x_n, t_n)\}_{n=1}^{\infty}$ is a sequence of points in the upper half plane $\mathbb{R} \times \mathbb{R}^+ = \{(x, t) : x \in \mathbb{R}, t > 0\}$ without finite limit points.

Before proceeding to the precise statement and proof of the sort of theorem just outlined, some preliminary discussion and several preparatory lemmas are needed.

We begin by recalling some detailed asymptotic properties of the Airy function $\mathrm{Ai}(z)$ defined in (4.1) which will find use in our development. These aspects of Ai are all well known, and may be found explained in Olver's text [23] or in the recent book of Fedoriouk [11]. First, the Airy function of a real argument is a bounded, real-analytic function that tends to zero at $\pm\infty$. On the positive real axis it decreases monotonicaly and exponentially to zero, and in fact, the Airy function and its first derivative satisfy the inequalities

$$0 < \mathrm{Ai}(x) \leqslant \frac{1}{2\pi^{1/2}x^{1/4}} e^{-\xi}, \qquad 0 < -\mathrm{Ai}'(x) \leqslant \frac{x^{1/4}}{2\pi^{1/2}} e^{-\xi}\left(1+\frac{7}{72\xi}\right) \tag{4.2}$$

for $x \geqslant 0$, where $\xi = \frac{2}{3}x^{3/2}$. On the other hand, as x tends to $-\infty$, Ai only decreases algebraically, but it oscillates fiercely, having the form

$$\begin{aligned} \mathrm{Ai}(-x) &= \frac{1}{2\pi^{1/2}x^{1/4}} \cos\left(\xi - \frac{1}{4}\pi\right)\left(1 + O\left(\frac{1}{\xi}\right)\right) \quad \text{and} \\ \mathrm{Ai}'(-x) &= \frac{x^{1/4}}{2\pi^{1/2}} \sin\left(\xi - \frac{1}{4}\pi\right)\left(1 + O\left(\frac{1}{\xi}\right)\right) \end{aligned} \tag{4.3}$$

as $x \to +\infty$, where $\xi = \frac{2}{3}x^{3/2}$ as before. More generally, the nth derivative $\mathrm{Ai}^{(n)}$ of the Airy function has the asymptotic form

$$\begin{aligned} \mathrm{Ai}^{(n)}(x) &\sim c x^{n/2 - 1/4} e^{-\xi} \\ \mathrm{Ai}^{(n)}(-x) &\sim c x^{n/2 - 1/4} \sin(\xi + \varphi_n) \end{aligned} \tag{4.4}$$

as $x \to +\infty$, where ξ is as above, $c = 1/2\pi^{1/2}$, and $\varphi_n = -\frac{3}{4}\pi$ for n even and $\varphi_n = -\frac{1}{4}\pi$ for n odd. The relative error committed in using any of these asymptotic forms rather than the Airy function itself is of order $x^{-3/2}$ just as in (4.3). Even though the Airy function does not lie in $L_1(\mathbb{R})$ on account of its slow decay to zero at $-\infty$, it is nonetheless improperly integrable over $\mathbb{R}$ because of its rapid oscillation, with

$$\int_{-\infty}^{\infty} \mathrm{Ai}(z)\, dz = 1.$$

Finally, the Airy function has a Fourier transform at least within the class $\mathscr{S}'(\mathbb{R})$ of tempered distributions, and this transform is known explicitly to be

$$\mathscr{F}(\mathrm{Ai})(y) = e^{-iy^3}, \tag{4.5}$$

as follows more or less immediately from its definition in (4.1).

The relationship between the Airy function and the GKdV equation that is our primary focus of interest becomes apparent in formula (4.6) in the next lemma.

LEMMA 4.1. *Let $T>0$ be given and suppose that u is a weak solution of the GKdV equation corresponding to initial data $\psi \in L_2(\mathbb{R})$, constructed in Theorem 2.2 so that $u \in L_\infty(0, T; L_2(\mathbb{R}))$ and suppose in addition that $u^{p+1} \in L_\infty(0, T; L_1(\mathbb{R}))$. Then u satisfies the integral equation*

$$u(x, t) = \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) \psi(y)\, dy + \frac{1}{p+1} \int_0^t \int_{-\infty}^{\infty} \frac{1}{(t-s)^{2/3}} \mathrm{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^{p+1}(y, s)\, dy\, ds \tag{4.6}$$

for $(x, t) \in \mathbb{R} \times [0, T]$, where Ai *is the Airy function discussed above. This formula holds in the class of tempered distributions $\mathscr{S}'(\mathbb{R})$, in $L_\infty(0, T; L_2(\mathbb{R}))$, and pointwise almost everywhere.*

Proof. Because $u \in L_\infty(0, T; L_2(\mathbb{R}))$, then $u_{xxx} \in L_\infty(0, T; H^{-3}(\mathbb{R}))$. Moreover, $u^p u_x \in L_\infty(0, T; W_1^{-1}(\mathbb{R}))$, and so the differential equation implies that $u_t \in L_\infty(0, T; H^{-3}(\mathbb{R})) + W_1^{-1}(\mathbb{R})) \subsetneq L_\infty(0, T; \mathscr{S}'(\mathbb{R}))$. It follows that u lies in $C(0, T; \mathscr{S}'(\mathbb{R}))$ and thus the value of u at $t=0$ is assumed at least in $\mathscr{S}'(\mathbb{R})$. However, from our previous theory, we know that $u(\cdot, t) \to \psi$ as $t \to 0$ at least in $H_{\mathrm{loc}}^{-1/2}(\mathbb{R})$, and hence in $\mathscr{S}'(\mathbb{R})$. Consequently $u(\cdot, t) \to \psi$ as $t \to 0$ in $\mathscr{S}'(\mathbb{R})$.

Applying the Fourier transform in the spatial variable x to the GKdV equation, we obtain an $\mathscr{S}'(\mathbb{R})$-valued ordinary differential equation satified by the Fourier transform $\hat{u} \in C(0, T; \mathscr{S}'(\mathbb{R}))$ of u, namely

$$\frac{d}{dt} \hat{u} - i\xi^3 \hat{u} + \frac{i\xi}{p+1} u^{p+1} = 0, \qquad \hat{u}(\xi, 0) = \hat{\psi}(\xi).$$

Treating u^{p+1} as a forcing term, this equation is easily and uniquely solved by Duhamel's principle since both $e^{-i\xi^3 t}$ and $\xi e^{-i\xi^3 t}$ are multipliers in $\mathscr{S}'(\mathbb{R})$. One obtains that

$$\hat{u}(\xi, t) = e^{-i\xi^3 t} \hat{\psi}(\xi) - \frac{i}{p+1} \int_0^t e^{i\xi^3 (t-s)} \xi u^{p+1}\, ds, \tag{4.7}$$

which holds at least in $\mathscr{S}'(\mathbb{R})$, though in fact each term has a pointwise almost-everywhere meaning. Applying the inverse Fourier transform and using (4.5) leads to (4.6), where in the first instance the convolutions are interpreted as convolutions in the class $\mathscr{S}'(\mathbb{R})$ of tempered distributions. However, both the left-hand side and the first term on the right-hand side (see Lemma 4.3) lie in $L_\infty(0, T; L_2(\mathbb{R}))$, and hence so does the second integral in (4.6).

This concludes the discussion of the proof of Lemma 4.1. ∎

Having derived the integral equation (4.6), it is now natural to investigate properties of the two integrals on its right-hand side. The next two results report on estimates for these integrals that lie at the heart of our theory.

LEMMA 4.2. *Let $k \geqslant 0$ and $\psi \in H^k(\mathbb{R}; w_\sigma)$ where $\sigma \geqslant 1/16$ and the weight w_σ is as defined in (3.1). Let $T > 0$ and suppose that $u \in L_\infty(0, T; H^k(\mathbb{R}, w_\sigma)) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$ is the solution of the GKdV equation corresponding to the initial data ψ constructed in Theorem 3.1. If $k = 0$ and $p = 1$ or if $k \geqslant 1$ and p is arbitrary, then the integral*

$$\begin{aligned}\wedge(x, t) &= \wedge_p(x, t)\\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^{p+1}(y, s)\, dy\, ds \qquad (4.8)\end{aligned}$$

is k times differentiable with respect to x for (x, t) in the strip $\mathbb{R} \times (0, T)$ and $\partial_x^j \wedge(x, t)$ is a continuous function of both variables for $0 \leqslant j \leqslant k$.

Proof. Consider first the case $k = 0$ and $p = 1$. Break the integral into parts as follows:

$$\begin{aligned}\left|\wedge(x, t)\right| \leqslant &\left|\int_0^t \int_{-\infty}^{-m} \frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^2(y, s)\, dy\, ds\right|\\ &+ \left|\int_0^t \int_{-m}^{m} \frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^2(y, s)\, dy\, ds\right|\\ &+ \left|\int_0^t \int_{m}^{\infty} \frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^2(y, s)\, dy\, ds\right|. \qquad (4.9)\end{aligned}$$

From (4.2), one sees immediately that

$$\left|\frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right)\right| \leqslant \frac{C}{(t-s)^{2/3}}$$

for all $t \geqslant s$ and $x - y \geqslant 0$. Hence it transpires that

$$\begin{aligned}&\left|\int_0^t \int_{-\infty}^{x} \frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^2(y, s)\, dy\, ds\right|\\ &\quad \leqslant C \int_0^t \frac{1}{(t-s)^{2/3}} \int_{-\infty}^{x} u^2(y, s)\, dy\, ds \leqslant C T^{1/3} \|u\|^2_{L_\infty(0, T; L_2(\mathbb{R}))}, \qquad (4.10)\end{aligned}$$

and the latter combination is finite by assumption. For the second integral on the right side of (4.9), use is made of (4.3) in the following way:

$$\begin{aligned}
&\left|\int_0^t \int_x^\infty \frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^2(y,s)\, dy\, ds\right| \\
&\quad \leqslant C \int_0^t \int_x^\infty \frac{1}{(t-s)^{3/4}} (y-x)^{1/4} u^2(y,s)\, dy\, ds \\
&\quad = C \int_0^t \frac{1}{(t-s)^{3/4}} \int_x^\infty \frac{(y-x)^{1/4}}{w_\sigma^2(y)} w_\sigma^2(y)\, u^2(y,s)\, dy\, ds \\
&\quad \leqslant C \sup_{y \geqslant x} \left(\frac{(y-x)^{1/4}}{w_\sigma^2(y)}\right) \int_0^t \frac{1}{(t-s)^{3/4}} \int_{-\infty}^\infty w_\sigma^2(y)\, u^2(y,s)\, dy\, ds \\
&\quad \leqslant C T^{1/4} \sup_{y \geqslant x} \left(\frac{(y-x)^{1/4}}{w_\sigma^2(y)}\right) \|u\|^2_{L_\infty(0,\, T;\, L_2(w_\sigma))}. \qquad (4.11)
\end{aligned}$$

Notice that if $\sigma \geqslant 1/16$, then

$$\sup_{y \geqslant x} \frac{(y-x)^{1/4}}{w_\sigma^2(y)} \leqslant \begin{cases} C, & x \geqslant 0, \\ C\,|x|^{1/4}, & x \leqslant 0. \end{cases} \qquad (4.12)$$

Combining (4.11) and (4.10), we find that $\bigwedge_1$ is indeed locally bounded if $k=0$ and $p=1$, as claimed in the statement of the lemma. The continuity of $\bigwedge$ in this case follows since its defining integral has been shown above to converge uniformly for (x, t) in bounded subsets of $\mathbb{R} \times \mathbb{R}^+$.

We turn now to the case $k \geqslant 1$ and arbitrary $p \geqslant 1$. Since the result for general values of k follows from induction and the arguments that come to the fore for $k=1$, we shall concentrate on this case.

Because $u \in L_\infty(0, T; H^1(\mathbb{R}))$, it follows that u is uniformly bounded on $\mathbb{R} \times [0, T]$, say by a constant M. Hence it follows that

$$\left|\bigwedge\nolimits_p(x,t)\right| \leqslant M^{p-1} \int_0^t \int_{-\infty}^\infty \frac{1}{(t-s)^{2/3}} \left|\operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right)\right| u^2(y,s)\, dy\, ds,$$

and the argument used for bounding $\bigwedge_1$ in the case $k=0$ shows the right-hand side of this inequality to be locally bounded.

By standard results regarding convolution, the partial derivative of $\bigwedge_p$ with respect to x may be written in the form

$$\partial_x \bigwedge\nolimits_p(x,t) = (p+1) \int_0^t \int_{-\infty}^\infty \frac{1}{(t-s)^{2/3}} \operatorname{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^p(y,s) u_x(y,s)\, dy\, ds.$$

Bounding the right-hand side now proceeds as in the case $k=0$, $p=1$, except that $u^p u_x$ replaces u^2. Breaking the spatial integral again at $y=x$, we see that the integral over the interval $(-\infty, x)$ gives no problem. The remaining integral is treated as follows,

$$\begin{aligned}
&\int_0^t \int_x^\infty \frac{1}{(t-s)^{2/3}} \left| \mathrm{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^p(y,s) u_x(y,s) \right| dy\, ds \\
&\quad \leqslant \int_0^t \int_x^\infty \frac{1}{(t-s)^{3/4}} \frac{(y-x)^{1/4}}{w_\sigma^2(y)} w_\sigma^2(y)\, |u^p(y,s)\, u_x(y,s)|\, dy\, ds \\
&\quad \leqslant \sup_{y \geqslant x} \frac{(y-x)^{1/4}}{w_\sigma^2(y)} T^{1/4} M^{p-1} \int_{-\infty}^{\infty} w_\sigma^2(y)\, |u(y,s) u_x(y,s)|\, dy,
\end{aligned}$$

and because of (4.12), this last expression is seen to be locally bounded, uniformly for t in $[0, T]$. Again, continuity follows from the uniform convergence of the integral in (4.8) with respect to (x, t) provided (x, t) runs over a bounded subset of $\mathbb{R} \times \mathbb{R}^+$.

The case $k>1$ follows similar lines and so its detailed proof is not presented. ▮

Next, attention is fixed upon the first integral on the right-hand side of (4.6). We propose to give explicit initial data ψ so that its convolution with the Airy-function kernel develops particular singularities at a given point in space-time. Here is the result in view, the somewhat technical proof of which is presented in Appendix B.

LEMMA 4.3. *Let k be a non-negative integer and let*

$$\psi_k(y) = \frac{\mathrm{Ai}^{(k)}(-y)}{(1+y^2)^m}, \tag{4.13}$$

where m lies in the interval $1/8 + k/2 < m \leqslant 1/4 + k/2$. Then $\psi_k \in H^k(\mathbb{R}; w_\sigma) \cap C^\infty(\mathbb{R})$ for any $\sigma < m - k/2 - 1/8$, the function

$$\Psi_k(x, t) = \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) \psi_k(y)\, dy \tag{4.14}$$

lies in $C(\mathbb{R}^+; H^k(\mathbb{R}; w_\sigma))$, its kth derivative with respect to x is continuous everywhere except at the point $(0, 1)$, and at this point

$$\lim_{(x,t) \to (0,1)} \partial_x^k \Psi_k(x, t) = +\infty. \tag{4.15}$$

Moreover, $\Psi_k(x, t) \to \psi_k(x)$ *as* $t \to 0$, *in* $H^k(\mathbb{R}) \cap W^k_\infty(\mathbb{R})$, *and* $\partial_x^k \Psi_k$ *is uniformly bounded on any set of the form*

$$\{(x, t) : 0 \leqslant t \leqslant T, |x| + |t-1| \geqslant \delta\}$$

where $T < +\infty$ *and* $\delta > 0$.

A simple change of variable gives the following corollary to Lemma 4.3.

Corollary 4.4. *Let* k *be a non-negative integer and let* $x_* \in \mathbb{R}$ *and* $t_* > 0$ *be given. Define the function* $\psi(x; x_*, t_*, k)$ *by*

$$\psi(x; x_*, t_*, k) = \frac{\mathrm{Ai}^{(k)}(-\beta(x - x_*))}{[1 + \beta^2(x - x_*)^2]^m} = \psi_k(\beta(x - x_*)), \tag{4.16}$$

where $\beta = t_*^{-1/3}$. *If* m *satisfies the restriction* $1/8 + k/2 < m \leqslant 1/4 + k/2$ *of Lemma* 4.3, *then* $\psi \in H^1(\mathbb{R}; w_\sigma) \cap C^\infty(\mathbb{R})$ *for any* $\sigma < m - k/2 - 1/8$ *and the function*

$$\begin{aligned} \Psi(x, t; x_*, t_*, k) &= \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) \psi(y; x_*, t_*, k)\, dy \\ &= \Psi_k(\beta(x - x_*), t/t_*) \end{aligned} \tag{4.17}$$

lies in $C(\mathbb{R}^+; H^k(\mathbb{R}; w_\sigma))$ *and its* k*th derivative with respect to* x *is continuous everywhere except at the point* (x_*, t_*) *where*

$$\lim_{(x, t) \to (x_*, t_*)} \partial_x^k \Psi(x, t; x_*, t_*, k) = +\infty.$$

Moreover, $\Psi(x, t; x_*, t_*, k) \to \psi(x; x_*, t_*, k)$ *as* $t \to 0$ *in* $H^k(\mathbb{R}, w_\sigma) \cap W^k_\infty(\mathbb{R})$ *and* $\partial_x^k \Psi(x, t; x_*\ t_*, k)$ *is uniformly bounded on any set of the form*

$$\{(x, t) : 0 \leqslant t \leqslant Tt_*, \text{ and } |x - x_*| \geqslant \delta t_*^{1/3} \text{ or } |t - t_*| \geqslant \delta t_*\},$$

where T, δ *are positive.*

The preceding corollary may be extended in the following fairly obvious manner.

Lemma 4.5. *Let* k *be a non-negative integer and let* $\{(x_n, t_n)\}_{n=1}^{\infty}$ *be a sequence of points in* $\mathbb{R} \times (0, \infty)$ *without finite accumulation points and such that* $\{t_n\}_{n=1}^{\infty}$ *does not cluster at zero. Then there exists a function* $\psi \in H^k(\mathbb{R}; w_\sigma)$ *where* σ *is as in Lemma* 4.3 *such that the integral*

$$\Psi(x, t) = \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) \psi(y)\, dy \tag{4.18}$$

lies in $C(\mathbb{R}^+; H^k(\mathbb{R}; w_\sigma))$ *and its* k*th derivative* $\partial_x^k \Psi$ *is continuous everywhere in* $\mathbb{R}\times\mathbb{R}^+$ *except at the points* $\{(x_n, t_n)\}_{n=1}^\infty$ *where*

$$\lim_{(x,t)\to(x_n,t_n)} \partial_x^k \Psi(x,t) = +\infty, \qquad n=1,2,\ldots.$$

Proof. For short, we write

$$\psi_n(x) \qquad \text{for} \quad \psi(x; x_n, t_n, k),$$

for $n=1,2,\ldots$, and define ψ by the sum

$$\psi(x) = \sum_{n=1}^{\infty} \varepsilon_n \psi_n(x), \tag{4.19}$$

where the ε_n will be specified momentarily. Because of the last corollary, each $\psi_n \in H^k(\mathbb{R}; w_\sigma)$ for any $\sigma < m - k/2 - 1/8$ where m is as in Lemma 4.3. Hence, if we specify that $\varepsilon_n \leqslant 1/(2^n \|\psi_n\|_{H^k(\mathbb{R};w_\sigma)})$ for $n=1,2,\ldots$, then the series on the right-hand side of formula (4.19) converges in $H^k(\mathbb{R}; w_\sigma)$. If Ψ_n is defined to be

$$\Psi_n(x,t) = \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) \psi_n(y)\, dy,$$

then it follows that

$$\Psi(x,t) = \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) \psi(y)\, dy = \sum_{n=1}^{\infty} \varepsilon_n \Psi_n(x,t),$$

where the convergence of the series is at least in $C(0, T; H^k(\mathbb{R}; w_\sigma))$ for each $t>0$.

Attention is now turned to $\partial_x^k \Psi$ and its constituent parts $\partial_x^k \Psi_n$, which are denoted $\Psi^{(k)}$ and $\Psi_n^{(k)}$, respectively, $n = 1, 2, \ldots$. By the result of Corollary 4.4, for any positive constants T and δ, the function $\Psi_n^{(k)}$ is uniformly bounded on the set

$$A_n^{\delta,T} = \{(x,t): 0 \leqslant t \leqslant Tt_n,\ |x-x_n|\, t_n^{2/3} + |t-t_n| \geqslant \delta t_n\},$$

with a bound that depends on T and δ, but not on n. Consider a point $(x_0, t_0) \neq (x_n, t_n)$ for all $n=1,2,\ldots$ and a rectangle $R = \{(x,t): |x| \leqslant X$ and $0 \leqslant t \leqslant T\}$ such that $(x_0, t_0) \in R$. Consider an index n such that $(x_n, t_n) \notin \tilde{R}$ where $\tilde{R} = \{(x,t): |x| \leqslant 2X$ and $0 \leqslant t \leqslant 2T\}$. The quantities $\mu = \inf_n t_n$ and $\nu = \mathrm{distance}\{(x_0, t_0), \{(x_n, t_n)\}_{n=1}^\infty\}$ are strictly positive on account of the assumptions of the lemma. We claim that if T is chosen large enough and δ small enough, then there is an $r>0$ such that $\{(x,t)$:

$|(x, t)-(x_0, t_0)|<r\}$ lies in $A_n^{\delta, T}$ for any such index n. Indeed, if T is taken so that $T\mu > t_0$, then certainly $0 \leqslant t \leqslant Tt_n$ for all n and all t with $(x, t) \in B_r(x_0, t_0)$ for r small. Moreover, since $(x_n, t_n) \notin \tilde{R}$ and $(x_0, t_0) \in R$, it follows that either

$$|t_0 - t_n| \geqslant \tfrac{1}{2} t_n,$$

or else that

$$|x_0 - x_n|\, t_n^{2/3} \geqslant \frac{X}{2^{4/3} T^{1/3}}\, t_n.$$

By insisting that $\delta \leqslant \frac{1}{2} \min\{\frac{1}{2}, X/2^{4/3} T^{1/3}\}$, the stated conclusion is ensured for small enough values of r. Now let n be such that $(x_n, t_n) \in \tilde{R}$. Because there can be only finitely many such indices, we may revise the choice of δ if necessary, based on the positive constant ν, so that for small r, $B_r(x_0, t_0) \subset A_n^{\delta, T}$ for these values of n as well. Fixing positive values of T, δ, and r such that $B_r(x_0, t_0) \subset A_n^{\delta, T}$ for all n, we conclude that $\Psi_n^{(k)}$ is bounded and continuous on $B_r(x_0, t_0)$ with a bound that is independent of n. If it is also required that $\varepsilon_n \leqslant 2^{-n}$, for $n = 1, 2, \ldots$, then it follows that $\Psi^{(k)}$ is a bounded, continuous function on $B_r(x_0, t_0)$.

On the other hand, if we consider points (x, t) coming arbitrarily close to a particular point (x_m, t_m), then the same argument as presented above allows the conclusion that

$$\Psi^{(k)}(x, t) = \Psi_m^{(k)}(x, t) + \Psi_{(m)}^{(k)}(x, t),$$

where

$$\Psi_{(m)}^{(k)}(x, t) = \sum_{n \neq m} \varepsilon_n \Psi_n^{(k)}(x, t)$$

is a convergent sum of functions uniformly bounded in a neighborhood of (x_m, t_m). Consequently, $\Psi^{(k)}(x, t)$ has the same singularity as does $\Psi_m^{(k)}(x, t)$ as $(x, t) \to (x_m, t_m)$.

The conclusions of the lemma are thus verified. ∎

With these preliminary results in hand, the main results of the paper may be established.

THEOREM 4.6. *Let $T > 0$ be given and let $x_* \in \mathbb{R}$ and $t_* \in (0, T)$ be chosen. Suppose $k \geqslant 0$ and if $k = 0$, suppose $p = 1$, otherwise the positive integer p is unrestricted. There exists $\psi \in H^k(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap C_b^k(\mathbb{R})$ and a solution u of* (1.1) *corresponding to the initial value ψ such that*

(i) $u \in L_\infty(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$,

(ii) *$\partial_x^k u$ is continuous on $\mathbb{R} \times [0, T] \setminus \{(x_*, t_*)\}$, and*

(iii) $\lim_{x \to x_*, t \to t_*} \partial_x^k u(x, t) = +\infty$.

Remark. The symbol $C_b^k(\mathbb{R})$ connotes those C^k-functions defined on $\mathbb{R}$ which, along with their derivatives up to order k, are uniformly bounded.

Proof. Define the initial data ψ by

$$\psi(x) = \mu\psi(x; x_*, t_*, k), \tag{4.20}$$

where $\psi(x; x_*, t_*, k)$ is defined in (4.15), $\mu > 0$, and

$$\frac{1}{8} + \frac{k}{2} < m \leqslant \frac{1}{4} + \frac{k}{2}. \tag{4.21}$$

Because of the choice of m and the asymptotic properties of Ai in (4.4), it follows that $\psi \in H^k(\mathbb{R})$ and clearly $\psi \in C^\infty(\mathbb{R})$. Referring to Theorem 2.1 and Theorem 2.2, if $\mu > 0$ is chosen sufficiently small, it will follow that a solution of (1.1) with initial value ψ exists at least on the time interval $[0, T]$ and lies in

$$C(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$$

if $k \geqslant 2$ and in

$$L_\infty(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$$

if $k = 0$ or 1.

Our hypotheses on k and p are framed so that the representation (4.6) is valid for the solution u. By an application of Corollary 4.4 it is deduced that the linear term on the right-hand side of (4.6) has exactly the properties (i), (ii), and (iii).

On the other hand, if condition (4.21) on m is strengthened by th requirement that $3/16 + k/2 < m$, then it follows again from Corollary 4. that $\psi \in H^k(\mathbb{R}; w_\sigma)$ where $\sigma \geqslant 1/16$. In this case, Theorem 3.1 combine with Lemma 4.2 assures that the non-linear term $\wedge_p(x, t)$ on the righ hand side of (4.6) is bounded and continuous in $\mathbb{R} \times [0, T]$, along with i spatial derivatives of order at most k.

Combining the outcome of the last two paragraphs yields the desir state of affairs regarding u, and the proof of the theorem is complete. ∎

Corresponding to Corollary 4.5 of Lemma 4.4, Theorem 4.6 has associated corollary which is now stated and proved.

COROLLARY 4.7. *Let $T>0$ be given and let $\{(x_n, t_n)\}_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}\times(0, T)$ without finite limit points and such that $\{t_n\}_{n=1}^{\infty}$ is bounded below by a positive constant. Let either $k=0$ and $p=1$ or $k\geqslant 1$ and $p\geqslant 1$ be an arbitrary integer. Then there exists $\psi\in H^k(\mathbb{R})\cap C^\infty(\mathbb{R})$ and a solution u of (1.1) with initial value ψ such that*

(i) *u lies in $L_\infty(0, T; H^k(\mathbb{R}))\cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$, or in $C(0, T; H^k(\mathbb{R}))$ $\cap L_2(0, T; H^{k+1}_{\text{loc}}(\mathbb{R}))$, if $k\geqslant 2$,*

(ii) *$\partial_x^k u$ is continuous on $\mathbb{R}\times[0, T]\backslash\bigcup_{n=1}^{\infty}\{(x_n, t_n)\}$ and*

(iii) *$\lim_{(x,t)\to(x_n,t_n)} u(x, t) = +\infty$, for $n=1, 2, \ldots$.*

Proof. The proof follows the line put forth in the last theorem. First, appeal to Theorem 2.1 or Theorem 2.2 for the existence of a solution u, then to Lemma 4.1 for the validity of the representation (4.6) for u. Provided ψ is chosen as in Corollary 4.5 with $3/16+k/2<m\leqslant 1/4+k/2$, we may again rely on Theorem 3.1 and Lemma 4.1 to assure that the non-linear term $\bigwedge_p(x, t)$ in (4.6) is bounded and continuous in the relevant strip $\mathbb{R}\times[0, T]$. But, Corollary 4.5 shows that the linear term on the right-hand side of (4.6) has the properties (i), (ii), and (iii).

The corollary is thereby proved. ∎

Remark. It is worth noting that if ψ is initial data in H^k that leads to a solution u of the linearized initial-value problem which exhibits dispersive blowup in C^k, then $a\psi$ has the same property for any $a\neq 0$. This obvious point has the consequence that the initial data in H^k for which the associated solutions form singularities in the C^k-norm are dense. For consider an initial datum g such that the solution u of the linearized initial-value problem ((1.1) with $p=0$) lies in C^k for all t in the range $[0, T]$. Let $\varphi_k = g+(1/k)\psi$ and let u_k be the associated solution of the linearized initial-value problem. Then because the problem is linear, $u_k = u+(1/k)v$, this function exhibits dispersive blowup, and $u_k\to u$ in $C(0, T; H^k)$. A more complicated argument yields a similar result in the non-linear case, but this point will not be pursued here.

5. REMARKS AND EXTENSIONS

In this final section of the paper, we initiate some discussion based on the preceding results. First, it will be shown how our theory bears upon the well-known property that the KdV flow exchanges decay of the initial data for smootheness of the solution at later times. Second, we indicate an area of potential extension of the present results to more general dispersion relations than that envinced by the GKdV equations, by working out briefly the theory for the next order of approximation to the full dispersion relation corresponding to the two-dimensional Euler equations for surface waves in water at a critical value of the depth.

It has been known for some time (see, for example, Cohen [10], Kato [15], Sachs [27]) that the KdV flow exchanges decay and smoothness properties. For instance, if the initial data ψ belongs to $L_2(\mathbb{R})$ and decays exponentially to zero at $+\infty$, then the solution $u(\cdot, t)$ is C^∞ in the spatial variable (and hence in the temporal variable) for $t>0$. It is therefore of interest to look for the minimal decay assumption on $\psi \in L_2(\mathbb{R})$ which ensures that u is, say, a continuous function on $\mathbb{R} \times \mathbb{R}^+$. A result in this direction is given in the following theorem in which for the sake of simplicity, we deal only with the case $p=1$ of the standard KdV equation.

THEOREM 5.1. *Let $w = w_\sigma$ be defined as in* (3.1). *Then the following holds.*

(i) *If $\sigma > 1/8$ and $\psi \in L_2(\mathbb{R}; w_\sigma)$, every solution u of* (1.1) *with $p=1$ corresponding to ψ is continuous on $\mathbb{R} \times \mathbb{R}^+$.*

(ii) *If $1/16 < \sigma < 1/8$, then there exists $\psi \in L_2(\mathbb{R}, w_\sigma)$ and a corresponding solution u of* (3.1) *which is continuous for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ except at a given point (x_*, t_*), where $t_* > 0$.*

Proof. By Lemma 4.1 we may write

$$u(x, t) = \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right) \psi(y)\, dy + \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} \frac{1}{(t-s)^{2/3}} \mathrm{Ai}'\left(\frac{x-y}{(t-s)^{1/3}}\right) u^2(y, s)\, ds\, dy. \quad (5.1)$$

To understand the validity of assertion (i) it suffices to notice that, in consequence of the results in Section 3 and 4, the assumption $\psi \in L_2(\mathbb{R}, w_\sigma)$ with $\sigma > 1/8$ ensures that both terms in (5.1) are continuous in $\mathbb{R} \times \mathbb{R}^+$.

Part (ii) is a direct consequence of Theorem 4.1. In fact, if $1/16 \leqslant \sigma < 1/8$, one can always choose m with $3/16 < m < 1/4$ such that $(1+x^2)^\sigma \mathrm{Ai}(-x)/(1+x^2)^m \in L_2(\mathbb{R})$. Hence ψ defined by $\psi(x) = \mathrm{Ai}(-\beta(x-x_*))/(1+x^2)^m$, where $\beta = t_*^{-1/3}$, belongs to $L_2(\mathbb{R}, w_\sigma)$ for $1/16 < \sigma < 1/8$ and according to Lemma 4.2 one can control the nonlinear term in (5.1) while the linear term blows up at $(x, t) = (x_*, t_*)$. ∎

Remarks. (i) It should be noted that for $0 \leqslant \sigma \leqslant 1/16$ the linear portion of (5.1) may blow up in finite time (choosing $\psi(x) = \mathrm{Ai}(-x)/(1+x^2)^m$ with a suitable m), but in this regime we have not been able to control the nonlinear term. Thus, although the C^0 norm of the solution u given by (5.1) is likely to blow up, we have not thus far formed a proof that this is so.

(ii) Consider now Eq. (1.1) with $p \geqslant 4$, the case where non-linear blowup is likely to occur. Let $\psi \in H^1(\mathbb{R}, w_\sigma)$ be the initial data. Then an argument similar to that given above shows that, if $\sigma > 1/8$, any solution u corresponding to ψ will be C^1 in x at least in the entire temporal interval wherein u lies in $H^1(\mathbb{R})$.

(iii) A proof similar to that of Theorem 5.1 shows that if $\psi \in H^k(\mathbb{R}, w_\sigma)$ with $\sigma > 1/8$, then every solution u of the KdV equation corresponding to ψ is C^k in space for every $t > 0$.

So far we have considered equations having a rather specific dispersion relation, namely (1.2). It would be interesting to deal with the more general dispersion relation

$$\omega = \omega(k) = kP(k) \tag{5.2}$$

where P is some polynomial of even degree. For instance $P(k) = 1 - k^2 + k^4$ corresponds to the next order approximation of the linearized dispersion relation for the full system of equations for surface water waves from which the KdV equation is derived.

The very same phenomena we investigate in this paper hold true when the dispersion is given by (5.2). We will exemplify this by considering the initial-value problem for a fifth-order equation,

$$u_t + u^p u_x + u_{xxxxx} = 0, \tag{5.3}$$

where again p is a positive integer. When $p = 1$, this equation arises as the approximation to small-amplitude, long waves on the surface of shallow water having the critical depth 0.54 cm. (At this particular depth, the quadratic term in the Taylor expansion $k = 0$ of the dispersion relation has a zero coefficient owing to the compensatory effects of surface tension. Because of this, the first, non-trivial effect of dispersion is felt at a higher order than appears in the KdV model.)

The initial-value problem for this equation is always locally well posed, and it is globally well posed provided $p < 8$, as the following theorem asserts.

THEOREM 5.2. *Let initial data* $\psi \in H^k(\mathbb{R})$ *be specified for* Eq. (5.3). *Then the following is true.*

(i) *If* $k = 0$ *and* $p < 8$, *there exists a solution* u *of* (5.3) *with initial data* ψ *which, for any positive* T *and* R, *lies in* $L_\infty(\mathbb{R}^+; L_2(\mathbb{R})) \cap L_2(0, T; H^2([-R, R]))$.

(ii) *If* $k \geqslant 2$ *then there exists a unique solution* u *of* (5.3) *corresponding to* ψ *and a positive, possibly infinite* $T^* = T^*(\psi)$ *such that for all* T *with* $0 < T < T^*$ *and all* $R > 0$, $u \in C(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+2}([-R, R]))$.

Moreover, for each positive value of R and $T<T^*$, *the correspondence* $\psi \mapsto u$ *is continuous from* $H^k(\mathbb{R})$ *into* $C(0, T; H^k(\mathbb{R})) \cap L_2(0, T; H^{k+2}([-R, R]))$. *If* $k-5j>-5$, *the solution u also has the property that for the same range of T and* $R>0$, $\partial_t^j u \in C(0, T; H^{k-5j}(\mathbb{R})) \cap L_2(0, T; H^{k+2-5j}([-R, R]))$ *and the continuity of the map* $\psi \mapsto u$ *extends to these function classes as well.*

(iii) *In* (ii), *if* $p<8$, *then* $T^*=+\infty$. *If* $p=8$ *and* $\|\psi\|_0$ *is sufficiently small, then* $T^*=+\infty$. *If* $p>8$ *and* $\|\psi\|_2$ *is sufficiently small, then* $T^*=+\infty$. *In each of these cases,* $\|u(\cdot, t)\|_2$ *is bounded in terms of* $\|\psi\|_2$, *independently of t.*

Proof. The proof follows standard lines except for the smoothing aspect. We content ourselves with a derivation of the a priori bound needed for deducing the smoothing result for the case $k=0$ and $p<8$.

Let r be the weight function used in Section 2. We will need r to possess the additional property

$$|r'''(x)| \leqslant c r'(x) \tag{5.4}$$

for all $x \in \mathbb{R}$, where c is some fixed constant. For sufficiently small $\varepsilon>0$, the function $r(x)=1+\tanh(\varepsilon x)$ has all the desired properties. We multiply (5.3) by ru and perform several integrations by parts to get the equation

$$\begin{aligned}\frac{5}{2}\int_{-\infty}^{\infty} r_x u_{xx}^2\,dx &= \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} ru^2\,dx + \frac{5}{2}\int_{-\infty}^{\infty} r''' u_x^2\,dx \\ &\quad -\frac{1}{2}\int_{-\infty}^{\infty} r^{(5)}u^2\,dx - \frac{1}{p+2}\int_{-\infty}^{\infty} r_x u^{p+2}\,dx. \end{aligned}\tag{5.5}$$

Using (5.4) and the elementary inequality

$$\int_{-\infty}^{\infty} r_x u_x^2\,dx \leqslant c\int_{-\infty}^{\infty} r_x u^2\,dx + \left(\int_{-\infty}^{\infty} r_x u^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty} r_x u_{xx}^2\,dx\right)^{1/2}, \tag{5.6}$$

the second term on the right-hand side of (5.5) is majorized by

$$\tfrac{1}{2}\int r_x u_{xx}^2\,dx + C\int r_x u^2\,dx$$

for a sufficiently large constant C.

The last term in (5.5) can be handled as in the proof of Theorem 2.2 part (i), using (5.6).

Finally we integrate (5.5) in t and get, by making suitable choices of r and using the identity $\|u(\cdot, t)\|_0 = \|\psi\|_0$, that for arbitrary $R > 0$ and $T > 0$

$$\int_0^T \int_{-R}^R u_{xx}^2 \, dx \, dt \leqslant C(R; \|\psi\|_0). \quad \blacksquare$$

Remark. It should be acknowledged that the two invariants

$$I(u) = \int_{-\infty}^{\infty} u^2(x, t) \, dx$$

and

$$J(u) = \int_{-\infty}^{\infty} \left[u_{xx}^2(x, t) + \frac{2}{(p+1)(p+2)} u^{p+2}(x, t) \right] dx$$

play an important role in the proof of Theorem 5.2. As is easily deduced, if u is a smooth solution of (5.3) that decays to zero at infinity, along with its first few derivatives, then the functionals I and J are independent of t.

Remark. We digress for a moment to show how the last theorem and the following simple calculation serves to cast light on a problem considered by Yoshimura and Watanabe [35]. In their paper, attention was given to Eq. (5.3) with $p = 1$. A numerical scheme for the periodic initial-value problem was proposed and used to investigate properties of solutions. A number of interesting things were observed, but one aspect to their study gives credence to a conclusion which is demonstrably false.

Yoshimura and Watanabe claim to discover chaos in this initial-value problem, and present the outcome of a number of numerical simulations as evidence. One way they attempt to demonstrate the existence of chaos is by showing that the distance between two trajectories started from slightly different initial data grows exponentially. Indeed, they present graphs of the L_∞-distance between two trajectories started from data which is exactly the same save for slightly different amplitudes, and this quantity appears to grow exponentially.

As we will now show, this is impossible for the true solutions of the differential equation, and leads us to the tentative conclusion that what one is witnessing is an instability in the numerical scheme. Indeed, Eq. (5.3) is notoriously difficult to integrate accurately with an explicit scheme such as that outlined in [35], the reason being that one usually encounters a von Neumann stability condition that demands $\Delta t/(\Delta x)^5$ be not too large. In consequence, if Δx is small enough to resolve the spatial structure, then Δt must be very small indeed to maintain stability.

Here is the reason exponential growth between the difference in two solutions is impossible. Let u and v be solutions of (5.3) with $p=1$ corresponding to initial data g and h, respectively. Here, g, h, u, and v are all periodic of period L, say, in the spatial variable x, though the same result holds for the problem posed on the entire real line. By the periodic analog of Theorem 5.2, we know that if $g, h \in H^2_{\text{per}}(0, L)$ (for example, if g, h are C^2 functions which are periodic of period L), then for all t,

$$\|u(\cdot, t)\|_{H^2(0,L)} \leqslant C, \qquad \|v(\cdot, t)\|_{H^2(0,L)} \leqslant C,$$

where the constant C depends only on the H^2-norm of the initial data g and h. If $w = u - v$ is the difference, then w satisfies the initial-value problem

$$w_t + \tfrac{1}{2}[(u+v)w]_x + w_{xxxxx} = 0, \qquad w(x, 0) = g(x) - h(x), \tag{5.7}$$

and if $W = u + v$, then because of the just-mentioned H^2-bounds and the triangle inequality,

$$\|W(\cdot, t)\|_{H^2(0,L)} \leqslant 2C, \qquad \|w(\cdot, t)\|_{H^2(0,L)} \leqslant 2C$$

for all $t \geqslant 0$. If (5.7) is multiplied by w and integrated over $[0, L]$, then after integrations by parts, there appears the relation

$$\frac{1}{2}\frac{d}{dt}\int_0^L w^2(x, t)\, dx = \frac{1}{4}\int_0^L W_x(x, t)\, w^2(x, t)\, dx.$$

The right-hand side of the last equation is bounded above by $2C^3$. Hence it follows that if $C_0 = 4C^3$, then

$$\int_0^L w^2(x, t)\, dx \leqslant \int_0^L (g(x) - h(x))^2\, dx + C_0 t,$$

thus demonstrating that the L_2-norm of the difference grows at most linearly in time. A similar energy estimate for higher L_2-based norms leads to the same conclusion for the seminorms

$$\int_0^L w_x^2(x, t)\, dx \leqslant \int_0^L (g'(x) - h'(x))^2\, dx + C_1 t$$

and

$$\int_0^L w_{xx}^2(x, t)\, dx \leqslant \int_0^L (g''(x) - h''(x))^2\, dx + C_2 t.$$

This in turn implies by interpolation that

$$\sup_{0 \leqslant x \leqslant L} |u(x, t) - v(x, t)| \leqslant A + Bt,$$

where B is a constant that depends only on g and h, and A depends only on $g-h$.

We now return to the main line of development and prove a blow-up result similar to that contained in Theorem 4.1.

THEOREM 5.3. *Let $p=1$ and x_* and $t_*>0$ be given. Then there exists $\psi \in L^2(\mathbb{R}) \cap C_b(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and a solution u of* (5.3) *in $L_\infty(0, T; L_2(\mathbb{R})) \cap L_2(0, T; H^2_{\text{loc}}(\mathbb{R}))$, where $T>t_*$, corresponding to the initial data ψ, such that u is continuous on $\mathbb{R} \times (0, T) \setminus \{(x_*, t_*)\}$ and*

$$\lim_{(x,t) \to (x_*, t_*)} |u(x, t)| = +\infty.$$

Proof. The first step is to show that one can construct initial data which leads to the dispersive blowup of the C^0-norm of the solution of the linearized initial-value problem

$$u_t + u_{xxxxx} = 0, \qquad u(x, 0) = \psi(x). \tag{5.8}$$

The fundamental solution of (5.8) is

$$A(x, t) = \frac{1}{t^{1/5}} B\left(\frac{x}{t^{1/5}}\right),$$

where $B(z)$ is a smooth (analytic) function which decays exponentially when $z \to +\infty$ and which decays like $(-z)^{-3/8}$ as $z \to -\infty$ (see Sidi *et al.* [31]).

Let ψ be defined by

$$\psi(x) = \frac{B(-x)}{(1+x^2)^m}, \tag{5.9}$$

where $1/16 < m \leqslant 1/8$. Then $\psi \in L_2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and it is easy to show, following the lines of Benjamin *et al.* [4] and Appendix B that the C^0-norm of the solution u corresponding to ψ blows up exactly at $t=1$ ($u(\cdot, 1)$ has a singularity at $x=0$).

The rest of the proof of Theorem 5.3 is similar to that of Theorem 4.1, part (a). We establish an existence theorem in a weighted L_2-space similar to Theorem 3.1. The result is stated in the following lemma whose proof differs from that of the case $k=0$ in Theorem 3.1 only in technical details.

LEMMA 5.4. *Let w be the weight defined in* (3.1) *and let $p<8$ be given. Suppose the initial data ψ in* (5.3) *to lie in $L_2(\mathbb{R}, w)$. Then there exists a solution u of* (5.3) *corresponding to the initial data ψ such that, for any $T>0$ u belongs to $L_\infty(0, T; L_2(\mathbb{R}, w)) \cap L^2(0, T; H^2_{\text{loc}}(\mathbb{R}))$.*

We are now in position to conclude the proof of Theorem 5.3. Let ψ be defined as in (5.8), where $1/16 < m < 1/8$. Then $\psi \in L_2(\mathbb{R}; w_\sigma)$ for any σ such that $2m - \sigma > -1/8$. Let u be the solution corresponding to ψ given by Lemma 5.2. We present u in the following way by using Duhamel's formula:

$$u(x, t) = \frac{1}{t^{1/5}} \int_{-\infty}^{\infty} B\left(\frac{x-y}{t^{1/5}}\right) \psi(y)\, dy - \int_0^t \int_{-\infty}^{\infty} \frac{1}{(t-s)^{1/5}} B\left(\frac{x-y}{(t-s)^{1/5}}\right) uu_x(y, s)\, dy\, ds. \tag{5.10}$$

Because of the choice of m, the linear term in (5.9) blows up at $x = 0$, $t = 1$. On the other hand, an analysis similar to that used in proving Lemma 4.2 shows that we can keep control of the C^0-norm of the non-linear term on the right-hand side of (5.9). In this, use is made of the behavior of B' at $-\infty$, namely

$$|B'(-x)| = O(x^{-1/8}) \qquad \text{as} \quad x \to +\infty.$$

and of B'' at $-\infty$,

$$|B''(-x)| = O(x^{1/8}) \qquad \text{as} \quad x \to +\infty.$$

In fact the C^0-norm of the non-linear term is controlled assuming only $u \in L_\infty(\mathbb{R}^+; L_2(\mathbb{R}))$. The C^1 norm of the non-linear term is controlled provided m is restricted more severly so that $3/32 < m < 1/8$.

Remark. We expect similar results regarding dispersive blowup to hold even if the dispersion relation P in (5.2) is not a polynomial. Indeed, such non-polynomial dispersion relations arise frequently in practice (cf. [4, 28]). In case the symbol P is homogeneous, there is a strong indication that the above analysis can be carried through more or less intact. However, if P is not homogeneous, extra difficulties arise. In any case, the existence theory in weighted spaces along the lines spelled out in Section 3 is considerably more challenging. These issues will be the subject of a subsequent paper where, in addition, the results of the present paper will be related to the fact that strongly dispersive equations like the GKdV equation are ill-posed in L_p-spaces.

APPENDIX A

The continuity with respect to variations of the initial data in H^k of solutions of the GKdV equation in $L_2(0, T; H^{k+1}_{\text{loc}})$ is established here. We will need the approximation scheme for solutions of the Korteweg–de Vries equation used by Bona and Smith [7]. Recall briefly that for any positive value of ε, they define a special $H^\infty(\mathbb{R})$-approximation ψ_ε associated to initial data $\psi \in H^k(\mathbb{R})$ such that

$$\|\psi - \psi_\varepsilon\|_r = o(\varepsilon^{k-r}), \qquad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad r \leqslant k, \tag{A.1}$$

$$\|\psi_\varepsilon\|_r = O(\varepsilon^{k-r}), \qquad \text{as} \quad \varepsilon \to 0 \quad \text{if} \quad r > k. \tag{A.2}$$

Moreover the convergence of (A.1) is uniform on compact subsets of $H^k(\mathbb{R})$ while that in (A.2) is uniform on bounded subsets of $H^k(\mathbb{R})$. Using these special approximations, it has been shown in [7] that if u_ε connotes the smooth solution of (1.1) corresponding to ψ_ε, then for any $T > 0$

$$\|u(\cdot, t) - u_\varepsilon(\cdot, t)\|_{C(0,T;H^r)} = o(\varepsilon^{(k-r)/6}) \tag{A.3}$$

as $\varepsilon \to 0$, uniformly on compact subsets of $H^k(\mathbb{R})$, if $r \leqslant k$, and

$$\|u_\varepsilon(\cdot, t)\|_{C(0,T;H^r)} = O(\varepsilon^{(k-r)/6}) \tag{A.4}$$

for any r, as $\varepsilon \to 0$. (Actually these results have been established in [7] only for the KdV equation, but the proof given there can be easily extended to the generalized equation (1.1) for any $T < T^*$ where $(0, T^*)$ is the maximal interval of existence of the particular solution u.)

We can now prove the continuity result advertised in Theorem 2.1.

THEOREM A.1. *For a fixed value of $k \geqslant 2$, the mapping that associates to $f \in H^k(\mathbb{R})$ the solution u of* (1.1) *with initial data f is, for arbitrary $T < T^*$, continuous from $H^k(\mathbb{R})$ to $L_2(0, T; H^{k+1}_{\mathrm{loc}}(\mathbb{R}))$.*

Proof. For the sake of simplicity the proof will be given only for the case $p = 1$. The proof in the general case is very similar.

Let $\{f_n\}_{n=1}^\infty$ be a sequence from $H^k(\mathbb{R})$ and suppose that $f_n \to f$ in $H^k(\mathbb{R})$. Let u_n and u be the solution of (1.1) with initial data f_n and f, respectively, $n = 1, 2, \ldots$. Let T and R be fixed and positive. It will be demonstrated that $u_n \to u$ in $L_2(0, T; H^{k+1}(-R, R))$, a result that implies the desired conclusion since T and R are arbitrary.

Let $p(x)$ be a smooth, increasing, bounded, real-valued function of a real variable such that the derivatives of p are all bounded and such that $p'(x) \geqslant 1$ for $-R \leqslant x \leqslant R$, say.

We will use the aforementioned approximation scheme for solutions of the Korteweg–de Vries equation.

It is first established that if v is an $H^k(\mathbb{R})$-solution of (1.1) corresponding to the initial data ψ and v_ε is the solution corresponding to ψ_ε, then

$$\int_0^T \int_{-R}^R |\partial_x^{k+1}[v(x, t) - v_\varepsilon(x, t)]|^2 \, dx \, dt$$

tends to zero as $\varepsilon \to 0$, uniformly on compact subsets of $H^k(\mathbb{R})$. To this end, let $W = v - v_\varepsilon$, so that W satisfies the initial-value problem

$$W_t + WW_x + (v_\varepsilon W)_x + W_{xxx} = 0, \qquad W(x, 0) = \psi(x) - \psi_\varepsilon(x). \quad \text{(A.5)}$$

Differentiating Eq. (A.5) k times with respect to x, multiplying the result by $p\,\partial_x^k W$, integrating over $\mathbb{R} \times [0, T]$, and integrating by parts several times, there appears the relation

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty} p\,|\partial_x^k W|^2\,dx - \frac{1}{2}\int_{-\infty}^{\infty} p\,|\partial_x^k(\psi - \psi_\varepsilon)|^2\,dx \\
&\quad + \frac{3}{2}\int_0^T\int_{-\infty}^{\infty} p_x |\partial_x^{k+1} W|^2\,dx\,dt - \frac{1}{2}\int_0^T\int_{-\infty}^{\infty} p_{xxx} |\partial_x^k W|^2\,dx\,dt \\
&\quad + \left(k - \frac{1}{2}\right)\int_0^T\int_{-\infty}^{\infty} p\,|\partial_x^k W|^2\,W_x\,dx\,dt - \frac{1}{2}\int_0^T\int_{-\infty}^{\infty} p_x W |\partial_x^k W|^2\,dx\,dt \\
&\quad + \sum_{j=1}^{k-1}\binom{k}{j}\int_0^T\int_{-\infty}^{\infty} p\,\partial_x^k W\,\partial_x^{k+1-j} W\,dx\,dt \\
&\quad + \int_0^T\int_{-\infty}^{\infty} pW\,\partial_x^{k+1} v_\varepsilon\,\partial_x^k W\,dx\,dt + \int_0^T\int_{-\infty}^{\infty} p v_\varepsilon\,\partial_x^{k+1} v_\varepsilon\,\partial_x^k W\,dx\,dt \\
&\quad + \sum_{j=1}^{k}\binom{k}{j}\int_0^T\int_{-\infty}^{\infty} p\,\partial_x^k W[\partial_x^{k+1-j} v_\varepsilon\,\partial_x^j W + \partial_x^j v_\varepsilon\,\partial_x^{k+1-j} W]\,dx\,dt \\
&= 0. \qquad \text{(A.6)}
\end{aligned}$$

Rearranging terms and estimating in a straightforward way, we come to the inequality

$$\begin{aligned}
&\int_0^T\int_{-R}^{R} |\partial_x^{k+1} W|^2\,dx\,dt \\
&\quad \leqslant \int_0^T\int_{-\infty}^{\infty} p_x |\partial_x^{k+1} W|^2\,dx\,dt \\
&\quad \leqslant T\|\psi - \psi_\varepsilon\|_k^2 + c(T; \|W\|_{C(0,T;H^2)})\,\|W\|^2_{L_\infty(0,T;H^k)} + o(1), \quad \text{(A.7)}
\end{aligned}$$

where the first constant depends only on p while the second constant depends on T and W in the way indicated. The $o(1)$ term in (A.7) appears from estimating the terms in (A.6) using (A.3) and (A.4). Thus one concludes that

$$\int_R^T\int_{-R}^{R} |\partial_x^{k+1} W|^2\,dx\,dt = o(1) \qquad \text{(A.8)}$$

as $\varepsilon \to 0$, and that this relation holds uniformly on compact subsets of $H^k(\mathbb{R})$.

With this latter result in hand, the proof of Theorem A.1 is now straightforward if we argue in the following manner. First, note that

$$\int_0^T \int_{-R}^R |\partial_x^{k+1}(u_n - u)|^2 \, dx \, dt \leqslant 2 \int_0^R \int_{-R}^R |\partial_x^{k+1}(u_n - u_{n\varepsilon})|^2 \, dx \, dt$$
$$+ 2 \int_0^T \int_{-R}^R |\partial_x^{k+1}(u_{n\varepsilon} - u_\varepsilon)|^2 \, dx \, dt$$
$$+ 2 \int_0^T \int_{-R}^R |\partial_x^{k+1}(u_\varepsilon - u)|^2 \, dx \, dt. \quad \text{(A.9)}$$

Since $\{f_n\}_{n=1}^\infty \cup \{f\}$ comprises a compact subset of $H^k(\mathbb{R})$, it follows that the first and third term on the right-hand side of (A.9) are $o(1)$, as $\varepsilon \to 0$, uniformly in n. Hence, given $\gamma > 0$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_0$, then

$$\int_0^T \int_{-R}^R |\partial_x^{k+1}(u_n - u)|^2 \, dx \, dt \leqslant \gamma + \int_0^T \int_{-R}^R |\partial_x^{k+1}(u_{n\varepsilon} - u_\varepsilon)|^2 \, dx \, dt,$$

and this holds for all n. Fix a positive value of $\varepsilon \leqslant \varepsilon_0$, say. For this fixed ε, $\psi_{n\varepsilon} \to \psi_\varepsilon$ as $n \to \infty$ in $H^{k+1}(\mathbb{R})$. Hence $u_{n\varepsilon} \to u_\varepsilon$ in $C(0, T; H^{k+1}(\mathbb{R}))$. Thus it transpires that

$$\limsup_{n \to \infty} \int_0^T \int_{-R}^R |\partial_x^{k+1}(u_n - u)|^2 \, dx \, dt \leqslant \gamma,$$

and since $\gamma > 0$ was arbitrary, the advertised result follows. ∎

Appendix B

In this appendix we will be concerned with some detailed properties of the solution of the linearized KdV equation when the initial data has the special form studied in Section 4. As spelled out there, interest will therefore be focused upon the integral

$$J(x, x_0, t, \beta) = \frac{1}{t^{1/3}} \int_{-\infty}^\infty \mathrm{Ai}\left(\frac{x - x_0 - y}{t^{1/3}}\right) \mathrm{Ai}(-\beta y)(1 + \beta^2 y^2)^{-m} \, dy. \quad \text{(B1)}$$

A simple change of variables removes two of the four parameters appearing in the definition of J, and we are then left to analyze the integral

$$I(x, s) = \frac{1}{s} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{s}\right) \mathrm{Ai}(-y)(1+y^2)^{-m}\, dy \tag{B.2}$$

where $s = t^{1/3}$. The result in view is the conclusion of Lemma 4.3, namely that for suitable values of the parameter m, the integral I is a C^k-function of x at all points (x, t) in the upper half-plane $\mathbb{R} \times \mathbb{R}^+$ except at the point $(\bar{x}, \bar{s}) = (0, 1)$ where the kth derivative of I with respect to x does not exist, and in fact the limit of $\partial_x^k I(x, s)$ as (x, s) tends to $(0, 1)$ is infinite.

These facts about I are established next. In the exposition to follow, we shall concentrate on the special case $k = 0$ where the ideas are most transparent. Once the method of proof is appreciated, the case of general k present no extra difficulty in principle.

Proof (of Lemma 4.3). As already mentioned, we consider in detail the representative case $k = 0$. A few remarks about the general case appear at the end of the proof.

Because of (4.2), it follows that the Airy function lies in $L_1([k, \infty))$ for any finite value of k. In particular, the integral

$$\frac{1}{s} \int_{-\infty}^{x} \mathrm{Ai}\left(\frac{x-y}{s}\right) \mathrm{Ai}(-y)(1+y^2)^{-m}\, dy$$
$$= \int_{0}^{\infty} \mathrm{Ai}(z)\, \mathrm{Ai}(sz - x)(1 + (sz-x)^2)^{-m}\, dz$$

converges absolutely and uniformly in the upper half plane $\mathbb{R} \times \mathbb{R}^+ = \{(x, s) \colon x \in \mathbb{R}, s \geqslant 0\}$, and represents a bounded, continuous function there. Note also that

$$\frac{1}{s} \int_{x}^{x+s} \mathrm{Ai}\left(\frac{x-y}{s}\right) \mathrm{Ai}(-y)(1+y^2)^{-m}\, dy$$
$$= \int_{0}^{1} \mathrm{Ai}(-z)\, \mathrm{Ai}(-sz - x)(1 + (sz+x)^2)^{-m}\, dz$$

is also a bounded, continuous function of $(x, s) \in \mathbb{R} \times \mathbb{R}^+$. It is thus left to consider the integral in (B.2) where the range of integration is $[x+s, \infty)$ rather than the entire real line.

Using the relation (4.3) one may write

$$\frac{1}{s}\int_{x+s}^{\infty} \mathrm{Ai}\left(\frac{x-y}{s}\right) \mathrm{Ai}(-y)(1+y^2)^{-m}\,dy$$
$$= \frac{1}{2\pi^{1/2}s^{3/4}}\int_{x+s}^{\infty} \cos\left(\frac{2}{3}\left(\frac{y-x}{s}\right)^{3/2}-\frac{\pi}{4}\right)$$
$$\times (y-x)^{1/4}\,\mathrm{Ai}(-y)(1+y^2)^{-m}\left[1+O\left(\frac{s^{3/2}}{1+(x-y)^{3/2}}\right)\right]dy.$$

The term on the right-hand side containing $O(s^{3/2}/(1+(x-y)^{3/2}))$ clearly converges absolutely and represents a continuous function of (x, s) in $\mathbb{R}\times\mathbb{R}^+$ which is uniformly bounded on any bounded time interval. The remaining term may be integrated by parts and put into the form

$$\frac{1}{2\pi^{1/2}}\left[\int_{x+s}^{\infty} \sin\left(\frac{2}{3}\left(\frac{y-x}{s}\right)^{3/2}-\frac{\pi}{4}\right)\frac{s^{3/4}\,\mathrm{Ai}(-y)}{(y-x)^{3/4}\,(1+y^2)^m}\,dy\right.$$
$$-\int_{x+s}^{\infty} \sin\left(\frac{2}{3}\left(\frac{y-x}{s}\right)^{3/2}-\frac{\pi}{4}\right)\frac{s^{3/4}}{(y-x)^{1/2}}\,\partial_y\left(\frac{\mathrm{Ai}(-y)}{(1+y^2)^m\,(y-x)^{1/4}}\right)dy$$
$$\left.+\sin\left(\frac{2}{3}\left(\frac{y-x}{s}\right)^{3/2}-\frac{\pi}{4}\right)\frac{s^{3/4}\,\mathrm{Ai}(-y)}{(y-x)^{3/4}\,(1+y^2)^m}\Bigg|_{y=x+s}^{y=\infty}\right]. \tag{B.3}$$

The boundary term in (B.3) is zero at $y=+\infty$ and at $y=x+s$ it equals

$$-\frac{1}{2\pi^{1/2}}\sin\left(\frac{2}{3}-\frac{\pi}{4}\right)\frac{\mathrm{Ai}(-x-s)}{(1+(x+s)^2)^m}$$

which is certainly a bounded, continuous function of (x, s). Since $m>0$, the first integral in (B.3) converges absolutely to a continuous function of (x, s) which is bounded on $\mathbb{R}\times[0, T]$ for any finite value of T.

As for the second integral in (B.3), an application of Leibniz' rule for the differentiation of a product yields a sum of three integrals, two of which converge absolutely to continuous functions on $\mathbb{R}\times\mathbb{R}^+$ which are uniformly bounded on bounded time intervals. Interest thus attaches to the remaining, troublesome term, namely

$$\frac{1}{2\pi^{1/2}}\int_{x+s}^{\infty} \sin\left(\frac{2}{3}\left(\frac{y-x}{s}\right)^{3/2}-\frac{\pi}{4}\right)\frac{s^{3/4}\,\mathrm{Ai}'(-y)}{(1+y^2)^m\,(y-x)^{3/4}}\,dy, \tag{B.4}$$

which requires more exacting analysis.

We proceed to study the integral in (B.4). Note first that if $x+s<0$, then the portion of the integral in (B.4) corresponding to integrating over the interval $[x+s, 0]$ is a uniformly bounded, continuous function on all of $\mathbb{R}\times\mathbb{R}^+$ since $\mathrm{Ai}(-y)$ has the exponentially decaying upper bound in (4.2) for negative values of y. So, if one defines $\alpha=\max\{0, x+s\}$, then it is only necessary to consider the integrand in (B.4) integrated over $[\alpha, \infty)$. In this range, we may use (4.3) again to write the last-mentioned integral as

$$\frac{1}{4\pi}\int_\alpha^\infty \sin\left(\frac{2}{3}\left(\frac{y-x}{s}\right)^{3/2}-\frac{\pi}{4}\right)\sin\left(\frac{2}{3}\,y^{3/2}-\frac{\pi}{4}\right)\frac{y^{1/4}s^{3/4}}{(y-x)^{3/4}\,(1+y^2)^m}\,dy$$
$$+\text{ bounded continuous function.} \tag{B.5}$$

If we let γ be defined by

$$\gamma(y)=\gamma(y, x, s)=\frac{1}{4\pi}\frac{y^{1/4}s^{3/4}}{(y-x)^{3/4}\,(1+y^2)^m}, \tag{B.6}$$

then elementary trigonometric identities reduce the integral displayed in (B.5) to

$$-\frac{1}{2}\int_\alpha^\infty \cos(\theta_+(y))\,\gamma(y)\,dy+\frac{1}{2}\int_\alpha^\infty \cos(\theta_-(y))\,\gamma(y)\,dy, \tag{B.7}$$

where

$$\theta_+(y)=\theta_+(y, x, s)=\frac{2}{3}\left(\left(\frac{y-x}{s}\right)^{3/2}+y^{3/2}-\frac{\pi}{2}\right) \tag{B.8}$$

and

$$\theta_-(y)=\theta_-(y, x, s)=\frac{2}{3}\left(\left(\frac{y-x}{s}\right)^{3/2}-y^{3/2}\right). \tag{B.9}$$

Note that if M is any fixed, positive number, then

$$\int_I \cos(\theta_\pm(y))\,\gamma(y)\,dy$$

is a continuous function of (x, s) which is uniformly bounded on bounded time intervals, where our convention is that $I=[\alpha, M]$ provided $M\geqslant\alpha$, and $I=\varnothing$ otherwise. Thus up to a continuous function bounded on bounded time intervals, we might as well suppose that in fact $\alpha=\max\{x+s, 2\}$, say. In particular, y gets nowhere near the value zero on the interval $[\alpha, \infty)$. Provided the relevant denominators are well enough behaved, the integrals in (B.7) can be integrated by parts thusly:

$$\int_\alpha^\infty \cos(\theta_\pm(y))\,\gamma(y)\,dy$$
$$= +\int_\alpha^\infty \sin(\theta_\pm(y))\frac{\partial_y^2\theta_\pm(y)}{(\partial_y\theta_\pm(y))^2}\gamma(y)\,dy$$
$$-\int_\alpha^\infty \sin(\theta_\pm(y))\frac{\partial_y\gamma(y)}{\partial_y\theta_\pm(y)}\,dy + \frac{\sin(\theta_\pm(y))}{\partial_y\theta_\pm(y)}\gamma(y)\Bigg|_{y=\alpha}^{y=\infty}. \tag{B.10}$$

For $\theta_+(y)$ it is readily deduced that all the terms on the right-hand side of (B.10) are continuous functions of (x, s) on $\mathbb{R}\times\mathbb{R}^+$ which are bounded on bounded time intervals. This follows since $\partial_y\theta_+(y)$ grows like $y^{1/2}$ as $y\to+\infty$ and is bounded below on $[\alpha, \infty)$. For the term involving θ_-, more detailed analysis is needed. Henceforth, θ_- will be written unadorned as simply θ.

It is worth remarking at this point that if $x=0$ and $s=t^{1/3}=1$, then $\theta=\theta_-\equiv 0$, so that the integral involving θ_-in (B.7) reduces to

$$\int_\alpha^\infty \gamma(y)\,dy,$$

which is a divergent integral since $m<1/4$, diverging to $+\infty$ in fact. This is in accord with the statement of the lemma, and, indeed, it is easy to ascertain that

$$\lim_{(x,s)\to(0,1)}\int_\alpha^\infty \cos(\theta(y))\,\gamma(y)\,dy = +\infty \tag{B.11}$$

once it is known that the integral under the limit in (B.11) converges for $(x, s)\neq(0, 1)$.

Thus interest is now centered upon the integral in (B.11) for values $(x, s)\neq(0, 1)$. Indeed, let $\delta>0$ be given and let (x, s) be such that

$$|x|+|1-t|\geqslant\delta. \tag{B.12}$$

The argument proceeds in two parts. First, suppose that $|x|<\delta/2$ so that $|1-t|>\delta/2$. In this regime, one quickly verifies that $\theta_y(y)$ grows like $y^{1/2}$ as $y\to+\infty$ and that $\theta_y(y)\geqslant c>0$ uniformly for $y\in[\alpha, \infty)$ and $0\leqslant t\leqslant T$, where T is any finite, positive number. Hence an appeal to formula (B.10) assures that the integral comprises a continuous function which is bounded on bounded time intervals.

For the second part, suppose now that $|x|\geqslant\delta/2$. An examination of θ reveals that it has a single critical point at the value $y=y_0$ where

$$y_0=\frac{x}{1-s^3}=\frac{x}{1-t}. \tag{B.13}$$

This point may or may not lie in the interval $[\alpha, \infty)$ of integration of

$$\int_{\alpha}^{\infty} \cos(\theta(y))\, \gamma(y)\, dy. \tag{B.14}$$

If it lies well outside this range, then partial integration as in (B.10) yields a representation of the integral in (B.14) that is clearly a continuous function bounded on $\mathbb{R} \times [0, T]$ for any finite value of T. If y_0 is in or close to the range of integration then we proceed to break up the integral as in the classical method of stationary phase. Notice that only values of t between 0 and 2 (and non-zero values of x) lead to critical points in the range of integration. For definiteness, take $0 < x$ and $t < 1$. We also continue to suppose that $|x| > \delta/2$. A simple change of variables converts the integral in (B.14) to the form

$$\int_{\tilde{\alpha}}^{\infty} e^{i\lambda\omega(z)}\, g(z)\, dz, \tag{B.15}$$

where it is only the real part of this integral that is of interest here. In the integral in (B.15), ω, λ, and g are given by

$$\omega(z) = \frac{2}{3}\left[\frac{(z+t-1)}{t^{1/2}} - z^{3/2}\right], \qquad \lambda = \left(\frac{x}{1-t}\right)^{3/2},$$

and

$$g(z) = \frac{t^{1/4}(1-t)^{1/2}}{4\pi x^{1/2}} \frac{z^{1/4}}{(z+t-1)^{3/4}\,(1+\lambda^{4/3}z^2)^m}. \tag{B.16}$$

The critical point ω is now located at $z = 1$, independently of x and t. The integral in (B.15) is broken into two parts, namely

$$\int_{\tilde{\alpha}}^{2} e^{i\lambda\omega(z)} g(z)\, dz \qquad \text{and} \qquad \int_{\max\{2, \tilde{\alpha}\}}^{\infty} e^{i\lambda\omega(z)} g(z)\, dz, \tag{B.17}$$

with the understanding that if $\tilde{\alpha} \geqslant 2$, the first integral is zero. Since $\tilde{\alpha} \geqslant 0$ and $|x| > \delta/2$, the first integral in (B.17) is a bounded, continuous function of (x, t). For the second integral, we simply integrate by parts once again to reach the formula

$$\int_{z_0}^{\infty} e^{i\lambda\omega(t)} g(z)\, dz$$

$$= g(t) \frac{e^{i\lambda\omega(z)}}{i\lambda\omega_z(z)}\Bigg|_{z=z_0}^{z=\infty} - \int_{z_0}^{\infty} \frac{e^{i\lambda\omega(z)}}{i\lambda} \frac{g_z(z)}{\omega_z(z)}\, dz, \tag{B.18}$$

where $z_0 = \max\{2, \tilde{\alpha}\}$. The boundary term vanishes at $z = +\infty$, while at $z = z_0$ we obtain a function which is clearly continuous as long as both $x \neq 0$ and $t \neq 1$. Moreover, we have that for some constant c,

$$\left| g(z_0) \frac{e^{i\lambda\omega(z_0)}}{i\lambda\omega_z(z_0)} \right| \leqslant c \frac{t^{1/4}(1-t)}{x},$$

and since we are dealing with the situation in which $|x| > \delta/2$, it follows that the boundary term is continuous even at $t = 1$, and uniformly bounded on bounded time intervals. As for the integral on the right-hand side of (B.18), the absolute value of the integrand is bounded by an expression of the form

$$c \frac{t^{1/4}(1-t)}{x^2} \left[\frac{1}{z^{3/2}} \left(1 + \frac{1}{\lambda^{4m/3}} \right) \right]$$

and since x is bounded away from zero and λ is likewise bounded below on finite time intervals, we may again conclude that this integral converges absolutely and uniformly for $|x| \geqslant \delta/2$ and t bounded, therefore representing a continuous function that is bounded on bounded time intervals.

It is thus established that $I(x, s)$ is a convergent integral if $(x, s) \neq (0, 1)$. Moreover, $I(x, s)$ is jointly continuous in this domain and uniformly bounded on any region of the form

$$\{(x, s) : |x| + |1 - s| \geqslant \delta > 0, 0 \leqslant s \leqslant \delta^{-1} < +\infty\}.$$

We now turn briefly to the convergence of $I(x, s)$ to the initial datum $\psi(x) = \mathrm{Ai}(-x)/(1 + x^2)^m$ as $s \to 0$. Notice first that since $\psi \in L_2(\mathbb{R})$ and the Airy kernel generates a C_0-semigroup in any space $H^s(\mathbb{R})$, $s \geqslant 0$, it follows that $I(x, s)$ converges to $\psi(x)$ in $L_2(\mathbb{R})$ as $s \to 0$. Indeed, this fact is obvious upon consideration of the Fourier transform of I in the variable x. It is also the case that $I(x, s)$ converges to $\psi(x)$ as $s \to 0$ in $L_\infty(\mathbb{R})$, a fact which is now verified.

First, since Ai is improperly integrable with total mass one, it follows that the difference $I(x, s) - \mathrm{Ai}(-x)(1 + x^2)^{-m}$ can be written as

$$\frac{1}{s} \int_{-\infty}^{\infty} \mathrm{Ai}\left(\frac{x-y}{s}\right) \left[\frac{\mathrm{Ai}(-y)}{(1+y^2)^m} - \frac{\mathrm{Ai}(-x)}{(1+x^2)^m} \right] dy. \tag{B.19}$$

This latter, improper integral is broken up into three parts I_1, I_2, and I_3 corresponding, respectively, to the intervals of integration $(-\infty, x-\alpha]$, $(x-\alpha, x+\alpha)$, and $[x+\alpha, \infty)$, where $\alpha = \alpha(s)$ will be specified momentarily. The three integrals I_1, I_2, and I_3 will be estimated separately.

First, by the mean-value theorem we have that

$$I_2 = \frac{1}{s}\int_{x-\alpha}^{x+\alpha} \mathrm{Ai}\left(\frac{x-y}{s}\right) \omega(x, y)(y-x)\, dy,$$

where

$$\omega(x, y) = -\left[\frac{\mathrm{Ai}'(-z)}{(1+z^2)^m} + 2m\frac{z\,\mathrm{Ai}(-z)}{(1+z^2)^{m+1}}\right],$$

and, for each $y \in [x-\alpha, x+\alpha]$, $z = z(x, y)$ lies between x and y. Referring to (4.2) and (4.3), one sees easily that ω is a uniformly bounded function of x and y provided that $m \geqslant 1/8$. Hence the estimate

$$|I_2| \leqslant \frac{|\omega|_\infty |\mathrm{Ai}|_\infty}{s}\int_{-\alpha}^{\alpha} |z|\, dz \leqslant \frac{c\alpha^2}{s}$$

follows readily, where the constant c is independent of both x and s. Choosing $\alpha = \alpha(s) = s^\nu$ where $\nu > 1/2$ then leads to the upper bound

$$|I_2| \leqslant cs^{2\nu - 1}$$

which tends to zero uniformly in x as s tends to zero.

Turning now to I_3, we proceed as follows:

$$I_3 = \int_{x+\alpha}^{\infty} \frac{1}{s}\mathrm{Ai}\left(\frac{x-y}{s}\right) \mathrm{Ai}(-y)(1+y^2)^{-m}\, dy - \int_{x+\alpha}^{\infty} \frac{1}{s}\mathrm{Ai}\left(\frac{x-y}{s}\right) dy\, \mathrm{Ai}(-x)(1+x^2)^{-m}. \tag{B.20}$$

The second integral on the right-hand side of (B.20) may be written as

$$-\mathrm{Ai}(-x)(1+x^2)^{-m}\int_{-\alpha/s}^{\infty} \mathrm{Ai}(z)\, dz$$

and if it insisted that $\alpha = s^\nu$ where $\nu < 1$, then $\alpha/s \to \infty$ as $s \to 0$ and therefore, because Ai is improperly integrable, this term is seen to tend to 0 as s tends to zero, uniformly for $x \in \mathbb{R}$. As for the other term, the combination $(y-x)/s$ is larger than $s^{\alpha-1}$ on the interval of integration and this tends to infinity as s tends to zero. Hence, one is naturally inclined to use the asymptotic form in (4.3) of $\mathrm{Ai}((x-y)/s)$ in the following way. First, write

$$\int_{x+\alpha}^{\infty} \frac{1}{s} \mathrm{Ai}\left(\frac{x-y}{s}\right) \mathrm{Ai}(-y)(1+y^2)^{-m}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{x+\alpha}^{\infty} \frac{1}{s^{3/4}} \frac{1}{(y-x)^{1/4}} \cos\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right)$$

$$\times \frac{\mathrm{Ai}(-y)}{(1+y^2)^m} \left[1 + O\left(\frac{s^{3/2}}{1+(y-x)^{3/2}}\right)\right] dy.$$

Clearly the term containing $O(s^{3/2}/[1+(y-x)^{3/2}])$ tends to zero uniformly in x as s tends to zero. The other term is integrated by parts to reach the expression

$$\frac{1}{2}\int_{x+\alpha}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \frac{s^{3/4}}{(y-x)^{3/4}} \frac{\mathrm{Ai}(-y)}{(1+y^2)^m}\, dy$$

$$- \int_{x+\alpha}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right)$$

$$\times \frac{s^{3/4}}{(y-x)^{1/2}} \partial_y \left[\frac{\mathrm{Ai}(-y)}{(1+y^2)^m (y-x)^{1/4}}\right] dy$$

$$+ \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \frac{s^{3/4}}{(y-x)^{3/4}} \frac{\mathrm{Ai}(-y)}{(1+y^2)^m} \Bigg|_{y=x+\alpha}^{y=\infty}. \tag{B.21}$$

The boundary term at $y = +\infty$ is zero and the boundary term at $y = x+\alpha$ tends to zero as s tends to zero, uniformly in x since $\nu < 1$. By writing the first term in (B.21) in two parts wherein the integrand is integrated separately over the intervals $[x+\alpha, x+1]$ and $[x+1, \infty)$, and using again the restriction $\nu < 1$, it is deduced that this expression, too, tends to zero as s tends to zero, uniformly for $x \in \mathbb{R}$. Consider now the second term in (B.21) and use Leibniz' rule to write the derivative with respect to y as a sum of three terms. Two of the resulting integrals clearly tend to zero as s tends to zero, uniformly for all real x. The third term, namely

$$\int_{x+\alpha}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \frac{s^{3/4}}{(y-x)^{3/4}} \frac{\mathrm{Ai}'(-y)}{(1+y^2)^m}\, dy, \tag{B.22}$$

requires more exacting consideration.

In fact, we argue very much like we did earlier following formula (B.5) to obtain the desired conclusion regarding the integral in (B.22). First since Ai′ lies in $L_1(m, \infty)$ for any finite value of m, it follows that

$$\left| \int_{x+\alpha}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \frac{s^{3/4}\,\mathrm{Ai}'(-y)}{(y-x)^{3/4}\,(1+y^2)^m}\, dy \right| \leqslant \left(\frac{s}{\alpha}\right)^{3/4} \|\mathrm{Ai}'\|_{L_1((-1,\infty))},$$

and the right-hand side of this inequality tends to zero as $s \to 0$ since, again, $v < 1$. Of course, if $x \geqslant 1$, this integral does not appear in our appraisal of the integral in (B.22). Letting $\bar{x} = \max\{1, x+\alpha\}$, we are left to consider the integral

$$\int_{\bar{x}}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \frac{s^{3/4}\,\mathrm{Ai}'(-y)}{(y-x)^{3/4}\,(1+y^2)^m}\, dy.$$

Since $\bar{x} > 0$, use may again be made of the asymptotic formula (4.3) to write

$$\begin{aligned} \int_{\bar{x}}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \frac{s^{3/4}\,\mathrm{Ai}'(-y)}{(y-x)^{3/4}\,(1+y^2)^m}\, dy &= \frac{1}{2\pi^{1/2}} \int_{\bar{x}}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \\ &\quad \times \sin\left(\frac{2}{3}\, y^{3/2} - \frac{\pi}{4}\right) \frac{s^{3/4} y^{1/4}}{(y-x)^{3/4}\,(1+y^2)^m}\, dy \\ &\quad + \frac{1}{2\pi^{1/2}} \int_{\bar{x}}^{\infty} \sin\left(\frac{2(y-x)^{3/2}}{3s^{3/2}} - \frac{\pi}{4}\right) \\ &\quad \times \frac{s^{3/4} y^{1/4}}{(y-x)^{3/4}\,(1+y^2)^m}\, O\left(\frac{1}{y^{3/2}}\right) dy. \end{aligned} \tag{B.23}$$

The second term on the right hand side of (B.23) converges absolutely and tends to zero as $s \to 0$, uniformly for $x \in \mathbb{R}$. The first integral on the right-hand side of (B.23) is written, up to a constant, as

$$\int_{\bar{x}}^{\infty} \cos(\theta_+(y))\, \gamma(y)\, dy + \int_{\bar{x}}^{\infty} \cos(\theta_-(y))\, \gamma(y)\, dy, \tag{B.24}$$

where θ_+, θ_-, and γ are defined in (B.8), (B.9), and (B.6), respectively. Using the formula for integration by parts in (B.10) leads quickly to the conclusion that the integral in (B.24) involving θ_+ tends to zero as $s \to 0$, uniformly for $x \in \mathbb{R}$.

Turning now to the second integral in (B.24), and abbreviating θ_- by simply θ, we proceed as in the analysis of the second integral in (B7). First consider the case wherein $\tilde{x} \leqslant 1/2$, say. Since $x \geqslant 1$, it follows that if $t \leqslant 1/2$, then for all $y \geqslant \tilde{x}$, $\theta_y(y) \geqslant cy^{1/2}/s^{3/2}$ for some positive constant c. This inequality combined with formula (B.10) assures that for such values of x, the second integral in (B.24) converges absolutely and tends uniformly to zero as $s \to 0$.

If $\tilde{x} > 1/2$, we proceed in a manner like that appearing earlier near formula (B.15), making a change of variables to put the integral in question into the form

$$\int_{\tilde{x}}^{\infty} e^{i\lambda\omega(z)} g(z)\, dz \tag{B.25}$$

where λ, ω, and g are defined in (B.16), $\tilde{x} = \max\{(1-t)/x, (\tilde{x}/x)(1-t)\}$, and it is only the real part of the latter integral that is of direct interest here. This integral is broken into two parts, namely

$$\int_{\tilde{x}}^{2} e^{i\lambda\omega(z)} g(z)\, dz \quad \text{and} \quad \int_{\max\{2,\tilde{x}\}}^{\infty} e^{i\lambda\omega(z)} g(z)\, dz, \tag{B.26}$$

where it is understood that the first integral is to be ignored if $\tilde{x} \geqslant 2$. Since $\tilde{x} \geqslant 0$, the absolute value of the first integral in (B.26) is bounded above by $ct^{1/4}$ where c is a constant which is independent of $x \geqslant 1/2$ and $t \leqslant 1/2$. The second expression in (B.26) is integrated by parts as in (B.18) to reach the alternate form

$$g(z)\frac{e^{i\lambda\omega(z)}}{i\lambda\omega_z(z)}\bigg|_{z=\max\{2,\tilde{x}\}}^{z=\infty} - \frac{1}{i\lambda}\int_{\max\{2,\tilde{x}\}}^{\infty} e^{i\lambda\omega(t)}\frac{g_z(z)}{\omega_z(z)}\, dz.$$

But $\omega_z(z) \geqslant cz^{1/2}/s^{3/2}$ if $z \geqslant 2$ and $t \leqslant 1/2$, and so it follows at once that both the terms in the last display tend to zero as $s \to 0$, uniformly for $x \geqslant 1/2$.

The proof of the lemma is now complete in the case $k = 0$.

As for the cases wherein $k > 0$, the arguments are identical except that the kth derivative $\mathrm{Ai}^{(k)}(-y)$ replaces $\mathrm{Ai}(-y)$ in the integral expression in (B.2), the power m is taken to be larger in compensation, and the general asymptotic formulas (4.4) are used in place of those in (4.2) and (4.3). ∎

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