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MODELS FOR PROPAGATION OF BORES I. TWO DIMENSIONAL THEORY

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Dedicated to the memory of Peter Hess

Abstract. Considered here are model equations for the propagation of bores on the surface of a canal or river. Interest will be focused on both the Korteweg-deVries equation and the regularized long-wave equation with a Burgers-type dissipative term appended. Existence, uniqueness and continuous dependence results are established using the techniques of Bona and Smith. We then consider the steady bores whose existence was established by Bona and Schonbek and show these to be stable solutions of the evolution equation. This latter result extends the earlier work of Pego on this problem.

1. Introduction. The physical phenomenon underlying the present study is the propagation of a surge of liquid in a channel. Such motions have attracted attention for centuries, being readily observed in nature. The first careful and extensive laboratory study of such wave motion appears to have been made by Favre [7]. In his experiments, a gate separating different levels of water in a channel is abruptly removed. This is the same experiment as that performed by Scott Russell [28] nearly a century earlier, but the impetus for this previous study was the newly discovered solitary wave, and thus a different physical regime was in view. By selecting suitable geometries and varying the levels of the water on either side of the gate, two general classes of waves may be observed, both of which are termed bores. So-called strong bores have a rapid, turbulent change of water level, whilst weak or undular bores have a gently sloping or oscillatory transition between the different levels. The matter may be viewed on a much grander scale, for example in the Tsein-Tang river where the motion is driven by the tides (cf. Stokes [32] for a photograph and commentary concerning bores).

The present study will be concerned with an idealized, two-dimensional situation in which an undular bore is uniform across the channel in which it propagates and the

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channel itself is level and uniform with a rectangular cross-section, say. A forthcoming report will deal with the case when there is cross-channel variation of the wave. The three-dimensional situation is more realistic of course, but there is much to be learned from the simpler case considered here that is somewhat obscured by the complexity inherent in the three-dimensional models. For both the two-dimensional and the three-dimensional situations, our theory will rely upon approximate, long-wave models to describe the fluid motion.

Both the Korteweg-deVries equation

$$U_t + U_x + UU_x + U_{xxx} = 0 \tag{1.1}$$

(Korteweg & deVries [20], henceforth referred to as the KdV equation) and the regularized long-wave equation

$$U_t + U_x + UU_x - U_{xxt} = 0 \tag{1.2}$$

proposed by Peregrine [27] and Benjamin *et al.* [2] have been suggested as models for the propagation of bores. Both these models incorporate the nonlinear and dispersive effects inherent in bore propagation, but neither takes account of dissipative effects.

On laboratory scales, there is little doubt that dissipative effects must be included to have any hope of accurate prediction (cf. the experimental studies of Hammack [10], Hammack & Segur [11] and Bona *et al.* [3]). Even on the larger scales often appearing in nature, dissipative effects need to be considered, at least near the front of the bore.

The modeling of dissipation in surface water waves at the same level of accuracy as nonlinearity and dispersion appear in KdV-type equations is somewhat complicated (cf. Kakutani & Matsuuchi [15], Mei & Liu [21], Miles [22], Ott and Sudan [24]). In practice, one often incorporates some relatively simple dissipative term into one of the models (1.1) or (1.2) (see Bona & Smith [5], Grad & Hu [9], Johnson [13], [14]) in the hope that the resulting solutions will still reflect the essentials of what occurs in the field. As pointed out by Bona *et al.* [3], as long as only a relatively small number of wave numbers are present in the disturbance, one has reason for optimism regarding this point of view.

This just-mentioned approach is adopted here. Thus the principal aim of this work is to demonstrate the classical well-posedness of the initial-value problems

$$U_t + U_x + UU_x - \nu U_{xx} + U_{xxx} = 0 \tag{1.3}$$

and

$$V_t + V_x + VV_x - \nu V_{xx} - V_{xxt} = 0, \tag{1.4}$$

when initial data corresponding to bore propagation is provided, namely

$$U(x,0) = g(x) \quad \text{or} \quad V(x,0) = g(x). \tag{1.5}$$

In (1.3) and (1.4), it is assumed that $\nu \geq 0$ whilst in (1.5) it is supposed that g is sufficiently smooth with

$$g(x) \longrightarrow C_+ \text{ as } x \longrightarrow +\infty \text{ and } g(x) \longrightarrow C_- \text{ as } x \longrightarrow -\infty, \tag{1.6}$$

where $C_- > C_+$. (In both equations (1.3) and (1.4), three physically important but mathematically irrelevant constants have been scaled out.)

The KdV equation (1.1) has attracted considerable attention in the last couple of decades, partly because of the inverse-scattering theory introduced by Gardner *et al.* [8] and partly because of the large number of important physical situations that it or its generalizations model (cf. Benjamin [1], Jeffery and Kakutani [12], Scott Chu and McLaughlin [29] for a partial list). Rigorous theory for the KdV equation began with the work of Sjöberg [31] and Temam [34] on the periodic initial-value problem and was extended and refined by Bona & Smith [5] and Kato [16]. The latter authors studied the initial-value problem on all of $\mathbb{R}$ with data lying in Sobolev classes, and so tending to zero as the spatial variable x tends to $\pm\infty$. They showed that solutions exist, are unique, and define a continuous mapping of the temporal interval into the function class from which the data was derived. Moreover, they demonstrated that the solution depends continuously on the initial data. More recent work has combined the ideas in Bona & Smith [5], Kato [17] and Strichartz [33] to obtain well-posedness corresponding to rather rough initial data and some surprising smoothing results for the solutions of the KdV equation (cf. Kenig *et al.* [18] for a review of this theory). Similar results were obtained somewhat earlier (Benjamin *et al.* [2]) for the regularized long wave equation (1.2). The mathematical status of equations (1.3) and (1.4) is less well understood when bore-like initial data is in question as in (1.5)–(1.6). One problem is that such data has an infinite amount of energy, thus leading to difficulties in obtaining *a-priori* bounds on solutions. Benjamin *et al.* [2] were able to circumvent this problem and thereby concluded a satisfactory theory for the initial-value problem (1.4)–(1.5) with $\nu = 0$. A theory for the initial-value problem (1.3)–(1.5) has not been put forward heretofore save for the case $\nu = 0$ for which Cohen [6] obtained some preliminary results using the inverse-scattering transform.

The plan of the paper is as follows. After briefly reviewing our notational conventions in Section 2, the initial-value problem (1.4)–(1.5) is considered in Section 3. The theory presented is a straightforward generalization of that presented in Benjamin *et al.* [2] and consequently our presentation is abbreviated. The more difficult case of the initial-value problem (1.3)–(1.5) is considered in Section 4. Here we follow the lead of Bona & Smith [5] in regularizing equation (1.3) with the additional dispersive term $-\epsilon U_{xxt}$. The regularized equation thus obtained is reduced via a change of variables to equation (1.4) and the theory in Section 3 then comes to our aid in establishing global solutions for the regularized equation. The remaining part of the section is devoted to obtaining *a priori* bounds on the solutions of the regularized initial-value problem. In Section 5 the strong convergence as ϵ tends to zero of the solutions to the regularized problem is established. The limiting functions are shown to be the desired solutions to the problem (1.3)–(1.5). It follows readily from the strong convergence of the regularized solutions that solutions of (1.3)–(1.5) depend continuously on the initial data. In Section 6 the theory developed in the earlier sections is put to use in considering the stability of the steady bore solutions of (1.3) or (1.4). The existence of these traveling-waves was shown by Bona & Schonbek [4]. Pego [25] has demonstrated the stability of these steady bores. However the stability is only established in the special circumstance that the perturbation has zero total mass. This assumption is somewhat artificial and is shown to be unnecessary. The crucial ingredient is a theory for (1.3) and (1.4) in suitable weighted Sobolev spaces.

2. Notation. The notation to be used is mostly standard, but worth briefly reviewing. For $1 \leq p \leq \infty$, we denote by $L_p = L_p(\mathbb{R})$ the Banach space of measurable real-valued functions defined on the real line $\mathbb{R}$ which are p-th power Lebesgue integrable (essentially bounded in the case $p = \infty$). The usual norm is denoted by $|\cdot|_p$. For a non-negative integer s, $H^s = H^s(\mathbb{R})$ is the Sobolev space of functions in L_2 whose generalized derivatives up to order s also belong to L_2. If s is an integer, this space is equipped with the norm

$$\|f\|_s^2 = \sum_{j=0}^{s} |f^{(j)}|_2^2.$$

For non-integer values of s, the norm is defined via Fourier transforms in the usual way. Of course $H^0 = L_2$ and the L_2-norm $|\cdot|_2 = \|\cdot\|_0$ will be denoted by the symbol $|\cdot|_2$. The symbol H^∞ is reserved for $\cap_{s\geq 0} H^s$, but no topology will be needed for this space. The symbol $C_b = C_b(\mathbb{R})$ connotes the Banach space of bounded continuous functions defined on $\mathbb{R}$ with the supremum norm. Similarly $C_b^k = C_b^k(\mathbb{R})$ is the subspace of C_b consisting of functions whose first k derivatives lie in C_b with the usual norm. For T a positive real number or $+\infty$, let

$$C_b(0, T; H^s) = \big\{u : \mathbb{R} \times [0, T] \to \mathbb{R} : u(\cdot, t) \in H^s \;\forall t \in [0, T] \text{ and the mapping } t \longmapsto u(\cdot, t) \text{ from } [0, T] \text{ into } H^s \text{ is bounded and continuous}\big\}.$$

In case T is finite, the correspondence $t \mapsto u(\cdot, t)$ is automatically bounded and the superfluous subscript b is therefore dropped. The norm is the obvious one given by

$$\|u\|_{C(0,T;H^s)} = \sup_{0\leq t\leq T} \|u(\cdot, t)\|_s.$$

The space $C(0, T; C_b^k)$ is defined similarly.

3. Existence theory.

3.1. The regularized long-wave equation. Giving precision to the assumptions about the initial data g in (1.5), it will be assumed that it satisfies the following conditions:

$$\begin{aligned} &\text{(i)} \quad g(x) \longrightarrow C_\pm \text{ as } x \longrightarrow \pm\infty, \\ &\text{(ii)} \quad g' \in H^k, \\ &\text{(iii)} \quad (g - C_+) \in L_2([0, \infty)) \text{ and } (g - C_-) \in L_2((-\infty, 0]), \end{aligned} \tag{3.1}$$

for some non-negative integer k. It follows immediately from (ii) that g is continuous and this fact coupled with (i) implies g is bounded. The following elementary result will be used at an early stage in our analysis.

Lemma 1. *Let h satisfy (3.1) (i), (ii). Then there is a C^∞ function ϕ such that $\phi' \in H^\infty$, ϕ satisfies (3.1) (i), (ii) and $h - \phi$ lies in H^{k+1}. The function ϕ can be chosen such that*

$$|\phi - h|_2 \le 2|h'|_2 \quad \text{and} \quad |\phi|_\infty \le |h|_\infty, \tag{3.2a}$$

and so that there are constants C_k for which,

$$\|\phi'\|_k \le C_k |h'|_2 \tag{3.2b}$$

for $k = 0, 1, 2, \ldots$. Moreover, if h satisfies (3.1)(iii), then so does ϕ.

Proof. Let $\rho(x)$ be a non-negative C^∞-function with support in $[-1, 1]$, say, such that

$$\int_{-1}^{1} \rho(x)dx = 1 \quad \text{and} \quad \int_{-1}^{1} \rho^2(x)\,dx \le 1.$$

Define the function ϕ to be $\rho \star h$ where $\star$ denotes convolution. We see plainly that

$$|\phi|_\infty = \left| \int_{-\infty}^{\infty} h(x-y)\rho(y)dy \right|_\infty \le |h|_\infty,$$

since ρ has L_1-norm equal to 1. From the general properties of convolutions, one has, for $m \ge 1$,

$$\frac{d^m\phi}{dx^m} = \frac{d^{m-1}\rho}{dx^{m-1}} \star h'.$$

The right-hand side of this relation is the convolution of an L_1- and an L_2-function, and so is an L_2-function. In addition,

$$|\phi^{(m)}|_2 \le |\rho^{(m-1)}|_1 |h'|_2 = C_m |h'|_2,$$

which shows that ϕ' is an H^∞ function and that (3.2b) holds, for $k = 1, 2, \ldots$. In particular, $(\phi - h)'$ lies in H^k. (If $\phi - h$ were also an L_2-function, then one could adduce that $\phi - h \in H^{k+1} \subset H^1$.)

To prove (3.1)(i) holds for ϕ, consider

$$\lim_{x\to\pm\infty} \phi(x) = \lim_{x\to\pm\infty} \int_{-\infty}^{\infty} h(x-y)\rho(y)dy = \lim_{x\to\pm\infty} \int_{-1}^{1} h(x-y)\rho(y)dy$$
$$= \int_{-1}^{1} \lim_{x\to\pm\infty} h(x-y)\rho(y)dy = C_\pm,$$

since ρ has mass one. The interchange of the limiting procedure and the integration is justified since h is bounded and continuous and $\lim_{x\to\pm\infty} h(x-y) = C_\pm$, uniformly for $|y| \le 1$.

We now show that $\phi - h \in L_2$ and verify (3.2a). Since ρ has mass one, it follows that

$$\phi(x) - h(x) = \int_{-\infty}^{\infty} \rho(y)\,(h(x-y) - h(x))\,dy,$$

from which one concludes that

$$\int_{-\infty}^{\infty} (\phi(x) - h(x))^2\,dx = \int_{-\infty}^{\infty} \Big(\int_{-1}^{1} \rho(y)(h(x-y) - h(x))dy\Big)^2 dx$$
$$\leq \int_{-\infty}^{\infty} 2 \int_{-1}^{1} \rho^2(y)\,[h(x-y) - h(x)]^2\,dydx$$
$$= \int_{-1}^{1} 2\rho^2(y) \int_{-\infty}^{\infty} \Big\{\int_{0}^{-y} h'(x+t)dt \int_{0}^{-y} h'(x+s)ds\Big\} dxdy$$
$$= \int_{-1}^{1} 2\rho^2(y) \int_{0}^{-y} \int_{0}^{-y} \Big\{\int_{-\infty}^{\infty} h'(x+t)h'(x+s)dx\Big\} dsdtdy.$$

But, the Cauchy-Schwarz inequality implies that

$$\int_{-\infty}^{\infty} h'(x+t)h'(x+s)dx \leq |h'|_2^2,$$

and the latter quantity is bounded, independently of s and t. Hence, it transpires that

$$|\phi - h|_2^2 \leq 2|h'|_2^2 \int_{-1}^{1} \rho^2(y)y^2dy \leq 2|h'|_2^2.$$

If h satisfies (3.1)(iii), define H to be

$$H(x) = \begin{cases} C_+ & \text{for } x > 0, \\ C_- & \text{for } x \leq 0. \end{cases}$$

Then the function ϕ satisfies (3.1) (iii) if and only if $\phi - \rho \star H$ lies in L_2. But one notes that

$$\phi - \rho \star H = \rho \star (h - H),$$

and the right-hand side is an L_1-function convolved with an L_2-function, and hence is an L_2-function.

The proof of the lemma is complete. □

Attention is now focused on the initial-value problem

$$U_t + U_x + UU_x - \nu U_{xx} - U_{xxt} = 0, \quad U(x,0) = g(x), \tag{3.3}$$

where g satisfies the conditions in (3.1). The requirement $C_- \geq C_+$, which is an appropriate assumption for bore propagation, plays no role in the theory presented now.

The initial-value problem (3.3) has already been treated by Benjamin *et al.* (1972, p.65) in the case $\nu = 0$ and the analysis is virtually unchanged if $\nu > 0$, as is now briefly indicated. First rewrite the equation in the form

$$(1 - \partial_x^2)(U_t + \nu U) = \nu U - \partial_x(U + \tfrac{1}{2}U^2),$$

and then formally invert the elliptic operator $(1 - \partial_x^2)$ subject to zero Neumann conditions at $\pm\infty$ to obtain the integro-differential equation

$$U_t(x,t) + \nu U(x,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|}\{\nu U(\xi,t) - \partial_\xi\big(U(\xi,t) + \tfrac{1}{2}U^2(\xi,t)\big)\}d\xi. \quad (3.4)$$

Integrating by parts the last two terms in the integrand, and viewing (3.4) as an ordinary differential equation of the form $U_t + \nu U = f$, we find U to satisfy the integral equation

$$\begin{aligned} U(x,t) = e^{-\nu t}g(x) + \int_0^t e^{-\nu(t-s)}\Big\{\int_{-\infty}^{\infty} K'(x-\xi)\big(U(\xi,s) + \tfrac{1}{2}U^2(\xi,s)\big)d\xi \\ + \nu\int_{-\infty}^{\infty} K(x-\xi)U(\xi,t)d\xi\Big\}ds, \end{aligned} \quad (3.5)$$

with $K(z) = \frac{1}{2}e^{-|z|}$, $K'(z) = -\mathrm{sgn}(z)e^{-|z|}$, and $g(x)$ the specified initial value of U. Supposing that $g \in C_b^k$ for some non-negative integer k, it is straightforward to view (3.5) as an operator equation of the form $U = A(U)$ and, by use of the contraction-mapping principle, conclude the existence of a solution in the subspace of $C(0,T;C_b^k)$ of functions satisfying (3.1) (i), provided $T > 0$ is sufficiently small. If $k \geq 2$, it follows that this solution of (3.5) is also a classical solution of (3.3), and that U_x, U_{xx} and U_{xt} all tend to zero as $x \to \pm\infty$. Even for $k = 0$ or 1, U defines a distributional solution of (3.3) (cf. Benjamin *et al.* [2]). The local existence result may then be extended to a global result by the use of energy-type estimates. The main step in obtaining a global result is an L_∞-bound on the local solution which is finite on any finite time interval $[0,T]$ on which the solution is known to exist. Thus suppose U is a solution of (3.3) lying at least in $C(0,T;C_b^k)$. Let $V = U - \phi$, where ϕ is defined as in Lemma 1, relative to g. Then we find that V satisfies the equation

$$\begin{aligned} &V_t + V_x + \phi_x + (V+\phi)(V+\phi)_x - \nu(V+\phi)_{xx} - V_{xxt} = 0, \\ &V(x,0) = g(x) - \phi(x). \end{aligned} \quad (3.6)$$

By Lemma 1, it follows that $(g - \phi) \in L_2$. Multiply equation (3.6) by V and integrate over $\mathbb{R}$ to obtain the relation,

$$\begin{aligned} \tfrac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(V^2 + V_x^2)\,dx = -\int_{-\infty}^{\infty}(VV_x + \phi_x V)dx \\ - \int_{-\infty}^{\infty} V(V+\phi)(V+\phi)_x dx + \nu\int_{-\infty}^{\infty} V(V_{xx} + \phi_{xx})dx. \end{aligned}$$

Simplifying this equation, and estimating in a straightforward manner yields

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}(V^2+V_x^2)\,dx+2\nu\int_{-\infty}^{\infty}V_x^2dx\\ =\int_{-\infty}^{\infty}(V^2V_x-\phi_xV^2-2\phi\phi_xV+V\phi_{xx})\,dx\\ \le|\phi_x|_\infty|V|_2^2+2|\phi|_\infty|\phi_x|_2|V|_2+|\phi_{xx}|_2|V|_2.\end{aligned}$$

By an application of Gronwall's lemma, it follows that the H^1-norm of V is bounded on bounded time intervals. By the use of Sobolev's inequality, it then follows that the L_∞-norm of V is bounded, and since $V=U-g$, it follows that the L_∞-norm of U is bounded on bounded time intervals. With this bound in hand it is then straightforward to pass from the local solution to a global one. We summarize the state of affairs for equation (3.2) in the following theorem, which contains stronger regularity conclusions than those established by Benjamin *et al.* [2].

Theorem 1. *Let g satisfy the conditions in* (3.1) *for some $k\ge 0$. Then there is a unique solution U to the initial-value problem* (1.2) *which, for each fixed $T>0$, has $\partial_t^m U_x\in C(0,T;H^k)$ and $\partial_t^m(U-g)\in C(0,T;H^{k+1})$, for $m=0,1,2,\dots$.*

3.2. Approximation of solutions of the Korteweg-de Vries equation by solutions of the regularized long-wave equation. The initial-value problem (1.3) is now the focus of attention. Changing to a moving coordinate system wherein $V(x,t)=U(x+t,t)$ leads to the initial-value problem

$$V_t+VV_x-\nu V_{xx}+V_{xxx}=0,\qquad V(x,0)=g(x), \tag{3.7}$$

for V. The theory for (3.7) will be approached by way of the regularized initial-value problem

$$V_t+VV_x-\nu V_{xx}+V_{xxx}-\epsilon V_{xxt}=0,\qquad V(x,0)=g(x). \tag{3.8}$$

Here $\epsilon>0$ is fixed for the time being but eventually tends to zero. Now (3.8) is comparatively easy to handle. As noted in [5] the change of variables

$$U(x,t)=\epsilon V(\epsilon^{\frac{1}{2}}(x-t),\epsilon^{\frac{3}{2}}t)$$

transforms (3.8) into the initial-value problem

$$U_t+U_x+UU_x-\mu U_{xx}-U_{xxt}=0,\qquad U(x,0)=\epsilon g(\epsilon^{\frac{1}{2}}x), \tag{3.9}$$

where $\mu=\nu/\epsilon^{\frac{1}{2}}$. For ϵ a fixed positive number, Theorem 1 comes to our aid and the following result is the consequence.

Theorem 2. *Let g satisfy (3.1) for some $k \geq 0$. Then the initial-value problem (3.8) admits a unique solution U such that, for each fixed $T > 0$, $U_x \in C(0, T; H^k)$ and $U - \phi \in C(0, T; H^{k+1})$. Moreover,*

$$\partial_t^l U \in C(0, T; H^{k+1-l}) \quad \text{for } 1 \leq l \leq k+1. \qquad \square$$

To obtain a solution of (3.7) by passing to the limit as $\epsilon \downarrow 0$ in (3.8), *a-priori* bounds on the solution of (3.8) will be needed. For the present, let g satisfy the conditions in (3.1) with $k = \infty$. The solution U to the regularized problem will therefore be a C^∞-function of both its variables and any derivative of U is, for each $t \geq 0$, an L_2-function of the spatial variable. Let ϕ be as in Lemma 1 relative to g.

Proposition 1. *If $W = U - \phi$, where U and ϕ are as just described, then W admits the bound*

$$|W|_2 \leq M_0(t), \tag{3.10}$$

where $M_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and monotone non-decreasing. Moreover, M_0 is independent of $\nu \geq 0$ and ϵ in the range $(0, 1)$ say, but does depend on $|g'|_2$ and $|g|_\infty$.

Proof. The function W satisfies the initial-value problem

$$\begin{aligned} &W_t + (W+\phi)(W+\phi)_x + W_{xxx} - \nu W_{xx} - \epsilon W_{xxt} - \nu\phi_{xx} + \phi_{xxx} = 0, \\ &W(x,0) = g(x) - \phi(x). \end{aligned} \tag{3.11}$$

Multiply the above equation by W and integrate over $\mathbb{R}$. There appears, after appropriate integrations by parts, the equation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (W^2 + \epsilon W_x^2)\,dx + \nu \int_{-\infty}^{\infty} W_x^2 dx \\ &= -\int_{-\infty}^{\infty} \phi W_x W\,dx + \nu\int_{-\infty}^{\infty} \phi_{xx} W\,dx - \int_{-\infty}^{\infty} (W+\phi)\phi_x W\,dx - \int_{-\infty}^{\infty} \phi_{xxx} W\,dx. \end{aligned}$$

After estimation and simplification, this yields,

$$\begin{aligned} &\frac{d}{dt}\int_{-\infty}^{\infty} (W^2 + \epsilon W_x^2)\,dx + 2\nu\int_{-\infty}^{\infty} W_x^2 dx \\ &\leq |\phi_x|_\infty |W|_2^2 + 2|\phi|_\infty |\phi_x|_2 |W|_2 + 2|\phi_{xxx}|_2 |W|_2 + 2\nu|\phi_{xx}|_2 |W|_2. \end{aligned} \tag{3.12}$$

Define the functional E by

$$E(t) = \int_{-\infty}^{\infty} \left(W(x,t)^2 + \epsilon W_x(x,t)^2\right) dx.$$

Then equation (3.12) implies the inequality

$$\frac{dE}{dt} \leq C_1 E(t) + C_2 E^{\frac{1}{2}}(t),$$

from which it follows that

$$E^{\frac{1}{2}}(t) \leq E^{\frac{1}{2}}(0)e^{2C_2t} + \frac{C_1}{C_2}\left(e^{2C_2t} - 1\right) = M_0(t).$$

Here M_0 depends only on the norms of ϕ' in H^1 and ϕ in L_∞, and on $|g-\phi|_2$ and $\epsilon^{\frac{1}{2}}|g'-\phi'|_2$. Thus, because of the conclusion enunciated in Lemma 1, M_0 depends only on $|g'|_2$, $|g|_\infty$ and on t and the proposition is proved. □

Bounds on W in higher-order Sobolev norms will be needed. These are quite easy to obtain in case ν is positive, but are a little harder to derive when ν is allowed to be vanishingly small. The case $\nu > 0$ may be handled as follows.

Proposition 2. *Suppose that $\nu > 0$, and that g satisfies* (3.1) *for some $k \geq 1$. Let U be the solution of the associated regularized problem* (3.8) *and let $W = U - \phi$ as in Proposition* 1. *Then for $0 \leq j \leq k$ there is a non-decreasing, continuous function N_j such that*

$$|\partial_x^j W|_2 \leq N_j(t),$$

where N_j depends only on the H^j-norm of g' and the L_∞-norm of g.

Proof. Let ϕ correspond to the initial data g as in Lemma 1, continue to write $W = U - \phi$, and set $\psi = \phi\phi_x + \phi_{xxx} - \nu\phi_{xx}$. Multiply (3.11) by W_{xx} and integrate over $\mathbb{R}$. After suitable integrations by parts, it transpires that

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\left(W_x^2 + \epsilon W_{xx}^2\right)dx + \nu\int_{-\infty}^{\infty} W_{xx}^2 dx \tag{3.13}$$
$$= \int_{-\infty}^{\infty}\psi W_{xx} + \int_{-\infty}^{\infty}(W+\phi)W_x W_{xx} + \int_{-\infty}^{\infty} W\phi_x W_{xx} \leq |\psi_x|_2|W_x|_2 + I_1 + I_2.$$

Bounds on the terms I_1 and I_2 are needed to make further progress. A bound for I_1 is obtained as follows:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty}(\phi W_x W_{xx} - \tfrac{1}{2}W_x^3)dx = -\tfrac{1}{2}\int_{-\infty}^{\infty}\left(W_x^3 + \phi_x W_x^2\right)dx \\ &\leq \tfrac{1}{2}|\phi_x|_2|W_x|_2|W_x|_\infty + \tfrac{1}{2}|W_x|_\infty|W_x|_2^2 \leq C_1|W_x|_2^{\frac{3}{2}}|W_{xx}|_2^{\frac{1}{2}} + \tfrac{1}{2}|W_x|_2^{\frac{5}{2}}|W_{xx}|_2^{\frac{1}{2}} \\ &\leq C_1|W_x|_2^{\frac{3}{2}}|W_{xx}|_2^{\frac{1}{2}} + \tfrac{1}{2}|W|_2^{\frac{5}{4}}|W_{xx}|_2^{\frac{7}{4}} \leq \tfrac{\nu}{4}|W_{xx}|_2^2 + C_2|W_x|_2^2 + C_3|W|_2^{10}. \end{aligned}$$

In deriving this estimate, Young's inequality and simple Sobolev and interpolation inequalities have been used. Note that since $C_1 = \frac{1}{2}|\phi_x|_2$, C_2 can be chosen to equal $\frac{3}{4}(\frac{C_1^4}{\nu})^{\frac{1}{3}}$, and thus depends only on the L_2-norm of ϕ_x. The constant C_3 is an absolute constant times an inverse power of ν. The quantity I_2 can also be bounded using the same techniques as follows:

$$I_2 = \int_{-\infty}^{\infty} W\phi_x W_{xx}dx \leq |\phi_x|_\infty|W|_2|W_{xx}|_2 \leq C_4|W|_2^2 + \frac{\nu}{4}|W_{xx}|_2^2,$$

where C_4 depends only on the L_∞-norm of ϕ_x and $1/\nu$. Substituting the last two estimates into (3.13) leads to the inequality

$$\frac{d}{dt}\int_{-\infty}^{\infty}(W_x^2+\epsilon W_{xx}^2)\,dx+\nu\int_{-\infty}^{\infty}W_{xx}^2dx\le C_5\int_{-\infty}^{\infty}W_x^2dx+J_1(t),$$

where, using (3.10), one may take $J_1 = C_3M_0(t)^{10}+C_4M_0(t)^2$, which is a continuous function of t and which is bounded on bounded time intervals, independently of $\nu \ge 0$ and ϵ in the range (0, 1). Using Gronwall's lemma, it follows from the last inequality that,

$$|W_x(\cdot,t)|_2^2+\nu\int_0^t|W_{xx}(\cdot,s)|_2^2ds\le N_1(t)^2,$$

where N_1 is a continuous function depending only on the L_2-norm of g', the L_2-norm of $\epsilon^{\frac{1}{2}}g''$, the L_∞-norm of g, and some inverse power of ν, but is independent of ϵ in the range (0, 1).

The argument now proceeds by induction on k. At the kth stage, we multiply (3.11) by $W_{(2k)} = \partial_x^{2k}W$ and integrate over $\mathbb{R}$. The terms

$$\frac{d}{dt}\int_{-\infty}^{\infty}(W_{(k)}^2+\epsilon W_{(k+1)}^2)\,dx+2\nu\int_{-\infty}^{\infty}W_{(k+1)}^2dx$$

appear. The other terms may be bounded in terms of the bounds obtained at the $(k-1)$st stage, or they may be hidden in the quantity

$$\nu\int_{-\infty}^{\infty}W_{(k+1)}^2dx,$$

as in the case $k = 1$. □

The case $\nu = 0$ is considerably more challenging. We begin with an L_2-bound on W_x.

Proposition 3. *Let $W = U - \phi$ be as defined above in Proposition 1 and assume that $\nu = 0$ in (3.8). Then there is a continuous function $M_1(t)$ such that*

$$|W_x|_2 \le M_1(t),$$

and $M_1(t)$ depends only on $|g'|_2$ and $|g|_\infty$.

Proof. Multiply (3.11) by $(W_{xx} - \epsilon W_{xt} + \frac{1}{2}W^2 + W\phi)$ and integrate over $\mathbb{R}$. Let ψ denote the expression $\phi\phi_x + \phi_{xxx}$. After integrating by parts, and a couple of crucial cancellations, there appears

$$\frac{d}{dt}\int_{-\infty}^{\infty}(\tfrac{1}{2}W_x^2-\tfrac{1}{6}W^3-\tfrac{1}{2}\phi W^2)\,dx=-\int_{-\infty}^{\infty}\psi\,(W_{xx}+\tfrac{1}{2}W^2-2\epsilon W_{xt}+W\phi)\,dx.$$

Integrating this over $[0, t]$ yields,

$$\int_{-\infty}^{\infty} W_x^2 dx = \int_{-\infty}^{\infty} (\tfrac{1}{3}W^3 + \phi W^2) dx$$
$$-2\int_0^t \int_{-\infty}^{\infty} \psi(W_{xx} + \tfrac{1}{2}W^2 - 2\epsilon W_{xt} + W\phi) dx + V(0),$$

where the quantity $V(0)$ denotes the value of the integral $\int_{-\infty}^{\infty}(W_x^2 - \frac{2}{3}W^3 - \phi W^2)dx$ at $t = 0$. Estimating the right-hand side of the above inequality, it is found that

$$\int_{-\infty}^{\infty} W_x^2 dx = \tfrac{2}{3}\int_{-\infty}^{\infty} W^3 dx + \int_{-\infty}^{\infty} \phi W^2 dx$$
$$+\int_0^t \int_{-\infty}^{\infty} \left(2\psi_{xx}W + \psi W^2 + 2\psi\phi W\right) dx d\tau + V(0) \tag{3.15}$$
$$\leq \tfrac{2}{3}|W|_2^{\frac{5}{2}}|W_x|_2^{\frac{1}{2}} + C_1|W|_2^2 + \int_0^t (C_2|W|_2 + C_3|W|_2^2) d\tau + V(0) \leq \tfrac{1}{2}|W_x|_2^2 + J_1(t),$$

where $J_1(t)$ is a continuous functional of the L_2-norm of W, and hence by Proposition 1, is uniformly bounded on bounded time intervals, independently of ϵ in $(0, 1)$. Note that the coefficients in J_1 depend only on the L_2- and the L_∞-norms of ϕ and its derivatives and hence according to Lemma 1, depends only on $|g'|_2$ and $|g|_\infty$. □

The difficulty in obtaining *a-priori* bounds for the case $\nu = 0$ occurs at the next stage. The bound on the H^2-seminorm of the solution is established in the next proposition.

Proposition 4. *Let $W = U - \phi$ as in Proposition 1, and assume that $\nu = 0$. Given any $T > 0$, there is an $\epsilon_0 = \epsilon_0(T, \|g'\|_2, |g|_\infty) > 0$ and a continuous function $M_2(t)$ such that for $0 < \epsilon < \epsilon_0$, the inequality*

$$|W_{xx}|_2 \leq M_2(t)$$

holds for $0 \leq t \leq T$. The function M_2 depends only on T, $\|g'\|_2$ and $|g|_\infty$.

Proof. Multiply equation (3.11) by the expression $W_{xxxx} + (W + \phi)W_{xx} + W_x^2$ and integrate over $\mathbb{R}$. After several integrations by parts and cancellations, we are led to the equation

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left\{(1 - \epsilon(W + \phi))\, W_{xx}^2 + \epsilon W_{xxx}^2\right\} dx$$
$$= -\int_{-\infty}^{\infty} \left\{W W_{xx} W_t + W_x^2 W_t + \phi W_{xx} W_t + 2\epsilon W_x W_{xx} W_{xt} + \frac{\epsilon}{2} W_t W_{xx}^2\right\} dx$$
$$-\frac{5}{2}\int_{-\infty}^{\infty} \phi_x W_{xx}^2 dx - 3\int_{-\infty}^{\infty} \phi_{xx} W_x W_{xx} dx - \int_{-\infty}^{\infty} \phi_{xxx} W W_{xx} dx$$
$$-\int_{-\infty}^{\infty} \{\psi_{xx} W_{xx} + \psi W W_{xx} + +\psi\phi W_{xx} + \psi W_x^2\} dx. \tag{3.16}$$

Fix the time $T > 0$. According to Propositions 1 and 2, the H^1-norm of W is uniformly bounded on the temporal interval $[0, T]$ by the quantity $(M_0(T)^2 + M_1(T)^2)^{\frac{1}{2}}$, independently of $1 \geq \epsilon > 0$. By an elementary imbedding theorem, it is inferred that $|W(\cdot, t)|_\infty$ is also uniformly bounded for $0 \leq t \leq T$, independently of ϵ in the aforementioned interval. Consequently, there exists an $\epsilon_0 > 0$ such that if $0 < \epsilon \leq \epsilon_0$, then

$$\epsilon |W(\cdot, t) + \phi|_\infty \leq \tfrac{1}{2}$$

for $0 \leq t \leq T$. Using this fact and the H^1- and L_∞-bounds, the following inequality is inferred from (3.16):

$$\begin{aligned} &\tfrac{1}{4}\int_{-\infty}^{\infty} \left(W_{xx}^2 + \epsilon W_{xxx}^2\right)(x, t)dx - \tfrac{1}{4}\int_{-\infty}^{\infty} \left(W_{xx}^2 + \epsilon W_{xxx}^2\right)(x, 0)dx \qquad (3.17) \\ &\leq C_1 + \int_0^t \left[C_2|W_{xx}|_2^2 + \tfrac{\epsilon}{2}|W_t|_\infty |W_{xx}|_2^2 + 2\epsilon |W_x|_\infty |W_{xx}|_2 |W_{xt}|_2 + C_3|W_t|_2\right]ds, \end{aligned}$$

where the constants C_i, $1 \leq i \leq 3$, depend only on $M_0(T)$, $M_1(T)$ and on various norms of ϕ which can be referred back to $|g'|_2$ and $|g|_\infty$ via Lemma 1, and so are independent of ϵ in $(0, \epsilon_0]$.

Information is now derived from (3.17). For convenience, two auxiliary variables are introduced, namely,

$$A^2(t) = \int_{-\infty}^{\infty} W_{xx}^2 dx \quad \text{and} \quad B^2(t) = \int_{-\infty}^{\infty} \left(W_t^2 + \epsilon W_{xt}^2\right) dx. \qquad (3.18)$$

We aim at obtaining bounds for the terms on the right-hand side of (3.17) by using a combination of the functions $A(t)$ and $B(t)$. First note (see [5, Lemma 3])

$$\begin{aligned} &\text{(i)} \quad |W_t|_\infty \leq \epsilon^{-\frac{1}{4}} B(t), \\ &\text{(ii)} \quad |W_x|_\infty \leq C A^{\frac{1}{2}}(t), \\ &\text{(iii)} \quad |W_{xt}|_2 \leq \epsilon^{-\frac{1}{2}} B(t), \end{aligned}$$

from which follows

$$\epsilon |W_t|_\infty |W_{xx}|_2^2 \leq \epsilon^{\frac{3}{4}} B(t) A^2(t)$$

and

$$\epsilon |W_x|_\infty |W_{xt}|_2 |W_{xx}|_2 \leq \epsilon^{\frac{1}{2}} C A^{\frac{3}{2}}(t) B(t).$$

Some control on $B(t)$ is needed. To this end, differentiate (3.11) for $\nu = 0$ with respect to t and multiply by W_t. On integrating by parts over $\mathbb{R}$, the following relation emerges:

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} (W_t^2 + \epsilon W_{xt}^2)dx &= -\int_{-\infty}^{\infty} (W_x + \phi_x) W_t^2 dx \leq (|W_x|_\infty + |\phi_x|_\infty) \int_{-\infty}^{\infty} W_t^2 dx \\ &\leq (|W_x|_2^{\frac{1}{2}} |W_{xx}|_2^{\frac{1}{2}} + |\phi_x|_\infty) \int_{-\infty}^{\infty} W_t^2 dx \leq C(|W_{xx}|_2^{\frac{1}{2}} + 1) \int_{-\infty}^{\infty} W_t^2 dx, \end{aligned}$$

where the constant depends on $M_0(T)$, $M_1(T)$ and g, but is independent of $\epsilon \leq \epsilon_0$. With the notation introduced above, this inequality can be expressed in the form

$$B^2(t) \leq B^2(0) + 2C \int_0^t (A^{\frac{1}{2}}(\tau) + 1) B^2(\tau) d\tau,$$

since

$$\left(|W_{xx}|_2^{\frac{1}{2}} + 1\right) \leq 2\left(A^{\frac{1}{2}}(\tau) + 1\right).$$

To obtain a bound on the initial value $B(0)$, multiply the regularized equation (3.11) with $\nu = 0$ by W_t and integrate over $\mathbb{R}$. After a suitable integration by parts, the following relation appears:

$$\int_{-\infty}^{\infty} (W_t^2 + \epsilon W_{xt}^2)\, dx = -\int_{-\infty}^{\infty} W_t \,(W W_x + (W\phi)_x + W_{xxx} + \phi_{xxx})\, dx$$
$$\leq B(t)\,(|W|_\infty |W_x|_2 + |\phi|_\infty |W_x|_2 + |\phi_x|_\infty |W|_2 + |W_{xxx}|_2 + |\phi_{xxx}|_2).$$

Simplifying the above expression by canceling the factor $B(t)$ on both sides of the inequality, we are led to

$$B(t) \leq \|\phi\|_3 + \|W\|_3\,(C + \|W\|_1)\,,$$

where C depends only on norms of ϕ. Setting $t = 0$ in this last inequality, it is found that

$$B(0) \leq \|\phi\|_3 + \|g - \phi\|_3 (C + \|g - \phi\|_1).$$

Using the foregoing analysis and (3.17), we arrive at the following system of inequalities:

$$\begin{aligned} A^2 &\leq C_1 + \tfrac{1}{2}\epsilon^{\frac{1}{2}} C_2 \int_0^t (BA^2 + B + BA^{\frac{3}{2}}) d\tau + \tfrac{1}{2} C_3 \int_0^t (A^2 + A) d\tau, \\ B^2 &\leq C_4 + C_5 \int_0^t A^{\frac{1}{2}} B^2 d\tau + C_6 \int_0^t B^2 d\tau, \end{aligned} \tag{3.19}$$

in which the constants are independent of $\epsilon \leq \epsilon_0$, but depend on T, $M_0(T)$, $M_1(T)$ and g in the ways appearing previously. The analysis of the system (3.19) now presents itself. Define a new quantity D by $D^2 = A^2 + 1$. Using the fact that $A^{\frac{3}{2}}$ and $A^{\frac{1}{2}}$ are bounded above by $A^2 + 1$, it follows from (3.19) that

$$\begin{aligned} D^2 &\leq C_1 + \epsilon^{\frac{1}{2}} C_2 \int_0^t B D^2 d\tau + C_3 \int_0^t D^2 d\tau, \\ B^2 &\leq C_4 + C_5 \int_0^t D^{\frac{1}{2}} B^2 d\tau + C_6 \int_0^t B^2 d\tau, \end{aligned}$$

where the constants have the dependence mentioned below (3.19), and so are independent of $\epsilon \le \epsilon_0$ for $0 < t \le T$. Next, define G and H by the equations

$$G^2 = C_1 + \epsilon^{\frac{1}{2}} C_2 \int_0^t HG^2 d\tau + C_3 \int_0^t G^2 d\tau,$$
$$H^2 = C_4 + C_5 \int_0^t G^{\frac{1}{2}} H^2 d\tau + C_6 \int_0^t H^2 d\tau.$$

If (G, H) solves this latter system and (D, B) satisfies the related integral inequality, then clearly $D(t) \le G(t)$ and $B(t) \le H(t)$ for $0 \le t \le T$. To analyze the system satisfied by G and H, it is helpful to make the change $w = G^{\frac{1}{2}}$ and $z = H$ of the dependent variables. It is also convenient to choose the constants C_3 and C_6 above so that $C_3 = 2C_6 = \lambda$, say. The resulting system can then be written in the form

$$\begin{aligned} \frac{d}{dt}\left(we^{-\lambda t}\right) &= \epsilon^{\frac{1}{2}} A_1 e^{-\lambda t} wz, \\ \frac{d}{dt}\left(ze^{-\lambda t}\right) &= A_2 e^{-\lambda t} wz. \end{aligned} \tag{3.20}$$

Having arrived at this point, a little calculation reveals that

$$w = \epsilon^{\frac{1}{2}} \frac{A_1}{A_2} z + Ce^{\lambda t} \tag{3.21}$$

for some constant C determined by the initial data. When this expression is substituted in the second equation in (3.20), a quadratic differential equation in z results which can be solved explicitly in the form

$$z(t) = z(0) \frac{e^{R(t)}}{1 - \epsilon^{\frac{1}{2}} A_1 S(t) z(0)} \tag{3.22}$$

where

$$R(t) = \lambda t + CA_2 \frac{e^{\lambda t} - 1}{\lambda} \quad \text{and} \quad S(t) = \int_0^t R(s)ds.$$

Given $T > 0$, choose ϵ_0 so that the relation above (3.17) holds, and $\epsilon_0^{\frac{1}{2}} A_1 S(T) z(0) \le \frac{1}{2}$, say. Then if $0 \le \epsilon \le \epsilon_0$, it follows that $z(t) \le 2z(0)exp(R(T))$ for $0 \le t \le T$. Once z is known to be bounded on $[0, T]$, independently of $\epsilon \le \epsilon_0$, w is seen to have the same property because of (3.21).

Tracing these bounds back leads to the desired bound for W_{xx} which is independent of small ϵ. This completes the proof of the proposition. □

Finally, *a-priori* bounds on the higher-order derivatives are proved in the next proposition.

Proposition 5. *Let $T > 0$ be given and suppose initial data g is provided that satisfies* (3.1) *for a value $k = m+1$ where $m \geq 3$. Let ϵ_0 be as in Proposition* 4 *relative to g and T. Then for $0 < \epsilon \leq \epsilon_0$, the solution W of the regularized problem* (3.11) *is bounded in $C(0,T;H^m)$ with a bound that depends only on T, $|g|_\infty$, and $\|g'\|_{m-1}^2 + \epsilon|g^{(m+1)}|_2^2$.*

Proof. The proof is by induction. By Proposition 4, W is bounded in $C(0,T;H^2)$ with a bound depending only on T, $|g|_\infty$ and $\|g'\|_2$. Let $m > 2$ and suppose by induction that W is bounded in $C(0,T;H^{m-1})$ independently of $\epsilon \in (0, \epsilon_0]$ with a bound depending only on T, $\|g'\|_{m-1}$ and $|g|_\infty$.

Multiply the regularized equation (3.11) by $W_{(2m)} = \partial_k^{2m} W$ and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{\infty}\left\{W_{(m)}^2 + \epsilon W_{(m+1)}^2\right\}dx\\ &= -\int_{-\infty}^{\infty}\frac{1}{2}W_{(m)}[(W+\phi)^2]_{(m+1)}dx - \int_{-\infty}^{\infty}W_{(m)}(W+\phi)_{(m+3)}dx,\end{aligned} \tag{3.23}$$

after integrations by parts. By the induction hypothesis, we have,

$$\begin{aligned}|W_{(j)}|_2 &\leq C, \quad \text{for } j \leq m-1,\\ |W_{(j)}|_\infty &\leq C, \quad \text{for } j \leq m-2,\end{aligned} \tag{3.24}$$

where the constant C depends only on T, $|g'|_2$ and $|g|_\infty$. Straightforward analysis shows that, in light of the bounds in (3.24), the most troublesome terms on the right-hand side of (3.23) are

$$\int_{-\infty}^{\infty} W_{(m)}W_{(m+1)}W\,dx, \quad \int_{-\infty}^{\infty} W_{(m)}^2 W_x\,dx, \quad \int_{-\infty}^{\infty} W_{(m)}W_{(m-1)}W_{xx}\,dx. \tag{3.25}$$

The first term in (3.25) is equivalent to the second after an integration by parts. Since $m \geq 3$, $|W_x|_\infty$ is bounded because of the induction hypothesis, and so the first two terms are bounded by a constant, which is independent of $0 \leq t \leq T$ and $0 < \epsilon \leq \epsilon_0$, times $|W_{(m)}|_2^2$. The last term in (3.25) is bounded by a similar constant times $|W_{(m)}|_2^{3/2}$ since $m \geq 3$. The other terms are bounded outright, or bounded in terms of a lower power of $|W_{(m)}|_2$. When the dust settles, there appears a differential inequality of the form

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left(W_{(m)}^2 + \epsilon W_{(m+1)}^2\right)dx \leq C_1|W_{(m)}|_2^2 + C_2.$$

If we define

$$E_m(t) = \int_{-\infty}^{\infty}\left(W_{(m)}^2 + \epsilon W_{(m+1)}^2\right)dx,$$

then the above inequality implies that

$$\frac{d}{dt}E_m(t) \leq C_1 E_m(t) + C_2,$$

from which it follows at once that

$$E_m(t) \le E_m(0)e^{C_1 t} + \tfrac{C_2}{C_1}\left(e^{C_1 t} - 1\right),$$

where

$$E_m(0) = \int_{-\infty}^{\infty} \{(\phi^{(m)} - g^{(m)})^2 + \epsilon(\phi^{(m+1)} - g^{(m+1)})_2\}dx.$$

The inductive step is now complete and the advertised result is established. □

4. Convergence of the approximations. In this section the behavior as $\epsilon \downarrow 0$ of the solutions to the regularization of the KdV and the KdV-Burgers equation (3.8) is studied. It is shown that these solutions converge strongly to the solution of the unregularized initial-value problem. This result is obtained by dint of regularizing the initial data as well as regularizing the equation, following again the line of argument in [5].

In more detail, if g is the bore-like initial data satisfying (3.1), let ϕ connote the smooth version of g constructed as in Lemma 1. Form the difference $h = g - \phi$, which is an L_2-function according to Lemma 1. The function h is then smoothed in an ϵ-dependent way by convolution. The smooth version of h will then be set as initial data for the solution $W = W_\epsilon$ of (3.11). The directed family $\{W_\epsilon\}_{0<\epsilon\le\epsilon_0}$ is then shown to be Cauchy in an appropriate function class. Its limit W is such that $V = W + \phi$ solves the originally posed initial-value problem (3.7).

The program just outlined commences with some preliminary results. For $h \in H^s$ and $\epsilon > 0$, the regularization h_ϵ of h is defined by

$$\widehat{h_\epsilon}(k) = \psi_\epsilon(k)\widehat{h}(k) = \psi(\epsilon^{1/6}k)\widehat{h}(k), \tag{4.1}$$

where ψ is an even C^∞-function with compact support such that $0 \le \psi \le 1$ everywhere, $\psi(0) = 1$, and $\xi(k) = 1 - \psi(k)$ has a zero of infinite order at 0. It follows from these stipulations that the inverse Fourier Transform $\check{\psi}$ of ψ is, along with all its derivatives, a real-valued L_1-function. Moreover, $\check{\psi}$ has total mass $\int_{-\infty}^{\infty} \check{\psi}(x)dx$ equal to 1.

Lemma 2. *Let $h \in H^s$ and let h_ϵ be the smoothed version of h defined in* (4.1). *Then, as $\epsilon \downarrow 0$,*

$$\|h_\epsilon\|_{s+j} = \mathcal{O}(\epsilon^{-\frac{1}{6}j}), \qquad \text{for } j = 1, 2, \dots,$$
$$\|h - h_\epsilon\|_{s-j} = o(\epsilon^{\frac{1}{6}j}), \qquad \text{for } j = 1, 2, \dots,$$
$$\|h - h_\epsilon\|_s = o(1).$$

Furthermore, the first bound holds uniformly on bounded subsets of H^s, and the last two hold uniformly on compact subsets of H^s. The second bound holds uniformly on bounded subsets of H^s if $o(\epsilon^{\frac{1}{6}j})$ is replaced by $\mathcal{O}(\epsilon^{-\frac{1}{6}j})$.

For a proof, see [5, p. 569].

Corollary 1. *Let g satisfy* (3.1) *for some $s \geq 3$, let ϕ be as in Lemma* 1 *relative to g, set $h = g - \phi$ and, for $\epsilon > 0$, let h_ϵ be as defined in Lemma* 2. *Let W_ϵ connote the solution of the initial-value problem* (3.11) *with initial data h_ϵ. Then for any $T > 0$, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, W_ϵ is uniformly bounded in $C(0, T; H^s)$. Moreover, for any $m > 0$, $\epsilon^{m/6} W_\epsilon$ is uniformly bounded in $C(0, T; H^{s+m})$ for $0 < \epsilon \leq \epsilon_0$.*

Proof. By applying Propositions 3, 4, and 5, it is ascertained that there is a constant C such that for $0 \leq t \leq T$ and $0 < \epsilon \leq \epsilon_0$, where ϵ_0 is determined in Proposition 4, one has

$$\| W_\epsilon \cdot, t)\|_s \leq C = C(\|h_\epsilon\|_s,\ \epsilon^{1/2}\|h_\epsilon\|_{s+1},\ |g'|_2,\ |g|_\infty,\ T).$$

By Lemma 2, it is seen that

$$\|h_\epsilon\|_s \leq C\|h\|_s, \quad \text{and} \quad \epsilon^{\frac{1}{2}}\|h_\epsilon\|_{s+1} \leq C\epsilon^{\frac{1}{3}}\|h_\epsilon\|_s.$$

Hence, independently of $t \in [0, T]$ and for sufficiently small ϵ, we see that $\| W_\epsilon \|_s$ has a bound depending only on T, ϵ_0 and $\|h\|_s$. A similar argument yields the $C(0, T; H^{s+m})$-bounds on $\epsilon^{\frac{1}{6}m} W_\epsilon$. □

Remark. By use of Proposition 4 above, one infers that W_ϵ is bounded in $C(0, T; H^1)$ with a bound that is independent of ϵ altogether.

Corollary 2. *Let the hypotheses and notation be as in Corollary* 1. *Then for any $T > 0$ and all positive $\epsilon \leq \epsilon_0$, $\partial_t W_\epsilon$ is uniformly bounded in $C(0, T; H^{s-3})$. Moreover, for $m = 1, 2, 3, 4$ or 5, $\epsilon^{m/6}\partial_x^{m+s-3}\partial_t W_\epsilon$ is bounded in $C(0, T; L_2)$, independently of ϵ in the range* $(0, \epsilon_0]$.

Proof. From equation (3.11) it is seen that

$$\left(1 - \epsilon\partial_x^2\right) W_{\epsilon t} = -\frac{\partial}{\partial x}\{\tfrac{1}{2}(W_\epsilon + \phi)^2 + (W_\epsilon + \phi)_{xx} - \nu(W_\epsilon + \phi)_x\}. \tag{4.2}$$

Inverting the operator on the left-hand side gives

$$W_{\epsilon t} = -K_\epsilon \star F_\epsilon(x, t)$$

where $K_\epsilon(x) = \frac{1}{2\epsilon^{1/2}} e^{-|x|/\epsilon^{1/2}}$ and

$$F_\epsilon(x, t) = \frac{\partial}{\partial x}\{\tfrac{1}{2}(W_\epsilon + \phi)^2 + (W_\epsilon + \phi)_{xx} - \nu(W_\epsilon + \phi)_x\}.$$

It follows that, for any r,

$$\begin{aligned} \| W_{\epsilon t}\|_r^2 = \|K_\epsilon \star F_\epsilon\|_r^2 &= \int_{-\infty}^{\infty} (1 + k^{2r})|\hat{K}_\epsilon|^2 |\hat{F}_\epsilon(k, t)|^2 dk \\ &\leq \int_{-\infty}^{\infty} (1 + k^{2r})|\hat{F}_\epsilon(k, t)|^2 dk = \|F_\epsilon(\cdot, t)\|_r^2 \end{aligned} \tag{4.3}$$

since $\hat{K}_\epsilon(z) = 1/1 + \epsilon k^2 \leq 1$, for all k. Applying (4.3) with $r = s - 3$ yields

$$\begin{aligned}\|W_{\epsilon t}(\cdot, t)\|_{s-3} &\leq \|F_\epsilon(\cdot, t)\|_{s-3} \\ &\leq C_1\|(W_\epsilon + \phi)(W_\epsilon + \phi)_x + (W_\epsilon + \phi)_{xxx} - \nu(W_\epsilon + \phi)_{xx}\|_{s-3} \\ &\leq C_2 + C_3\|W_\epsilon\|_s + C_4\|W_\epsilon\|_{s-2}\|W_\epsilon\|_{s-3} \leq \text{ Constant.}\end{aligned}$$

Similarly, if (4.3) is applied with $r = m + s - 3$, we obtain

$$\begin{aligned}\epsilon^{m/6}\|W_{\epsilon t}\|_{m+s-3} &\leq C_2' + C_3'\epsilon^{m/6}\|W_\epsilon\|_{m+s} + C_4'\epsilon^{m/6}\|W_\epsilon\|_{m+s-3}\|W_\epsilon\|_{m+s-2} \\ &\leq \text{Constant,}\end{aligned}$$

for ϵ sufficiently small, because of Corollary 2 and the restriction $m \leq 5$. □

Proposition 6. *For* $0 < \epsilon \leq \epsilon_0$, *let* W_ϵ *be the solution of* (3.11) *with initial data* h_ϵ, *where* $h \in H^s$, $s \geq 3$, *and* h_ϵ *is the regularization of* h *defined in* (4.1). *Then for any* $T > 0$, $\{W_\epsilon\}_{\epsilon>0}$ *is Cauchy in* $C(0, T; H^s)$ *as* $\epsilon \downarrow 0$.

Proof. First notice that

$$\begin{aligned}h_\epsilon(x) &= \int_{-\infty}^{\infty} \check{\psi}(x-y)h(y)dy = \int_{-\infty}^{\infty} \check{\psi}(x-y)(g(y) - \phi(y))dy \\ &= \int_{-\infty}^{\infty} \check{\psi}(x-y)g(y)dy - \int_{-\infty}^{\infty} \check{\psi}(x-y)\phi(y)dy = g_\epsilon(x) - \phi_\epsilon(x),\end{aligned}$$

the latter two integrals making sense because $\check{\psi} \in L_1$ and g and ϕ are in L_∞. Indeed, because $\partial_x^j\check{\psi}$ lies in L_1 for all j, it follows that g_ϵ and ϕ_ϵ are both C^∞-functions, all of whose derivatives are bounded. Since $\phi' \in H^\infty$, $\partial_x^j\phi_\epsilon$ lies in H^∞ for $j > 0$.

Let $W = W_\epsilon$ and $V = W_\delta$ with $\delta \leq \epsilon$. It is intended to show that for any $\alpha > 0$ and $T > 0$ there is an $\epsilon_0 = \epsilon_0(\alpha)$ such that $\|W - V\|_s \leq \alpha$ for all $\epsilon \leq \epsilon_0$ and for all $t \in [0, T]$. Since W and V satisfy the initial-value problems

$$\begin{aligned}&W_t - \epsilon W_{xxt} + (W + \phi_\epsilon)(W_x + \phi_{\epsilon x}) - \nu W_{xx} - \nu\phi_{\epsilon xx} + W_{xxx} + \phi_{\epsilon xxx} = 0, \\ &W(x, 0) = h_\epsilon(x)\end{aligned}$$

and

$$\begin{aligned}&V_t - \delta V_{xxt} + (V + \phi_\delta)(V_x + \phi_{\delta x}) - \nu V_{xx} - \nu\phi_{\delta xx} + V_{xxx} + \phi_{\delta xxx} = 0, \\ &V(x, 0) = h_\delta(x),\end{aligned}$$

respectively where ϕ_ϵ is defined above, and similarly for ϕ_δ. We find that $Y = W - V$ satisfies

$$\begin{aligned}Y_t - \delta Y_{xxt} + Y_{xxx} - \nu Y_{xx} &= (\epsilon - \delta)W_{xxt} + (YY_x - (YW)_x) - (W(\phi_\epsilon - \phi_\delta))_x \\ &\quad + (Y\phi_\delta)_x - (\phi_\epsilon\phi_{\epsilon x} + \phi_{\epsilon xxx} - \nu\phi_{\epsilon xx}) + (\phi_\delta\phi_{\delta x} + \phi_{\delta xxx} - \nu\phi_{\delta xx}), \\ Y(x, 0) &= h_\epsilon(x) - h_\delta(x) = f(x). \end{aligned} \tag{4.4}$$

Upon multiplying the differential equation in (4.4) by $Y_{(2j)} = \partial_x^{2j} Y$, where $0 \le j \le s$, and integrating over $\mathbb{R} \times [0, T]$, and after some rearrangements and integrations by parts, there obtains the relation

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\left\{Y_{(j)}^2 + \delta Y_{(j+1)}^2\right\}dx + \nu\int_0^t\int_{-\infty}^{\infty} Y_{(j+1)}^2 dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\left\{f_{(j)}^2 + \delta f_{(j+1)}^2\right\}dx + (-1)^j(\epsilon - \delta)\int_0^t\int_{-\infty}^{\infty} W_{t,(j+2)} Y_{(j)}\,dx\,d\tau \\
&\quad + (-1)^j\int_0^t\int_{-\infty}^{\infty}(YY_x - (YW)_x)_{(j)}\, Y_{(j)}\,dx\,d\tau \\
&\quad + (-1)^j\int_0^t\int_{-\infty}^{\infty}((Y\phi_\delta)_x - (W(\phi_\epsilon - \phi_\delta))_x)_{(j)}\, Y_{(j)}\,dx\,d\tau \\
&\quad + (-1)^j\int_0^t\int_{-\infty}^{\infty}\Big[(\phi_\delta\phi_{\delta x} - \phi_\epsilon\phi_{\epsilon x}) + (\phi_{\delta xxx} - \phi_{\epsilon xxx}) \\
&\quad - \nu(\phi_{\delta xx} - \phi_{\epsilon xx})\Big]_{(j)} Y_{(j)}\,dx\,d\tau.
\end{aligned} \tag{4.5}$$

for $j = 1, 2, \ldots, s$. □

The argument now proceeds by induction on j as in [5, Proposition 5]. The idea is to start at $j = 0$, use Gronwall's lemma to obtain a bound on $|Y(\cdot, t)|_2$ in terms of ϵ, proceed to the case $j = 1$ similarly, but making full use of the bound already established for $j = 0$, and so on until reaching $j = s$. In carrying out this program, we will rely upon Corollaries 1 and 2 for bounds on $W = W_\epsilon$. For j near s, terms will appear that are not bounded, independently of ϵ, but in fact blow up as ϵ tends to zero (the case $m > 0$ in Corollary 1 or Corollary 2). These terms will be offset by other terms that are not only bounded, but converge to zero at a suitable rate as ϵ tends to zero.

A couple of preliminary remarks will simplify the subsequent, inductive argument. First, note that because of the result of Lemma 2 and the fact that $\phi' \in H^\infty$, it transpires that for $0 \le j \le s$, the inner integral in the last term on the right-hand side of (4.5) admits a bound of the form

$$\begin{aligned}
&C_j|Y_{(j)}(\cdot, t)|_2(\|\phi_\epsilon' - \phi'\|_{j+2} + \|\phi_\delta' - \phi'\|_{j+2})(1 + \|\phi_\epsilon'\|_j + \|\phi_\delta'\|_j + |\phi_\epsilon|_\infty + |\phi_\delta|_\infty) \\
&\qquad \le C_j V_j(t)^{1/2} D_r \epsilon^r,
\end{aligned} \tag{4.6}$$

as $\epsilon \downarrow 0$, for any $r > 0$, where C_j depends only on j and ν and D_r depends on the H^{6r+j-2}-norm of ϕ', and hence on the L_2-norm of g' (see Lemma 1). The function V_j is defined by the integral

$$V_j(t)^2 = \int_{-\infty}^{\infty}(Y_{(j)}^2 + \delta Y_{(j+1)}^2)dx.$$

It will appear shortly that for suitable choices of r, the upper bound in (4.6) implies the last term on the right-hand side of (4.5) to be negligible compared to the other terms in (4.5) for sufficiently small values of ϵ. Similarly, for any $T > 0$ there is an $\epsilon_0 > 0$ for which the term

$$\int_{-\infty}^{\infty} (W(\phi_\epsilon - \phi_\delta))_{(j+1)} Y_{(j)} dx$$

may be bounded in absolute value by an expression of the form

$$C_{j,r}\epsilon^r |Y_{(j)}(\cdot, t)|_2, \tag{4.7}$$

uniformly for t in $[0, T]$ and ϵ in $(0, \epsilon_0]$, where $C_{j,r}$ depends upon the norm of W in $C(0, T; H^{j+1})$ and the norm of ϕ' in H^k for suitable k, and so ultimately on the L_2-norm of the initial data g' (see again Lemma 1). Note that in obtaining the upper bound in (4.7), the following remark is needed. If $j \geq s - 1$, then $|W_{(j+1)}|_\infty$ is not known to be bounded as $\epsilon \downarrow 0$, but rather may blow up at the rate $\epsilon^{-\frac{j-s}{6}-\frac{1}{4}}$. This is compensated by the fact that $|\phi_\epsilon - \phi_\delta|_2 \leq |\phi_\epsilon - \phi|_2 + |\phi - \phi_\delta|_2$ tends rapidly to zero as ϵ tends to zero, again because of the smoothness of ϕ. The upper bound exhibited in (4.7) is, for small values of ϵ, also negligible in comparison to others that arise in the analysis.

Making use of these remarks together with the results of Corollaries 1 and 2, the equations in (4.5) imply the following collection of differential inequalities:

$$\begin{aligned}\tfrac{1}{2}V_j(t) \leq \tfrac{1}{2}V_j(0) &+ \epsilon \int_0^t V_j(\tau)^{1/2} |W_{t,(j+2)}|_2 d\tau \\ &+ \left| \int_0^t \int_{-\infty}^{\infty} Y_{(j)} \left(\tfrac{1}{2}Y^2 - YW + Y\phi_\delta\right)_{(j+1)} dx d\tau \right| + C'_{j,r}\epsilon^r \int_0^t V_j(\tau)^{1/2} d\tau,\end{aligned} \tag{4.8}$$

for $j = 0, 1, \cdots, s$.

The argument has now reached a point where all the ingredients are available to proceed exactly as in the proof of [5, Proposition 5]. Briefly, consideration is given consecutively to each of the differential inequalities in (4.8) starting with $j = 0$. Taking $r = 1$ in (4.6), say, and using the bounds available from Corollaries 1 and 2, one derives the differential inequality

$$V_0(t) \leq V_0(0) + \frac{1}{2}C_0\epsilon^{2/3} \int_0^t V_0(\tau)^{1/2} d\tau + \frac{1}{2}D_0 \int_0^t V_0(\tau) d\tau, \tag{4.9}$$

which is valid for $0 \leq t \leq T$ and the constants C_0 and D_0 depend only on T and the norms $\|g'\|_2$ and $|g|_\infty$ of the initial data g. It follows at once from Gronwall's lemma that

$$V_0(t)^{1/2} \leq V_0(0)^{1/2} e^{D_0 T} + \epsilon^{2/3} C_0 \frac{e^{D_0 T} - 1}{D_0}, \tag{4.10}$$

for $0 \leq t \leq T$. Combining this inequality with what we know from Lemma 2 about the dependence upon ϵ of the initial data h_ϵ, it is concluded that

$$|Y(\cdot, t)|_2 \leq E_0 \epsilon^{1/2} \tag{4.11}$$

for $0 \leq t \leq T$ and $0 < \epsilon \leq \epsilon_0$. If $s > 3$, a higher power of ϵ can be obtained. Consider next the case $j = 1$. Carrying out the same sort of estimates that led to (4.9) and making use of the already derived (4.11) yields the inequality

$$|Y_x(\cdot, t)|_2 \leq E_1 \epsilon^{1/3}, \tag{4.12}$$

valid for $0 \leq t \leq T$ and $0 < \epsilon \leq \epsilon_0$, where E_1 depends only on the H^1-norm of h. Continuing in this manner leads to the desired conclusion. We may safely rely upon [5, §5] for details. It deserves remark that when $j = s$, the inequality one derives in place of (4.12) has the form

$$|\partial_x^s Y(\cdot, t)|_2 \leq E_s \left[\epsilon^{1/6} + \|h_\epsilon - h\|_s + \|h_\delta - h\|_s\right], \tag{4.13}$$

where E_s depends only on the H^s-norm of h. □

Corollary 3. *The family* $\{\partial_t W_\epsilon\}_{\epsilon>0}$ *is Cauchy in* $C(0, T; H^{s-3})$ *as* $\epsilon \downarrow 0$.

Proof. As before, let $Y = W - V$ where $W = W_\epsilon$ and $V = W_\delta$ with $\epsilon \geq \delta$. Then Y satisfies equation (4.5). The convergence of δY_{xxt} and $(\epsilon - \delta) V_{xxt}$ follows by Corollary 2 and the rest of the terms on the right hand side of (4.5) converge by virtue of the last proposition. □

Theorem 3. *Let g be as in* (3.1) *and let $h = g - \phi$ where ϕ is as specified in Lemma* 1 *relative to g. Then there exists a unique solution W of the equation*

$$W_t + (W + \phi)(W + \phi)_x + (W + \phi)_{xxx} - \nu(W + \phi)_{xx} = 0,$$

with initial data

$$W(x, 0) = g(x) - \phi(x).$$

Moreover, $W \in C(0, T; H^s)$ for all finite $T > 0$, and $U = W + \phi$ is a solution of the KdV equation with initial data g.

Proof. Uniqueness is immediate, for if y and z are two solutions, define $w = y - z$ so that $w(x, 0) = 0$ and w satisfies the differential equation

$$w_t + w_{xxx} - \nu w_{xx} + (wy)_x - yw_x + (\phi w)_x = 0.$$

Multiply the above equation by w and integrate over $\mathbb{R}$ to reach the relation

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx + \nu \int_{-\infty}^{\infty} w_x^2 dx = -\int_{-\infty}^{\infty} w(wy)_x dx + \int_{-\infty}^{\infty} wyw_x dx - \int_{-\infty}^{\infty} w(w\phi)_x dx.$$

Upon simplification, we are led to the inequality,

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2 dx + \nu \int_{-\infty}^{\infty} w_x^2 dx \leq |y_x|_\infty |w|_2^2 + \tfrac{1}{2}|\phi_x|_\infty |w|_2^2 \leq C|w|_2^2,$$

and Gronwall's lemma then implies $w(\cdot, t) = 0$ in $L_2(\mathbb{R})$ for all $t \geq 0$, and since w is continuous, w is identically zero, whence $y \equiv z$.

Attention is now turned to the question of existence of the solution. Let ϕ_ϵ and g_ϵ be the regularizations of ϕ and g as defined in (3.11) and let w_ϵ be the solution of problem (3.14) with initial data $g_\epsilon - \phi_\epsilon$. Then for each fixed $T > 0$, as $\epsilon \downarrow 0$

$$\begin{aligned} w_\epsilon &\longrightarrow W && \text{in } C(0, T; H^s), \\ \partial_t w_\epsilon &\longrightarrow \partial_t W = \partial_t U && \text{in } C(0, T; H^{s-3}). \end{aligned} \tag{4.14}$$

It follows that

$$\begin{aligned} \partial_x w_\epsilon^2 &\longrightarrow \partial_x (W^2) && \text{in } C(0, T, H^{s-1}), \\ \partial_{xxx} w_\epsilon &\longrightarrow \partial_{xxx} W && \text{in } C(0, T; H^{s-3}), \end{aligned} \tag{4.15}$$

since $\partial_t w_\epsilon$ is bounded in $C(0, T; H^{s-3})$ and $\partial_x^2 \partial_t w_\epsilon$ is bounded in $C(0, T; H^{s-5})$. Hence, in the sense of distributions, $\epsilon \partial_x^2 \partial_t w_\epsilon \longrightarrow 0$, as $\epsilon \downarrow 0$, and (4.6) then implies that $w_\epsilon \longrightarrow W$ in the sense of distributions, so that $\partial_t w_\epsilon \longrightarrow \partial_t W$ as distributions and hence $U_t = \partial_t W$. This, combined with (4.14) and (4.15), shows that at least in the sense of distributions,

$$\begin{aligned} &W_t + (W + \phi)(W + \phi)_x + (W + \phi)_{xxx} - \nu (W + \phi)_{xx} = 0, \\ &W(x, 0) = g(x) - \phi(x). \end{aligned} \tag{4.16}$$

Since $W = U - \phi$, we have, in the sense of distributions

$$U_t + U U_x + U_{xxx} - \nu U_{xx} = 0, \quad U(x, 0) = g(x).$$

Since $W \in C(0, T; H^s)$ and $W_t \in C(0, T; H^{s-3})$, and $s \geq 3$, then W is an L_2 solution of (4.16). If $s > 3$ then W is a solution in the classical sense because then all the terms in the equation lie in $C(0, T; H^1(\mathbb{R}))$ and hence are bounded, continuous functions of (x, t). Hence in case $s > 3$, $U = W + \phi$ is a solution of the KdV equation in the classical sense. □

5. Continuous dependence of solutions on the initial data. In this section it will be shown that the mapping $\mathcal{U}$ which assigns to each continuous function g satisfying (3.1) the unique solution u of the KdV-Burgers equation with initial data g, is continuous in the following precise sense. For $s \geq 1$, define the function class

$$\mathcal{P}_s = \left\{ g : g \in C_b(\mathbb{R}), \quad g \longrightarrow C_\pm \text{ as } x \longrightarrow \pm\infty, g' \in H^{s-1} \right\}.$$

Then we have the following theorem.

Theorem 4. *Let $\nu \geq 0$, $s \geq 3$ and $T > 0$ be given and suppose $\{g_n\}_{n=1}^\infty$ and g all lie in $\mathcal{P}_s$. If*

$$\lim_{n \uparrow \infty} (g_n - g) = 0 \quad \text{in } H^s,$$

then,

$$\lim_{n\uparrow\infty}(U_n - U) = 0 \quad in \;\; C(0, T; H^s),$$

where U_n and U are the unique solutions of the KdV-Burgers equation with initial data g_n and g, respectively.

Proof. Let ϕ_n and ϕ be defined as in Lemma 1 relative to g_n and g, $n = 1, 2, \ldots$, and set $W_n = U - \phi_n$, $W = U_n - \phi$. The theorem will be proved if we show that

$$\lim_{n\to\infty} \|W_n - W\|_s = 0$$

uniformly for $t \in [0, T]$. For then

$$\|U_n - U\|_s \leq \|W_n - W\|_s + \|\phi_n - \phi\|_s$$

and

$$\|\phi_n - \phi\|_s = \|\rho_*(g_n - g)\|_s \leq C_s\|g_n - g\|_s,$$

so that $(U_n - U) \to 0$ in H^s, uniformly for $t \in [0, T]$.

Let W_ϵ^n and W_ϵ be the solutions of the regularized equation (3.11) with initial data $(g_\epsilon^n - \phi_\epsilon^n)$ and $(g_\epsilon - \phi_\epsilon)$, respectively. By the triangle inequality, we see that

$$\|W^n - W\|_s \leq \|W^n - W_\epsilon^n\|_s + \|W_\epsilon^n - W_\epsilon\|_s + \|W_\epsilon - W\|_s.$$

Since $g_n - g$ converges to zero in H^s, it follows as before that the regularizations $\{\phi_n\}_{n=1}^\infty$ of $\{g_n\}_{n=1}^\infty$ are such that $\phi_n - \phi$ also converges to zero in H^s. Hence the sequence $\{h_n\}_{n=1}^\infty$ defined by $h_n = g_n - \phi_n$ converges to $g - \phi$ in H^s and thus the set $\{h_n : n = 1, 2, \ldots\} \cup \{h\}$ comprises a compact subset of H^s. In consequence of this remark, the third bound in Lemma 2, and the outcome of Proposition 6, it is adduced that the families $\{W_\epsilon^n\}_{0<t\leq 1}$, $n = 1, 2, \ldots$, and $\{W^\epsilon\}_{0<\epsilon\leq 1}$ are uniformly Cauchy in $C(0, T; H^s)$. Therefore, given a $\delta > 0$, an $\epsilon_0 > 0$ can be chosen so that for all ϵ in $(0, \epsilon_0]$ and for all $n = 1, 2, \ldots$,

$$\|W^n - W_\epsilon^n\|_s \leq \tfrac{\delta}{3}, \qquad \|W - W_\epsilon\|_s \leq \tfrac{\delta}{3}.$$

Fixing ϵ in the range $(0, \epsilon_0)$, it then remains to show that there exists an n_0 such that for all $n \geq n_0$

$$\|W_\epsilon^n - W_\epsilon\|_s \leq \tfrac{\delta}{3}.$$

For this fixed ϵ, use is made of the transformation

$$V_\epsilon(x, t) = \epsilon U_\epsilon(\epsilon^{1/2}(x - t), \epsilon^{3/2}t)$$

used already in [5].

Since ϵ is fixed for the moment, the notation may be simplified by dropping the subscript ϵ in the ensuing calculations, so $W = W_\epsilon$, $U = U_\epsilon$, $V = V_\epsilon$, $\phi = \phi_\epsilon$, and $g = g_\epsilon$. With this proviso, we have

$$W(x,t) = U(x,t) - \phi(x) = \epsilon^{-1}[V((x+\epsilon^{-1}t)\epsilon^{-\frac{1}{2}}, t\epsilon^{-\frac{3}{2}}) - \tilde{\phi}(\epsilon^{-\frac{1}{2}}x)], \qquad (5.1)$$

where $\tilde{\phi}(x) = \epsilon\phi(\epsilon^{\frac{1}{2}}x)$. Define y to be $V - \tilde{\phi}$, so that y satisfies the initial-value problem

$$\begin{aligned} &y_t + (y+\tilde{\phi})_x - y_{xxt} + (y+\tilde{\phi})(y+\tilde{\phi})_x - \nu(y+\tilde{\phi})_{xx} = 0, \\ &y(x,0) = h(x) = \epsilon^{-1}[g(\epsilon^{\frac{1}{2}}x) - \tilde{\phi}(x)]. \end{aligned} \qquad (5.2)$$

Let $y^n = V^n - \tilde{\phi}^n$. We show that if $h^n - h \longrightarrow 0$ in H^s as $n \longrightarrow \infty$, then $y^n \longrightarrow y$ in $C(0,T;H^s)$. If $Z^n = y^n - y$, then Z^n satisfies

$$\begin{aligned} Z_t^n + Z_x^n - Z_{xxt}^n - \nu Z_{xx}^n + Z^n Z_x^n + (yZ^n)_x + (\tilde{\phi}^n Z^n)_x + ((\tilde{\phi}^n - \tilde{\phi})y)_x \\ + (\tilde{\phi}^n\tilde{\phi}_x^n) - (\tilde{\phi}\tilde{\phi}_x) + (\tilde{\phi}^n - \tilde{\phi})_x - \nu(\tilde{\phi}^n - \tilde{\phi})_{xx} = 0. \end{aligned} \qquad (5.3)$$

Letting $\beta^n = \tilde{\phi}^n - \tilde{\phi}$, we may write

$$\begin{aligned} (\tilde{\phi}^n Z^n)_x - (\tilde{\phi}y)_x &= [y\beta_x^n + y_x\beta^n] + \tilde{\phi}^n{}_x Z^n + \tilde{\phi}^n Z_x^n, \\ \tilde{\phi}^n\tilde{\phi}^n{}_x - \tilde{\phi}\tilde{\phi}_x &= \tilde{\phi}^n\beta_x^n + \tilde{\phi}_x\beta^n, \\ (\tilde{\phi}^n - \tilde{\phi})_x &= \beta_x^n, \\ (\tilde{\phi}^n - \tilde{\phi})_{xx} &= \beta_{xx}^n. \end{aligned}$$

The quantity

$$\gamma^n = \beta_{xx}^n + (y + \tilde{\phi}^n + 1)\beta_x^n + (y_x + \tilde{\phi}_x)\beta^n$$

will also appear presently. In the following computations, the superscript n is also dropped. Define the quantity E_j by

$$E_j(t) = \int_{-\infty}^{\infty} \{Z_{(j)}^2 + Z_{(j+1)}^2\}\, dx.$$

Multiply equation (5.3) by $Z_{(2j)}$ and integrate over $\mathbb{R}$. After appropriate integrations by parts, there appears

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E_j(t) &= \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\{Z_{(j)}^2 + Z_{(j+1)}^2\}\, dx \\ &= (-1)^{j+1}\int_{-\infty}^{\infty}\Big\{ZZ_x + (yZ)_x + \tilde{\phi}_x Z + \tilde{\phi}Z_x\Big\} Z_{(2j)}dx \\ &+ (-1)^{j+1}\int_{-\infty}^{\infty}(\gamma^n)_{(j)}Z_{(j)}dx - \nu\int_{-\infty}^{\infty} Z_{(j+1)}^2 dx. \end{aligned} \qquad (5.4)$$

Integrate over $[0, t]$ to obtain

$$E_j(t) = E_j(0) + 2(-1)^{j+1}\int_0^t\int_{-\infty}^{\infty}\{ZZ_x + (yZ)_x + \tilde{\phi}_x Z + \tilde{\phi} Z_x\} Z_{(2j)}\,dx\,d\tau$$
$$+ 2(-1)^{j+1}\int_0^t\int_{-\infty}^{\infty}(\gamma^n)_{(j)} Z_{(j)}\,dx\,d\tau - \nu\int_0^t\int_{-\infty}^{\infty} Z_{(j+1)}^2\,dx\,d\tau, \qquad (5.5)$$

where

$$\int_{-\infty}^{\infty}(\gamma^n)_{(j)} Z_{(j)}\,dx \le |\gamma^n_{(j)}|_2 |Z_{(j)}|_2 \le \text{const}\,|\beta^n_{(j)}|_2 |Z_{(j)}|_2$$

and $|\beta^n_{(j)}(\cdot, s)|_2 \longrightarrow 0$ as $n \to \infty$, uniformly for $0 \le s \le t$. Integrating by parts and using Leibnitz rule gives

$$E_j(t) \le E_j(0) + C\int_0^t\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{j+1} Z_{(j+1-k)} Z_{(k)} Z_{(j)} + \sum_{k=0}^{j+1} Z_{(j+1-k)} y_{(k)} Z_{(j)}\Big)\,dx\,d\tau$$
$$+ \int_0^t\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{j+1}\tilde{\phi}_{j+2-k} Z_{(k)} Z_{(j)} + \sum_{k=0}^{j+1} Z_{(j+1-k)}\tilde{\phi}_k Z_{(j)}\Big)\,dx\,d\tau \qquad (5.6)$$
$$+ \text{const.}\int_0^t |\beta_j^n|_2 |Z_j|_2\,d\tau,$$

where the constants depend only on j. If $j = 0$, we are led to the inequality

$$E_0(t) \le E_0(0) + C_1\int_0^t E_0(\tau)\,d\tau + C_2\int_0^t |\beta^n|_2 E_0^{\frac{1}{2}}(\tau)\,d\tau.$$

Using Gronwall's lemma to solve this inequality yields

$$E_0^{\frac{1}{2}}(t) \le E_0^{\frac{1}{2}}(0) e^{\frac{C_1 t}{2}} + \frac{C_2|\beta^n|_2}{C_1}\big(e^{\frac{C_1 t}{2}} - 1\big),$$

from which it follows that $\lim_{n\to\infty}\|Z\|_1 = 0$ uniformly on $[0, T]$. Now assume inductively that $|Z_{(j)}|_2 \longrightarrow 0$ as $n \to \infty$, uniformly on $[0, T]$. From (5.6), one obtains

$$E_j(t) \le E_j(0) + C\Big|\int_0^t \big(E_j(\tau) + a_n E_j(\tau)^{1/2}\,d\tau\big)\Big|,$$

where $a_n \longrightarrow 0$ as $n \to \infty$. Therefore for $t \in [0, T]$, we have,

$$E_j^{\frac{1}{2}}(t) \le E_j^{\frac{1}{2}}(0) e^{\frac{Ct}{2}} + a_n\big(e^{\frac{Ct}{2}} - 1\big). \qquad (5.7)$$

From (5.7) it follows immediately that $\|Z^n\|_{j+1} \longrightarrow 0$ uniformly for $t \in [0, T]$. The proof of the theorem is complete. □

6. Stability of steady Bores. In this section, attention is devoted to travelling-wave solutions of the KdV-Burgers equation

$$U_t + UU_x + U_{xxx} - \nu U_{xx} = 0. \tag{6.1}$$

Such solutions, having the form $U(x, t) = \Phi(x - ct)$, satisfy the ordinary differential equation

$$-c\Phi' + \Phi\Phi' + \Phi''' - \nu\Phi'' = 0. \tag{6.2}$$

They provide a family of models for steady bore propagation. Without loss of generality, a traveling-wave solution can be scaled so that $\Phi(z) \to 1$ as $z \to -\infty$ and $\Phi(z) \to 0$ as $z \to +\infty$. This is possible since the travelling-wave profile Φ only depends on the difference $C_- - C_+$ where $C_\pm = \lim_{z\to\pm\infty} \Phi(z)$. (cf. [4]).

Consider the initial-value problem (3.7) for the KdV-Burgers equation (6.1) and suppose that a solution U is a perturbation of a travelling-wave Φ, so that

$$U(x, t) = \Phi(x - ct) + V(x, t). \tag{6.3}$$

It will be supposed that $U(x, 0)$ satisfies the conditions in (3.1) with $C_- = 1$ and $C_+ = 0$. This latter specification is in no way crucial to our theory, but it is a convenient normalization. Since the travelling-wave profile Φ has the properties that $\Phi \in L_\infty$ and $\Phi' \in H^\infty \cap W_1^\infty$, this means that the initial perturbation $V(x, 0)$ has the property that $V(x, 0) \in H^{k+1}$, where k appears in (3.1). Because the solution U of the KdV-Burgers equation with initial value $U(x, 0) = \Phi(x) + V(x, 0)$ is such that for any $T > 0$, $U_x \in C(0, T; H^k)$ and $U - U(x, 0) \in C(0, T; H^{k+1})$, it therefore follows that $V \in C(0, T; H^{k+1})$ for any finite $T > 0$. In particular, for any $T > 0$, V and V_x are uniformly bounded on $\mathbb{R} \times [0, T]$ if $k \geq 1$.

Substituting (6.3) into (6.1) and simplifying, it is found that V satisfies the initial-value problem

$$V_t + (V\Phi)_x + VV_x + V_{xxx} - \nu V_{xx} = 0, \quad V(x, 0) = V_0(x). \tag{6.4}$$

Pego [25] has shown that perturbations of a travelling wave which have zero mass are asymptotically stable.

Theorem 5 (Pego). *Let $v(x, t)$ satisfy* (6.4) *and suppose that $V_0 = \partial_x v_0$ where $v_0 \in H^3$. If $\|v_0\|_3$ is sufficiently small, then the solution $V(x, t)$ of* (6.4) *exists in $C_b([0, \infty); H^2)$ and satisfies $\|V(\cdot, t)\|_2 \longrightarrow 0$.*

Our aim is to generalize the above result by dispensing with the zero-mass condition imposed on the initial data. We begin by proving the following lemma.

Lemma 3. *Let γ be positive number and let $V(x, t)$ satisfy* (6.4) *where $V(x, 0)$ lies in H^2, say, and is such that,*

$$(1 + x^2)^\gamma V(x, 0) \in L_2(\mathbb{R}).$$

Then for each fixed $T > 0$, there is a constant C depending on ν, T, the norm of the function $(1+x^2)^\gamma V(x,0)$ in $L_2(\mathbb{R})$, and the bounds on $V(\cdot,t)$ in H^1 such that

$$\left|(1+x^2)^\gamma V(x,t)\right|_2 \le C(T) \quad \textit{for } 0 \le t \le T.$$

Proof. Define a weight function ψ by $\psi(x) = (1+x^2)^\gamma$. Notice that $\psi(x) \ge 1$ everywhere and especially that $\psi'(x)/\psi(x) = 2\gamma x/(1+x^2)$ is bounded and tends to zero as $x \to \pm\infty$, like $2\gamma/x$.

An energy-type argument is now mounted that leads to the desired conclusion. In the process of deriving *a priori* bounds, derivatives of V of order higher than two will appear in intermediate computations, but are not featured in the final differential inequalities. Because of the continuous dependence results derived in Section 5, the calculations leading to the crucial differential inequality may be justified for more regular intial data and then, by passing to a limit, are seen to hold under the relatively mild assumptions set forth in the statement of the lemma.

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\psi^2V^2dx = \int_{-\infty}^{\infty}\psi^2VV_tdx \tag{6.5}$$
$$= -\int_{-\infty}^{\infty}\psi^2V(V\Phi)_xdx - \int_{-\infty}^{\infty}\psi^2V^2V_xdx - \int_{-\infty}^{\infty}\psi^2VV_{xxx}dx + \nu\int_{-\infty}^{\infty}\psi^2VV_{xx}dx$$
$$= I + II + III + IV.$$

Analysis of I :

$$\begin{aligned} I &= -\int_{-\infty}^{\infty}\psi^2V(V\Phi)_xdx = \int_{-\infty}^{\infty}\psi\psi_xV^2\Phi dx - \tfrac{1}{2}\int_{-\infty}^{\infty}\psi^2V^2\Phi_xdx \\ &\le 2\int_{-\infty}^{\infty}\psi^2V^2(\psi_x\Phi)dx + \tfrac{1}{2}\int_{-\infty}^{\infty}\psi^2V^2|\Phi_x|dx \\ &\le (2C_0|\Phi|_\infty + \tfrac{1}{2}|\Phi_x|_\infty)\int\psi^2V^2dx \le K_1\int_{-\infty}^{\infty}\psi^2V^2dx. \end{aligned}$$

Analysis of II :

$$II = \tfrac{2}{3}\int_{-\infty}^{\infty}\psi\psi_xV^3dx \le \tfrac{4}{3}C_0\|V\|_{L_\infty(\mathbb{R}\times[0,T])}\int_{-\infty}^{\infty}\psi^2V^2dx \le K_2\int_{-\infty}^{\infty}\psi^2V^2dx.$$

Analysis of III :

$$\begin{aligned} III &= -\int_{-\infty}^{\infty}\psi^2VV_{xxx}dx = 2\int_{-\infty}^{\infty}\psi\psi_xVV_{xx}dx + \int_{-\infty}^{\infty}\psi^2V_xV_{xx}dx \\ &= -3\int_{-\infty}^{\infty}\psi\psi_xV_x^2dx + \int_{-\infty}^{\infty}(\psi\psi_x)_{xx}V^2dx. \end{aligned}$$

For any $A > 0$,

$$-3\int_{-\infty}^{\infty}\psi\psi_x V_x^2 dx \le 3\int_{|x|\le A}|\psi\psi_x|V_x^2 dx + 3\int_{|x|>A}|\psi\psi_x|V_x^2 dx$$
$$\le 3\|\psi\psi_x\|_{L_\infty([-A,A]\times[0,T])}|V_x|_2^2 + 3\|\frac{\psi_x}{\psi}\|_{L_\infty\{x:|x|>A\})}\int_{-\infty}^{\infty}\psi^2 V_x^2 dx$$
$$\le K_3 + \alpha\int_{-\infty}^{\infty}\psi^2 V_x^2 dx,$$

where K_3 and α both depend on A and α may be taken as small as desired by choosing A large enough. As for the other term, we may simply rely on the fact that $|(\psi\psi_x)_{xx}| \le K_3\psi^2$ for some absolute constant K_3. In conclusion, one deduces that

$$III \le K_2 + \alpha\int_{-\infty}^{\infty}\psi^2 V_x^2 dx + K_3\int_{-\infty}^{\infty}\psi^2 V^2 dx.$$

Analysis of IV:

$$IV = \nu\int_{-\infty}^{\infty}\psi^2 V V_{xx} dx = -2\nu\int_{-\infty}^{\infty}\psi\psi_x V V_x dx - \nu\int_{-\infty}^{\infty}\psi^2 V_x^2 dx$$
$$\le \frac{\nu}{2}\int_{-\infty}^{\infty}\psi^2 V_x^2 dx + 2\nu\int_{-\infty}^{\infty}\psi_x^2 V^2 dx - \nu\int_{-\infty}^{\infty}\psi^2 V_x^2 dx$$
$$\le -\frac{\nu}{2}\int_{-\infty}^{\infty}\psi^2 V_x^2 dx + K_4\int_{-\infty}^{\infty}\psi^2 V^2 dx.$$

If the last four estimates are used in (6.5) and A is chosen large enough that $\alpha < \frac{1}{2}\nu$, one obtains the differential inequality

$$\frac{d}{dt}\int\psi^2 V^2 \le Q + P\int\psi^2 V^2.$$

An application of Gronwall's lemma then insures that

$$\int_{-\infty}^{\infty}\psi^2 V^2(x,t)dx \le e^{Pt}\int_{-\infty}^{\infty}\psi^2 V^2(x,0)dx + \frac{Q}{P}(e^{Pt}-1),$$

and the desired result follows. □

Now an interesting point arises. Because the evolution equation is translation invariant, if Φ is a travelling-wave profile, so are all its translates Φ_α for $\alpha \in \mathbb{R}$, where $\Phi_\alpha(x) = \Phi(x+\alpha)$. When initial data $U(x,0)$ is posed, we can think of it as a small perturbation of not just a given travelling-wave profile, but also of any of a one-parameter family of travelling-wave profiles $\{\Phi_\alpha\}_{|\alpha|<\delta}$, for δ small enough. Roughly speaking, if $U_0 - \Phi$ is small, then so is $U_0 - \Phi_\alpha$ for α small. If we can show that for an appropriate small value of α, the solution U of the KdV-Burgers equation

(6.1) corresponding to the initial data U_0 has the property that $U(x,t)-\Phi_\alpha(x-ct)$ is small in some translation-invariant norm, for all $t>0$, then by the triangle inequality so is $U(x,t)-\Phi(x-ct)$.

Adding precision to the foregoing comments, we now assume to be given a traveling-wave profile Φ of (6.1) such that $\Phi(x)\to 1$ as $x\to-\infty$ and $\Phi(x)\to 0$ as $x\to+\infty$. At a certain crucial point, we will also assume that Φ is monotone decreasing throughout its domain. From the results in [4], it is known that Φ is monotone decreasing exactly when $\nu\geq\sqrt{2}$. (When $\nu<\sqrt{2}$, the travelling-wave profile oscillates infinitely often as $x\to-\infty$.) Let $U_0(x)$ be initial data which is near to Φ in the sense that the norm of $U-\Phi$ in $X=L_2((1+x^2)^{2\gamma}dx)\cap H^k(\mathbb{R})$ is small, say less than ϵ, a parameter to be determined presently, where $k\geq 2$ and $\gamma>\frac{1}{2}$. It follows that $U_0-\Phi\in L_1(\mathbb{R})$ since

$$\begin{aligned}\int_{-\infty}^{\infty}|U_0(x)-\Phi(x)|dx &\leq \int_{-\infty}^{\infty}\frac{(1+x^2)^\gamma|U_0(x)-\Phi(x)|}{(1+x^2)^\gamma}dx \\ \leq \Big(\int_{-\infty}^{\infty}(1+x^2)^{2\gamma}|U_0(x)-\Phi(x)|^2dx\Big)^{1/2}&\Big(\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^{2\gamma}}dx\Big)^{1/2}\leq C_0\epsilon.\end{aligned} \tag{6.6}$$

Define $G(\alpha)=\int_{-\infty}^{\infty}(U_0(x)-\Phi_\alpha(x))dx=\int_{-\infty}^{\infty}(U_0(x)-\Phi(x+\alpha))dx$. Then G is a smooth function of α and $G'(\alpha)\equiv-1$ since $\Phi(-\infty)-\Phi(\infty)=1$. In consequence, there is a unique value α_0 such that

$$0=G(\alpha_0)=\int_{-\infty}^{\infty}(U_0(x)-\Phi_{\alpha_0}(x))dx. \tag{6.7}$$

The value of α_0 is small. Indeed, note that

$$0=G(\alpha_0)=G(0)+\int_0^{\alpha_0}G'(s)ds=G(0)-\alpha_0,$$

whence

$$|\alpha_0|=|G(0)|\leq C_0\epsilon,$$

according to (6.6). Since α_0 is small, it follows that $U_0-\Phi_{\alpha_0}$ is small in the X-norm. This is clear since

$$\begin{aligned}\|U_0-\Phi_{\alpha_0}\|_X &\leq \|U_0-\Phi\|_X+\|\Phi-\Phi_{\alpha_0}\|_X \\ &\leq \epsilon+\|\Phi-\Phi_{\alpha_0}\|_{H^k}+\|\Phi-\Phi_{\alpha_0}\|_{L_2((1+x^2)^{2\gamma}dx)};\end{aligned} \tag{6.8}$$

the two terms on the right-hand side of (6.8) are small because

$$\int_{-\infty}^{\infty}(1+x^2)^{2\gamma}(\Phi(x)-\Phi_{\alpha_0}(x))^2dx$$
$$=\int_{-\infty}^{\infty}(1+x^2)^{2\gamma}\int_0^{\alpha_0}\Phi'(x+y)dy\int_0^{\alpha_0}\Phi'(x+z)dz\,dx$$
$$=\int_0^{\alpha_0}\int_0^{\alpha_0}\int_{-\infty}^{\infty}(1+x^2)^{2\gamma}\Phi'(x+y)\Phi'(x+z)dx\,dy\,dz$$
$$=\int_0^{\alpha_0}\int_0^{\alpha_0}\int_{-\infty}^{\infty}\frac{(1+x^2)^{2\gamma}}{(1+(x+y)^2)^{\gamma}(1+(x+z)^2)^{\gamma}}$$
$$(1+(x+y)^2)^{\gamma}\Phi'(x+y)(1+(x+z)^2)^{\gamma}\Phi'(x+z)dx\,dy\,dz$$
$$\leq \sup_{\substack{x\in\mathbb{R},|y|\leq\alpha_0\\|z|\leq\alpha_0}}\left[\frac{(1+x^2)^{2\gamma}}{(1+(x+y)^2)^{\gamma}(1+(x+z)^2)^{\gamma}}\right]\int_0^{\alpha_0}\int_0^{\alpha_0}\|\Phi'\|^2_{L_2((1+x^2)^{2\gamma}dx)}dy\,dz$$
$$=c\alpha_0^2,$$

with a similar but simpler calculation applying to the $H^k(\mathbb{R})$-norm. Note the fact that $|\Phi'|$ tends exponentially fast to zero at $\pm\infty$ (see [4]) guarantees $\|\Phi'\|_{L_2((1+x^2)^{2\gamma}dx)}$ to be finite. In summary, in terms of ϵ, $\|U_0-\Phi_{\alpha_0}\|_X\leq C_1\epsilon$, where C_1 depends on Φ, γ and on α_0, but is independent of $|\alpha_0|\leq 1$, say.

With these preliminary considerations in hand, the principal result concerning asymptotic stability is now stated and proved.

Theorem 6. *Let Φ be a monotone decreasing, bore-like travelling-wave solution of the KdV-Burger's equation* (6.1) *with speed of propagation $c>0$. Let $X=L_2((1+x^2)^{2\gamma}dx)\cap H^k(\mathbb{R})$, where $k\geq 2$ and $\gamma>\frac{1}{2}$. There is an $\epsilon>0$ such that corresponding to any initial data U_0 with $\|U_0-\Phi\|_X\leq\epsilon$, a real number α exists with $|\alpha|\leq\epsilon$ having the property*

$$\|U(\cdot,t)-\Phi(\cdot-ct+\alpha)\|_{H^k(\mathbb{R})}\longrightarrow 0$$

as $t\to+\infty$, where U is the solution of (6.1) *with initial data U_0.*

Proof. Suppose $\|U_0-\Phi\|_X\leq\epsilon$, where ϵ will be determined momentarily. We continue to assume that $C_-=1$ and $C_+=0$ in the specifications in (3.1), but note again that this is an assumption of convenience only.

Let α_0 be determined so that

$$\int_{-\infty}^{\infty}(U_0(x)-\Phi_{\alpha_0}(x))dx=0.$$

It follows from our previous ruminations that $|\alpha_0|\leq C_0\epsilon$ and that $\|U_0-\Phi_{\alpha_0}\|\leq C_1\epsilon$, where C_0 and C_1 are the constants introduced in the preceding analysis.

It follows from Theorem 3 and Lemma 3 that the solution U emanating from U_0 has the property that if

$$V(x,t) = U(x,t) - \Phi_{\alpha_0}(x - ct), \tag{6.9}$$

then $V \in L_\infty(0,T;X)$ for each finite value $T \geq 0$. It follows from this that $V \in L_\infty(0,T;L_1(\mathbb{R}))$ for each T, and then a simple computation using the equation (6.4), with Φ_{α_0} replacing Φ, satisfied by V shows that

$$\int_{-\infty}^{\infty} V(x,t)dx = 0 \tag{6.10}$$

for all $t \geq 0$. If we define $W(x,t)$ by

$$W(x,t) = \int_{-\infty}^{x} V(y,t)dy, \tag{6.11}$$

then W is bounded since $V \in L_1(\mathbb{R})$ and $W(x,t) \to 0$ as $|x| \to +\infty$, because of (6.10). Moreover, since $(1+x^2)^\gamma V(x,t) \in L_\infty(0,T;L_2(\mathbb{R}))$ for any finite value of T, we can infer spatial decay rates for W via the following observation: if $x > 0$, then

$$\begin{aligned}|W(x,t)| &\leq \left| \int_x^\infty \frac{(1+y^2)^\gamma |V(y,t)|}{(1+y^2)^\gamma} dy \right| \\ &\leq \left(\int_x^\infty \frac{1}{(1+y^2)^{2\gamma}} dy \right)^{1/2} \left(\int_{-\infty}^\infty (1+y^2)^{2\gamma} V^2(y,t) dy \right)^{1/2} \\ &\leq \frac{C}{(1+x^2)^{\gamma - \frac{1}{4}}} \|V(\cdot,t)\|_X,\end{aligned} \tag{6.12}$$

where C is a constant that depends only on the value of $\gamma > \frac{1}{4}$. A similar calculation shows (6.12) to hold for $x < 0$ as well. A consequence of (6.12) is that $W \in L_\infty(0,T;L_2(\mathbb{R}))$ for finite values of T if $1 - 4\gamma < -1$, or what is the same, $\gamma > \frac{1}{2}$.

Substitute $V = W_x$ into equation (6.4) satisfied by V, with Φ replaced by Φ_α, to obtain the equation

$$\begin{aligned}0 &= W_{xt} + W_x W_{xx} + (W_x \Phi_\alpha)_x + W_{xxxx} - \nu W_{xxx} \\ &= \partial_x (W_t + \tfrac{1}{2} W_x^2 + W_x \Phi_\alpha + W_{xxx} - \nu W_{xx}).\end{aligned}$$

This implies the expression in parentheses is independent of x. Upon integrating the resulting relationship with respect to t and remembering that $W_x = V$, we come to

$$\begin{aligned}W(x,t) - W(x,0) + \int_0^t \big[\tfrac{1}{2} V^2(x,s) + V(x,s)\Phi_\alpha(x - cs) \\ + V_{xx}(x,s) - \nu V_x(x,s)\big] ds = G(t).\end{aligned}$$

Taking the limit as $x \to +\infty$, say, in this last equation shows $G(t)$ to be identically zero, and it follows that W is in $C(0, T; H^{k+1}(\mathbb{R}))$ and that it is a solution of the initial-value problem

$$W_t + \tfrac{1}{2}W_x^2 + W_x\Phi_\alpha + W_{xxx} - \nu W_{xx} = 0, \quad W(x,0) = W_0(x), \tag{6.13}$$

where

$$W_0(x) = \int_{-\infty}^{x} V(y,0)dy = \int_{-\infty}^{x} (U_0(y) - \Phi_\alpha(y))dy.$$

The aim now is to obtain the relation (6.1) by deriving decay results for W from energy bounds. To this end, we proceed to make a few formal computations which are easily justified either directly via the regularity of W, or by regularizing the initial data and making use of the continuous-dependence theory in Section 5. First, multiply the equation in (6.13) by W and integrate the result with respect to x over the real line. After suitable integrations by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} W^2dx + \nu\int_{-\infty}^{\infty} W_x^2dx + \frac{1}{2}\int_{-\infty}^{\infty} WW_x^2dx - \frac{1}{2}\int_{-\infty}^{\infty}\Phi_x W^2dx = 0,$$

where the subscript α on Φ has been temporarily dropped. Since Φ is monotone decreasing, so is Φ_α and hence $\partial_x\Phi_\alpha < 0$. It follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty} W^2dx + \int_{-\infty}^{\infty} W_x^2(2\nu - |W|_\infty)dx \le 0. \tag{6.14}$$

If (6.13) is multiplied by W_{xx}, then integrated over $\mathbb{R}$ and the result simplified, there appears

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} W_x^2dx + \nu\int_{-\infty}^{\infty} W_{xx}^2dx + \frac{1}{2}\int_{-\infty}^{\infty}\Phi_x W_x^2 = 0. \tag{6.15}$$

If equation (6.15) is multiplied by a constant $\lambda > 0$ and added to the inequality (6.14), and then further estimates made, we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}(W^2+\lambda W_x^2)dx + \int_{-\infty}^{\infty} W_x^2(2\nu - |W|_\infty - \lambda|\Phi_x|_\infty)dx + 2\nu\int_{-\infty}^{\infty} W_{xx}^2dx \le 0. \tag{6.16}$$

First choose $\lambda > 0$ so that

$$2\nu - \lambda|\Phi_x|_\infty \ge \nu. \tag{6.17}$$

For any function $f \in H^1(\mathbb{R})$, and $\lambda > 0$,

$$|f|_\infty^2 \le \frac{1}{\lambda^{1/2}}\|f\|_{1,\lambda}, \tag{6.18}$$

where

$$\int_{-\infty}^{\infty}(f^2(x) + \lambda f_x^2(x))dx = \|f\|_{1,\lambda}^2.$$

In consequence of the choice of λ and (6.17), the differential inequality (6.16) can be extended to

$$\frac{d}{dt}\|W\|_{1,\lambda}^2 + (\nu - \lambda^{-1/4}\|W\|_{1,\lambda}) \int_{-\infty}^{\infty} W_x^2 dx + 2\nu\lambda \int_{-\infty}^{\infty} W_{xx}^2 dx \leq 0. \tag{6.19}$$

As λ is now fixed, the time has come to restrict the parameter ϵ. If, for example, ϵ can be restricted so that $\|W(\cdot,0)\|_{1,\lambda} \leq \nu\lambda^{1/4}/2$, then it follows from (6.18) that

$$\frac{d}{dt}\|W\|_{1,\lambda}^2 + \frac{1}{2}\nu \int_{-\infty}^{\infty} W_x^2 dx + 2\nu \int_{-\infty}^{\infty} W_{xx}^2 dx \leq 0, \tag{6.20}$$

at least at $t = 0$. But then for t near 0, $\|W(\cdot,t)\|_{1,\lambda}$ is decreasing, and hence (6.20) continues to hold, and so is globally valid. It would then follow that

$$\int_{-\infty}^{\infty} (W^2 + \lambda W_x^2) dx \tag{6.21a}$$

is decreasing, and that

$$\int_0^{\infty}\int_{-\infty}^{\infty} W_x^2 dxdt \quad \text{and} \quad \int_0^{\infty}\int_{-\infty}^{\infty} W_{xx}^2 dxdt \tag{6.21b}$$

are bounded. In consequence, $|W(\cdot,t)|_2$ and $|W_x(\cdot,t)|_2$ are uniformly bounded in time.

The condition $\|W(\cdot,0)\|_{1,\lambda} \leq \nu\lambda^{1/4}/2$ is guaranteed by a suitable choice of ϵ. Indeed, using (6.12) at $t = 0$, it is seen that

$$\begin{aligned}\int_{-\infty}^{\infty} [W^2(x,0) + \lambda W_x^2(x,0)]dx &= \int_{-\infty}^{\infty} W^2(x,0)dx + \lambda \int_{-\infty}^{\infty} V^2(x,0)dx \\ &\leq C\|V(\cdot,0)\|^2 \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{2\gamma-1/2}} dx + \lambda\|V(\cdot,0)\|_X^2 \\ &\leq C'\|V_0\|_X^2 = C'\epsilon^2.\end{aligned}$$

With (6.21b) in hand, it is straightforward to show that $|W_x(\cdot,t)|_2 = |V(\cdot,t)|_2 \to 0$ as $t \to +\infty$. In fact, according to (6.15),

$$\frac{d}{dt}\int_{-\infty}^{\infty} W_x^2 dx = -\nu \int_{-\infty}^{\infty} W_{xx}^2 dx - \frac{1}{2}\int_{-\infty}^{\infty} \Phi_x W_x^2$$

lies in $L_1(0,\infty)$. An $L_1(\mathbb{R}^+)$-function whose derivative lies in $L_1(\mathbb{R}^+)$ necessarily tends to zero at infinity.

This establishes the desired result (6.11) for $k = 0$. A straightforward induction on k suffices to establish the general result. The arguments just put forth for the case $k = 0$ are all that is required, and consequently we may suitably pass over the details. □

Remarks. 1. An entirely similar theory of asymptotic stability of bore-like traveling waves can be worked out for the regularized long-wave-Burgers equation (1.4). The details differ in no essential aspect from those presented above.

2. Once an asymptotic stability result has been obtained for monotone profiles when $\nu \geq \sqrt{2}$, the argument can be extended to prove the stability of oscillatory profiles where $\nu < \sqrt{2}$, provided the oscillations are small. The results of [4] show that small oscillations correspond to ν smaller than, but near the value $\sqrt{2}$. This sort of result has been worked out recently by Khodja [19] and Naumkin & Shishmarev [23] for the cases where the perturbation has zero added mass.

3. It also deserves remark that for the related problem where the nonlinearity takes the form $u^p u_x$, Pego *et al.* [26] have shown using their Evans-function approach that the travelling waves are unstable for ν small enough .

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