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On the asymptotic behavior of solutions to nonlinear, dispersive, dissipative wave equations

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Abstract

The renormalization techniques for determining the long-time asymptotics of nonlinear parabolic equations pioneered by Bricmont, Kupiainen and Lin are shown to be effective in analyzing nonlinear wave equations featuring both dissipation and dispersion. These methods allow us to recover recent results of Dix in a way which is both transparent and has interesting prospects for generalization.

1. Motivation and statement of the main results

This study is focussed on the long-time behavior of solutions of the damped, non-linear wave equation

$$u_t - Mu + u_{xxx} + u^p u_x = 0, \quad (x \in \mathbb{R},\ t > 1), \tag{1.1}$$

with initial data

$$u(\cdot, 1) = f_0(\cdot), \quad (x \in \mathbb{R}). \tag{1.2}$$

Here M is a Fourier-multiplier operator, which in Fourier transformed variables has the form $\widehat{Mu}(k) = -|k|^{2\beta}\hat{u}$, where $\frac{1}{2} < \beta \leqslant 1$, and $p \geqslant 2$. The initial-value problem (1.1)–(1.2) is always locally well posed, but for larger values of p and large initial data, it appears that it may not be globally well posed (see Bona et al. [1,2]). If attention is restricted to small initial data, then (1.1)–(1.2) is globally well posed and it is not difficult to see in this case that solutions decay to the zero function as t becomes unboundedly large. It is our purpose here to determine the detailed structure of this evanescence.

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This program has already been carried out in the elegant paper of Dix [4] (see also Naumkin and Shishmarev [5] for the case $p = 1$). We will show how some of Dix's results follow readily from a suitable adaptation of the renormalization techniques put forward by Bricmont et al. [3] in the context of nonlinear parabolic equations. While the results obtained in this way are no more telling than those of Dix, they are here obtained in a transparent and appealing way. Furthermore, this method appears to be a promising avenue of approach for the derivation of more refined aspects of the long-time asymptotics of solutions.

The principle result derived herein is easily motivated by consideration of the linear initial-value problem

$$v_t - Mv + \eta v_{xxx} = 0,$$
$$v(x, 1) = f_0(x), \tag{1.3}$$

with parameter η, which is formally solved by the formula

$$v(x, t) = S(t)f_0 = S_\eta(t)f_0 = \int_{-\infty}^{\infty} e^{-(|k|^{2\beta}+i\eta k^3)(t-1)}\, e^{-ikx}\hat{f}_0(k)\, dk, \tag{1.4}$$

where $\hat{f}_0(k) = \int_{-\infty}^{\infty} e^{ikx} f_0(x)\, dx$ is the Fourier transform of f_0. For large t, the kernel in (1.4) is very small except where $|k|$ is near 0. For such values of k, $|k|^{2\beta} \gg \eta |k|^3$, and hence if $\hat{f}_0$ is sufficiently regular,

$$v \approx \int_{-\infty}^{\infty} e^{-|k|^{2\beta}(t-1)}\, e^{-ikx}\left(\hat{f}_0(0) + k\hat{f}_0'(0) + \mathcal{O}|k^2|\right) dk$$
$$\simeq \hat{f}_0(0)\int_{-\infty}^{\infty} e^{-ikx}\, e^{-|k|^{2\beta}(t-1)}\, dk. \tag{1.5}$$

Defining

$$f^*(x) = \int_{-\infty}^{\infty} e^{-ikx}\, e^{-|k|^{2\beta}(t-1)}\, dk, \tag{1.6}$$

and making the change of variables $k' = kt^{1/2\beta}$ in (1.5) yields

$$v(x, t) \simeq \frac{\hat{f}_0(0)}{t^{1/2\beta}} f^*(x/t^{1/2\beta}) \tag{1.7}$$

as $t \to \infty$. Our main result, here stated informally to provide the reader with a concrete goal, shows both that the approximations made to reach (1.7) are justified, and that they remain so even in the context of the non-linear initial-value problem (1.1)–(1.2).

Theorem 1. *For a sufficiently small and smooth initial condition f_0, and for $\frac{1}{2} < \beta \leqslant 1$, $p \geqslant 2$, and $\varepsilon > 0$, there exist constants A, C_2, $C_\infty > 0$ depending only on the given data such that*

$$\left\| u(\cdot, t) - \frac{A}{t^{1/2\beta}} f^*(\cdot/t^{1/2\beta}) \right\|_{L^2} \leqslant \frac{C_2}{t^{((3/4\beta)-\varepsilon)}}, \tag{1.8a}$$

$$\left\| u(\cdot, t) - \frac{A}{t^{1/2\beta}} f^*(\cdot/t^{1/2\beta}) \right\|_{L^\infty} \leqslant \frac{C_\infty}{t^{((1/\beta)-\varepsilon)}} \tag{1.8b}$$

where f^* is given in (1.6).

Remark. A careful look at the proof in Section 3 will convince the reader that the theory could be formulated in a sharper way, in which for a given β, the decay results in (1.8) are valid for any $p \geqslant 2\beta$. In applications of (1.1) to physical problems, the dependent variable is real-valued and consequently non-integer values of p are seen to be somewhat artificial. If non-integer p are contemplated, a complex-valued version of the theory for (1.1)–(1.2) would be required. In consequence, we have eschewed the extra precision that might be possible in favor of the simpler development that is available for integer values of p.

The plan of the paper is straightforward. In Section 2 detailed consideration is given to the linear initial-value problem (1.3). This discussion leads naturally to the introduction of a particular weighted norm that turns out to be very useful in the analysis. The theory for the linear problem is both instructive and useful when attention is turned to the nonlinear problem in Sections 3 and 4. The paper concludes with a short summary and suggestions for further inquiry.

2. Norms and the linear problem

The approach taken is to apply the renormalization group ideas of Bricmont et al. [3] to the equation (1.1). It is convenient to start with the linear equation (1.3) since the theory for this simple situation already deviates from that of Bricmont et al. [3], because the main ideas appear in bold relief, avoiding the technicalities associated with the nonlinear term, and because use will be made of the linear theory in studying the nonlinear initial-value problem.

Let $\mathscr{B}$ be the Banach space of functions f such that their Fourier transform $\hat{f}$ lies in $\mathscr{C}^1(\mathbb{R})$, and for which the norm

$$\| f \| = \sup_k (1 + |k|^3) |\hat{f}(k)| + \sup_k |\hat{f}'(k)| \tag{2.1}$$

is finite. Note that this norm controls both the L_2- and L_∞-norms of f. Note also that f belongs to $\mathscr{B}$ if, for example, f lies in $W^{3,1}(\mathbb{R})$ and xf is a member of $L_1(\mathbb{R})$.

The renormalization group map is now introduced. Taking the solution v given by (1.4), define

$$v_L(x, t) = Lv(Lx, L^{2\beta}t) \tag{2.2}$$

for $L \geqslant 1$. The (linear) renormalization group map is denoted $R_{L,\eta}$ and its action on a function f is

$$(R_{L,\eta} f)(x) = v_L(x, 1), \tag{2.3}$$

where v is the solution of (1.4) with initial data f, v_L is as in (2.2), and the second subscript on R connotes the coefficient of the dispersive term v_{xxx} in (1.3). Direct calculation shows that v_L satisfies

$$\partial_t v_L = M v_L - L^{2\beta - 3} \eta \partial_x^3 v_L. \tag{2.4}$$

Thus the rescaling accomplished in the definition of $R_{L,\eta}$ diminishes the dispersive term if

$\beta < 3/2$ and L is large. Moreover, because of the semi-group property of the evolution equation, it transpires that

$$R_{L^n,1} = R_{L,\alpha^{n-1}} \circ R_{L,\alpha^{n-2}} \circ \cdots \circ R_{L,\alpha} \circ R_{L,1}, \tag{2.5}$$

where $\alpha = L^{2\beta-3}$.

Observe that the putative similarity function f^* defined in (1.6), which according to the intuition provided by (1.5) captures the long-term asymptotics of solutions of (1.3), is a fixed point

$$R_{L,0} f^* = f^* \tag{2.6}$$

of the renormalization group map $R_{L,0}$ without dispersion. Assuming that the α in (2.5) is less than one, which will be true if $\beta < \frac{3}{2}$ and $L > 1$, it is a reasonable conjecture that R_{L,α^n} converges to $R_{L,0}$ as n becomes unboundedly large. It is possible then that successive applications of R_{L,α^n} for n large drive one toward f^*. Thus for a given initial datum f, the right-hand side of (2.5) applied to f may, in the right circumstances, converge to f^*. This in turn means that $R_{L^n,1} f \to f^*$ and this result, properly interpreted, is exactly what we are after.

The latter observation leads to a search for conditions under which $R_{L,\alpha}$ is a contractive mapping.

Lemma 1. *Let the power β in the dissipative operator be positive and suppose $g \in \mathscr{B}$ satisfies $\hat{g}(0) = 0$. Then there exists a positive constant $C = C(\beta)$ such that*

$$\| R_{L,\eta} g \| \leqslant C(\beta) L^{-1} \| g \|. \tag{2.7}$$

The constant $C(\beta)$ is independent of $\eta \in [0, 1]$.

Proof. The solution of (1.3) with initial condition g, written in Fourier-transformed variables, is

$$\hat{v}(k, t) = \mathrm{e}^{-(|k|^{2\beta} + ik^3)(t-1)} \hat{g}(k),$$

and therefore

$$\widehat{R_{L,\eta}(g)}(k) = \widehat{Lv}(Lx, L^{2\beta}) = \hat{v}\left(\frac{k}{L}, L^{2\beta}\right) = \mathrm{e}^{-(|k|^{2\beta} + iL^{2\beta-3}\eta k^3)(1-L^{-2\beta})} \hat{g}\left(\frac{k}{L}\right).$$

Since $\hat{g}(0) = 0$ and $\hat{g} \in C^1(\mathbb{R})$, the Mean-Value Theorem implies that for any k, there is a point $\xi = \xi_k$ with $|\xi_k| \leqslant k/L$ such that

$$\left| \hat{g}\left(\frac{k}{L}\right) \right| \leqslant \left| \frac{k}{L} \right| |\hat{g}'(\xi_k)|.$$

In consequence, we have that

$$\begin{aligned} \sup_k (1 + |k|^3) \left| \widehat{R_{L,\eta} g}(k) \right| &\leqslant \frac{1}{L} \sup_k (1 + |k|^3) |k| \mathrm{e}^{-|k|^{2\beta}} |\hat{g}'(\xi_k)| \\ &\leqslant \frac{1}{L} C(\beta) \| g \|. \end{aligned}$$

Similarly, one determines that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\left(\widehat{R_{L,\eta}g}\right)(k) &= \Big\{-\left(2\beta|k|^{2\beta-1}+3\mathrm{i}L^{2\beta-3}\eta k^2\right)(1-L^{-2\beta})\hat{g}(k/L) \\ &\quad + \frac{1}{L}\hat{g}'\left(\frac{k}{L}\right)\Big\}\ \mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}L^{2\beta-3}\eta k^3)}(1-L^{-2\beta}) \\ &\leqslant \frac{1}{L}C(\beta)\left|\hat{g}'\left(\frac{k}{L}\right)\right| \leqslant \frac{1}{L}C(\beta)\parallel g\parallel,\end{aligned}$$

and the result follows. □

It will also be useful to understand the action of $R_{L,\alpha}$ on the fixed point f^* as L becomes large. To this end, we state and prove another lemma.

Lemma 2. *Suppose that* $\frac{1}{2}<\beta<1$. *Then there is a constant* $L_0=L_0(\beta)$ *and a constant* $C=C(\beta)$ *such that for any* $L>L_0$, *one has*

$$\parallel R_{L,\alpha^n}f^*-f^*\parallel \leqslant C\frac{1}{L^n}, \tag{2.8}$$

for $n=1, 2,\dots,$ *where* $\alpha=L^{2\beta-3}$ *and* f^* *is defined in* (1.6).

Proof. First notice that

$$\left(\widehat{R_{L,\alpha^n}f^*}\right)(k)=\mathrm{e}^{-(|k|^{2\beta}+\mathrm{i}\alpha^n k^3)(1-L^{-2\beta})}\ \mathrm{e}^{-|k|^{2\beta}L^{-2\beta}},$$

from which is follows that

$$\left(\widehat{R_{L,\alpha}f^*}\right)(k)-\hat{f}^*(k)=\mathrm{e}^{-|k|^{2\beta}}\left(\mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-L^{-2\beta})}-1\right).$$

Attention is now turned to estimating the quantity

$$\sup_k\,(1+|k|^3)\left|\left(\widehat{R_{L,\alpha^n}f^*}\right)(k)-\hat{f}^*(k)\right|.$$

The estimate is made in two parts. Suppose first that k, L and n are such that $|k|^3<\alpha^{-n\mu}$ for some fixed positive $\mu<1$. Then it is easily seen that

$$\left|\mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-L^{-2\beta})}-1\right|\leqslant C\alpha^{n(1-\mu)},\quad \text{while } (1+|k|^3)\ \mathrm{e}^{-|k|^{2\beta}}\leqslant C(\beta)$$

for all k. If on the other hand k, L and n are such that $|k|^3\geqslant\alpha^{-n\mu}$, then

$$\left|\mathrm{e}^{-\mathrm{i}\alpha^n k^3(1-L^{-2\beta})}-1\right|\leqslant 2$$

and, therefore,

$$\begin{aligned}\sup_{|k|^3\geqslant\alpha^{-n\mu}}(1+|k|^3)\ \mathrm{e}^{-|k|^{2\beta}} &\leqslant \mathrm{e}^{-(1/2)\alpha^{-2\beta n\mu/3}}\sup_k\,(1+|k|^3)\ \mathrm{e}^{-1/2|k|^{2\beta}} \\ &\leqslant C(\beta)\ \mathrm{e}^{-(1/2)L^{(3-2\beta)2\beta n\mu/3}} \\ &\leqslant C\frac{1}{L^n}\end{aligned}$$

for L large enough since β, $\mu > 0$ and $3-2\beta > 0$. If μ is restricted to lie in the interval $(0, (2-2\beta)/(3-2\beta)]$, then

$$\alpha^{n(1-\mu)} = L^{(2\beta-3)(1-\mu)n} \leqslant \left(\frac{1}{L}\right)^n.$$

In consequence of the above inequalities, it is seen that there are constants C and L_0 depending only on β such that

$$\sup_k (1+|k|^3)|(\widehat{R_{L,\alpha^n} f^*})(k) - \hat{f}^*(k)| \leqslant C\left(\frac{1}{L}\right)^n$$

for $L \geqslant L_0$. A similar set of inequalities shows that

$$\sup_k \left|\frac{\mathrm{d}}{\mathrm{d}k}\left(\widehat{R_{L,\alpha^n} f}^*(k) - \hat{f}^*(k)\right)\right| \leqslant C\left(\frac{1}{L}\right)^n$$

for L large, where C again depends only on β.

The lemma is thereby established. □

Remark. If $\beta = 1$, the foregoing reasoning leads to an estimate of the form

$$\| R_{L,\alpha^n} f^* - f^* \| \leqslant C\left(\frac{1}{L}\right)^{(1-\varepsilon)n}$$

valid for any $\varepsilon > 0$, where C depends on ε as well as β.

These two lemmas provide the tools needed to show convergence of the composition of the renormalization group maps. More precisely, suppose $f = f_0 \in \mathscr{B}$ to be given and define f_n for $n \geqslant 1$ by the formula

$$f_n = R_{L,\alpha^n} \circ R_{L,\alpha^{n-1}} \circ \cdots \circ R_{L,1} f_0. \tag{2.9}$$

We aim to show that $\{f_n\}_{n=0}^{\infty}$ is a convergent sequence in $\mathscr{B}$ and to obtain the rate of convergence.

Turning to the just mentioned task, for each $n = 1, 2, \ldots,$ write

$$f_n = A_n f^* + g_n$$

where $\widehat{g_n}(0) = 0$ so that $A_n = \widehat{f_n}(0)$. Straightforward computations show that

$$\begin{aligned} f_{n+1} &= R_{L,\alpha^{n+1}} f_n = A_n R_{L,\alpha^{n+1}} f^* + R_{L,\alpha^{n+1}} g_n \\ &= A_n f^* + A_n(R_{L,\alpha^{n+1}} f^* - f^*) + R_{L,\alpha^{n+1}} g_n. \end{aligned}$$

It is easy to verify that $\widehat{R_{L,\eta} f}(0) = \hat{f}(0)$ (see Proposition 2.1 in Bona et al. [2]). Consequently, one has $A_n = A = \hat{f}_0(0)$ for all n and

$$g_{n+1} = A(R_{L,\alpha^{n+1}} f^* - f^*) + R_{L,\alpha^{n+1}} g_n.$$

Applying Lemmas 1 and 2 leads to the estimate

$$\| g_{n+1} \| \leqslant A\frac{C}{L^{n+1}} + \frac{C}{L}\| g_n \|$$

valid for all $n \geqslant 1$, where C, which is larger than 1 without loss of generality, depends only on β. A simple induction then shows that

$$\| g_{n+1} \| \leqslant (n+1) A \left(\frac{C}{L} \right)^{n+1} \| g_0 \| \leqslant (n+1) \left(\frac{C}{L} \right)^{n+1} \| f_0 \|,$$

from which is follows at once that for all n,

$$\| f_n - Af^* \| \leqslant n \left(\frac{C'}{L} \right)^{n} \| f_0 \|. \tag{2.10}$$

Recalling from the definition (2.3), that

$$f_n(x) = L^n v(L^n x, L^{2\beta n}),$$

where v is the solution of (1.3) with initial value $v(\cdot, 1) = f_0$, and setting $t = L^{2\beta n}$, we see that

$$f_n(x) = t^{1/2\beta} v(x t^{1/2\beta}, t)$$

and that

$$nL^{-n} \leqslant C(\log t) t^{-(1/2\beta)}.$$

We have thus established the following proposition.

Proposition 1. *For* $\frac{1}{2} < \beta < 1$ *and any initial condition* $f_0 \in \mathscr{B}$, *there exists a constant* $C > 0$ *and a time* $t_0 > 0$ *such that for* $t \geqslant t_0$, *the solution* v *of* (1.3) *with* $\eta = 1$ *satisfies*

$$\| v(\cdot t^{1/2\beta}, t) - A t^{-1/2\beta} f^*(\cdot) \| \leqslant C(t^{-1/\beta} \log t) \| f_0 \|, \tag{2.11}$$

where $A = \hat{f}_0(0)$ *and* f^* *is defined in* (1.6).

Remarks. The proof only showed convergence for the sequence of times $t_n = L^{2n\beta}$. However, the result is self improving if one notes that as in Bricmont et al. [3], the analysis is unchanged if $L^{2\beta}$ is replaced by $\tau L^{2\beta}$ throughout for $1 \leqslant \tau \leqslant L^{2\beta}$. One thereby infers (2.11) for all $t > L^{2\beta}$.

The above analysis may be adapted to the case $0 < \beta \leqslant \frac{1}{2}$ in the linear case. One has to compensate for the non-differentiability of $|k|^{2\beta}$ for β in this range by modifying the norm defining the space $\mathscr{B}$, replacing the C^1-norm on $\hat{f}$ by an appropriate Hölder norm.

3. The nonlinear problem

Attention is returned to the nonlinear problem (1.1)–(1.2) with the aim of showing that the term $u^p u_x$, $p \geqslant 2$, does not affect the results of Proposition 1 provided $\| f_0 \|$ is sufficiently small.

Let u be a solution of (1.1)–(1.2) and consider the renormalized version

$$u_L(x, t) = Lu(Lx, L^{2\beta} t) \tag{3.1}$$

of u. The renormalized solution u_L satisfies the evolution equation

$$\partial_t u_L = M u_L - L^{2\beta - 3} \partial_x^3 u_L - L^{2\beta - p - 1} u_L^p \partial_x u_L. \tag{3.2}$$

Introduce the notation $\alpha = L^{2\beta-3}$ as in Section 2, $\gamma = L^{2\beta-p-1}$, and let $R_{L,\eta,\rho}$ denote the renormalization map (3.1) corresponding to the semi-group for equation (1.1) with coefficient η for the dispersive term and ρ for the nonlinear term. Thus if $f \in \mathscr{B}$ is given, then $(R_{L,\eta,\rho} f)(x)$ is $u_L(x, 1)$ where u_L is as in (3.1) and u is the solution of the initial-value problem

$$\begin{aligned} &u_t + \rho u^p u_x + \eta u_{xxx} - Mu = 0, \\ &u(x, 0) = f(x), \end{aligned} \tag{3.3}$$

for $x \in \mathbb{R}$ and $t \geqslant 0$. Note that $(R_{L,1,1} f_0)(x) = Lu(Lx, L^{2\beta})$ where u solves (1.1)–(1.2).

To proceed with the program that was effective in Section 2 for the linear problem, estimates on the Fourier transform of the nonlinear term in the evolution equation are needed. The following lemma is helpful in this regard.

Lemma 3. *Let $p \geqslant 1$ be an integer. Then there is a constant C depending only on p such that for any $f \in \mathscr{B}$ and all $k \in \mathbb{R}$,*

$$|\widehat{f^{p+1}}(k)| \leqslant \frac{C}{1 + |k|^3} \| f \|^{p+1}, \tag{3.4a}$$

$$\left|\widehat{f^p f_x}(k)\right| \leqslant \frac{C}{1 + k^2} \| f \|^{p+1}, \tag{3.4b}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}k} \widehat{f^p f_x}(k)\right| \leqslant C \| f \|^{p+1}. \tag{3.4c}$$

Proof. These straightforward estimates follow immediately upon writing

$$\widehat{f^{p+1}}(k) = \hat{f} * \cdots * \hat{f}(k) = \int_{\mathbb{R}^p} \hat{f}\big(k - (k_1 + \cdots + k_p)\big)\hat{f}(k_1) \cdots \hat{f}(k_p)\, \mathrm{d}k_1 \cdots \mathrm{d}k_p,$$

$$\begin{aligned} \widehat{f^p f_x}(k) &= \hat{f} * \cdots * \hat{f_x}(k) \\ &= \int_{\mathbb{R}^p} \hat{f}\big(k - (k_1 + \cdots + k_p)\big)\hat{f}(k_1) \cdots \hat{f}(k_{p-1}) \mathrm{i} k_p \hat{f}(k_p)\, \mathrm{d}k_1 \cdots \mathrm{d}k_p \end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}k} \widehat{f^p f_x}(k) = \int_{\mathbb{R}^p} \hat{f}'\big(k - (k_1 + \cdots + k_p)\big)\hat{f}(k_1) \cdots \hat{f}(k_{p-1}) \mathrm{i} k_p \hat{f}(k_p)\, \mathrm{d}k_1 \cdots \mathrm{d}k_p. \qquad \square$$

A bound on the kernel of the linear propagator will also prove to be useful.

Lemma 4. *There are constants C_0 and C_1 such that for any $\eta \in [0, 1]$, $t \geqslant 1$, and for all k,*

$$\left| \int_0^{t-1} \mathrm{e}^{-s(|k|^{2\beta} + \mathrm{i}\eta k^3)}\, \mathrm{d}s \right| \leqslant \frac{C_0 t}{1 + |k|^{2\beta}}, \tag{3.5a}$$

$$|k| \int_0^{t-1} \left| \frac{\mathrm{d}}{\mathrm{d}k} \mathrm{e}^{-s(|k|^{2\beta} + \mathrm{i}\eta k^3)} \right| \mathrm{d}s \leqslant C_1 t (1 + k^2). \tag{3.5b}$$

Proof. These two inequalities follow by computing the integrals in question and making elementary estimates. $\square$

A little later in the analysis, Duhamel's principle will be used and it will then be helpful to have estimates on the functional $N = N_\eta$ defined for $u \in C(1, T; \mathscr{B})$ by

$$N(u)(x, t) = \int_1^t S(t-s+1)u^p(x, s)u_x(x, s)\,\mathrm{d}s, \tag{3.6}$$

where $\{S(r)\}_{r\geqslant 1} = \{S_\eta(r)\}_{r\geqslant 1}$ is the semigroup defined in (1.4) associated to the initial-value problem for the linear equation (1.3). By using the estimates in Lemmas 3 and 4, one deduces that if $u \in C(1, T; \mathscr{B})$, then for $1 \leqslant t \leqslant T$,

$$\begin{aligned}
&\sup_k (1+|k|^3)|\widehat{N(u)}(k, t)| \\
&\quad\leqslant \sup_k (1+|k|^3)\left|\int_1^t \mathrm{e}^{-(t-s)\theta(k)}\widehat{u^p u_x}(k, s)\,\mathrm{d}s\right| \\
&\quad\leqslant \sup_k \left\{(1+|k|^3) \sup_{1\leqslant s\leqslant T}\left(\frac{C}{1+k^2}\|u(\cdot, s)\|^{p+1}\right)\int_1^t \mathrm{e}^{-(t-s)\operatorname{Re}\theta(k)}\,\mathrm{d}s\right\} \\
&\quad\leqslant \|u\|_T^{p+1} \sup_k \left\{\frac{C(1+|k|^3)}{1+k^2}\int_0^{t-1} \mathrm{e}^{-r|k|^{2\beta}}\,\mathrm{d}r\right\} \\
&\quad\leqslant \|u\|_T^{p+1} \sup_k \left\{CC_0\frac{(1+|k|^3)}{(1+k^2)(1+|k|^{2\beta})}\right\} \\
&\quad\leqslant C\|u\|_T^{p+1},
\end{aligned}$$

provided that $\beta \geqslant \frac{1}{2}$. Here $\theta(k) = |k|^{2\beta} + \mathrm{i}\eta k^3$ and we have introduced the space-time norm

$$\|u\|_T = \|u\|_{C(1,T;\mathscr{B})} = \sup_{1\leqslant t\leqslant T}\|u(\cdot, t)\|. \tag{3.7}$$

In a similar vein, it transpires that for $u \in C(1, T; \mathscr{B})$,

$$\begin{aligned}
&\sup_k \left|\widehat{N(u)}'(k, t)\right| \\
&\quad= \sup_k \left|\frac{\mathrm{d}}{\mathrm{d}k}\int_1^t \mathrm{e}^{-(t-s)\theta(k)}\widehat{u^p u_x}(k, s)\,\mathrm{d}s\right| \\
&\quad\leqslant \sup_k \frac{1}{p+1}\left|\int_1^t \frac{\mathrm{d}}{\mathrm{d}k}(\mathrm{e}^{-(t-s)\theta(k)})\mathrm{i}k\widehat{u^{p+1}}(k, s)\,\mathrm{d}s\right| \\
&\qquad+ \sup_k \left|\int_1^t \mathrm{e}^{-(t-s)\theta(k)}\frac{\mathrm{d}}{\mathrm{d}k}\widehat{u^p u_x}(k, s)\,\mathrm{d}s\right| \\
&\quad\leqslant \|u\|_T^{p+1} \sup_k \frac{C}{1+|k|^3}\int_0^{t-1}|k|\left|\frac{\mathrm{d}}{\mathrm{d}k}\,\mathrm{e}^{-r\theta(k)}\right|\mathrm{d}r \\
&\qquad+C\|u\|_T^{p+1}\sup_k \int_1^t \mathrm{e}^{-(t-s)\operatorname{Re}\theta(k)}\,\mathrm{d}s \\
&\quad\leqslant Ct\|u\|_T^{p+1}
\end{aligned} \tag{3.8}$$

for $1 \leqslant t \leqslant T$, where C is independent of both T and u.

These estimates lead to the following result, which will find use immediately in the attack on the main result.

Proposition 2. *Suppose* $\frac{1}{2} < \beta \leqslant 1$ *and that* $u_1, u_2 \in C(1, T; \mathscr{B})$ *where* $T = L^{2\beta}$ *with* $L > 1$ *given. Then the following inequalities hold*:

$$\| N(u_1) \|_T \leqslant CT \| u_1 \|_T^{p+1}, \tag{3.9}$$

$$\| N(u_1) - N(u_2) \|_T \leqslant CT(\| u_1 \|_T^p + \| u_2 \|_T^p) \| u_1 - u_2 \|_T. \tag{3.10}$$

Proof. Inequality (3.8) shows that $\| N(u_1)(\cdot, t) \| \leqslant Ct \| u_1 \|_T^{p+1}$ for $1 \leqslant t \leqslant T$. Taking the supremum over $t \in [1, T]$ thus gives (3.9). Using the elementary relation

$$| u^{p+1} - v^{p+1} | = | u - v || u^p + u^{p-1}v + \cdot + v^p | \leqslant C | u - v | (| u^p | + | v^p |)$$

and the same sort of estimates that appear in (3.8), the inequality (3.10) is likewise verified. □

With these preliminaries in hand, we turn to the primary task of determining the asymptotic behavior of solutions of (1.1)–(1.2). Let f_0 in $\mathscr{B}$ be given. Much as in the linear case, we consider the sequence

$$f_n = R_{L^n,1,1} f_0, \quad n = 1, 2, \ldots,$$

where $L > 0$ will be specified later. Because of the semigroup property,

$$f_n = R_{L,\alpha^{n-1},\gamma^{n-1}} \circ \cdots \circ R_{L,\alpha,\gamma} \circ R_{L,1,1} f_0, \tag{3.11}$$

or what is the same,

$$f_n = R_{L,\alpha^{n-1},\gamma^{n-1}} f_{n-1},$$

for $n = 1, 2, \ldots$, where $\alpha = L^{2\beta-3}$ and $\gamma = L^{2\beta-p-1}$ as above.

To analyse the sequence $\{f_n\}_{n=0}^{\infty}$, it is convenient to consider the initial-value problem

$$\begin{aligned} &\partial_t u_n = Mu_n - \alpha^{n-1} \partial_x^3 u_n - \gamma^{n-1} u_n^p \partial_x u_n, \\ &u_n(\cdot, 1) = f. \end{aligned} \tag{3.12}$$

Duhamel's principle is applied to (3.12) to obtain the formulation

$$\begin{aligned} u_n(\cdot, t) &= S_n(t) f + \gamma^{n-1} \int_1^t S_n(t-s) u_n^p(\cdot, s) \partial_x u_x(\cdot, s) \, \mathrm{d}s \\ &= u_n^0(\cdot, t) + \gamma^{n-1} N_n(u_n)(\cdot, t), \end{aligned} \tag{3.13}$$

where $S_n(t) = S_{\alpha^{n-1}}(t)$ is the semigroup defined in (1.4) associated with the linear initial-value problem (1.3) with $\eta = \alpha^{n-1}$. Because of the previously derived results, the terms on the right-hand side of (3.13) can be estimated thusly:

$$\| u_n^0 \|_T \leqslant C_2 \| f \| \quad \text{and} \quad \gamma^{n-1} \| N_n(u_n) \|_T \leqslant C_3 T \| u_n \|_T^{p+1} \gamma^{n-1}, \tag{3.14}$$

for $T > 0$, where C_2 and C_3 are independent of n, T, f, α and γ.

Define a mapping T_n on functions $v \in C(1, T; \mathscr{B})$ by

$$T_n(v) = u_n^0 + \gamma^{n-1} N_n(v),$$

where u_n^0 is as in (3.13). Let $B_n^f = \{u \in C(1, T; \mathscr{B}): \|u - u_n^0\|_T \leqslant \|f\|\}$ and presume that f is drawn from $B_R = \{g \in \mathscr{B}: \|g\| \leqslant R\}$.

It will now be shown that if R is chosen small enough and L is sufficiently large, with $T = L^{2\beta}$, then T_n defines a contraction mapping of B_n^f, for all n. First, because of the second inequality in (3.14), T_n will map B_n^f into itself provided

$$C_3 T \|u\|_T^{p+1} \gamma^{n-1} \leqslant \|f\| \tag{3.15}$$

for $u \in B_n^f$, and the latter holds if

$$C_3 T \gamma^{n-1} \left(\|f\| + \|u_n^0\|_T\right)^{p+1} \leqslant \|f\|.$$

Because of the first inequality in (3.14), this holds provided

$$C_3 T \gamma^{n-1} (1 + C_2)^{p+1} \|f\|^p \leqslant 1.$$

Since $\gamma = L^{2\beta - p - 1}$, $T = L^{2\beta}$ and $\|f\| \leqslant R$, (3.15) will hold if

$$C_3 (1 + C_2)^{p+1} R^p L^{2n\beta - (p+1)(n-1)} \leqslant 1.$$

Since $p \geqslant 2$ and $\beta \leqslant 1$, the exponent is negative for $n \geqslant 3$. Moreover, the inequality (3.10) in Proposition 2 shows that T_n is a contraction on B_n^f, if $\|f\|$ is small enough and thus (3.12) has a unique solution there.

Taking $f = f_{n-1}$ in (3.12), then using (3.11) and (3.13) leads to the formula

$$\begin{aligned} f_n(\cdot) &= L u_{f_{n-1}}(\cdot L, L^{2\beta}) + \gamma^{n-1} L N(u_n)(\cdot L, L^{2\beta}) \\ &= R_{L,\alpha^{n-1}} f_{n-1}(\cdot) + w_n(\cdot) \end{aligned} \tag{3.16}$$

for f_n, where

$$\begin{aligned} \|w_n\| &\leqslant L \gamma^{n-1} \|N(u_n)(\cdot L, L^{2\beta})\| \\ &\leqslant L^3 \gamma^{n-1} \|N(u_n)\|_L \leqslant C L^{2\beta+3} \gamma^{n-1} \|u_n\|_L^{p+1} \\ &\leqslant C L^{2\beta+3} \gamma^{n-1} \|f_{n-1}\|^{p+1}. \end{aligned} \tag{3.17}$$

Writing $f_n = A_n f^* + g_n$ with a constant A_n chosen so that $\hat{g}_n(0) = 0$, we see from (3.14) that

$$\begin{aligned} f_n &= A_{n-1} R_{L,\alpha^{n-1}} f^* + R_{L,\alpha^{n-1}} g_{n-1} + w_n \\ &= A_{n-1} f^* + A_{n-1}\left(R_{L,\alpha^{n-1}} f^* - f^*\right) + R_{L,\alpha^{n-1}} g_{n-1} + w_n. \end{aligned}$$

So $A_n = A_{n-1} + \hat{w}_n(0)$ and

$$g_n = A_{n-1}\left(R^0_{L,\alpha^{n-1}} f^* - f^*\right) + R^0_{L,\alpha^{n-1}} g_{n-1} + w_n - \hat{w}_n(0) f^*. \tag{3.18}$$

As observed for the linear case, $A_n = A_{n-1} = A$, say, for all n. Furthermore, we see that

$$\begin{aligned} \|g_n\| &\leqslant |A| \frac{C}{L^n} + C \frac{1}{L} \|g_{n-1}\| + C \|w_n\| \\ &\leqslant |A| \frac{C}{L^n} + C \frac{1}{L} \|g_{n-1}\| + C L^{2\beta+3} \gamma^{n-1} \|f_{n-1}\|^{p+1}, \end{aligned} \tag{3.19}$$

where we used Lemmas 1 and 2 from the linear case and the bound on w_n from (3.17).

Assume inductively that for all $m \leqslant n-1$, we have
(i) $\| f_m \| \leqslant \| f_0 \|$, $\| f_0 \| \leqslant 1/L$, and
(ii) $\| g_m \| \leqslant CL^{-m(1-\varepsilon)} \| f_0 \|$.
Note that $\| g_0 \| \leqslant C \| f_0 \|$, and that (3.19) implies

$$\| g_n \| CL^{-n} \| f_0 \|^{p+1} + CL^{-\varepsilon} L^{-n(1-\varepsilon)} \| f_0 \| + CL^{2\beta+3} L^{(2\beta-p-1)(n-1)} \| f_0 \|^{p+1}$$

Since $p \geqslant 2$ and $\frac{1}{2} \leqslant \beta \leqslant 1$, if follows that $2\beta - p - 1 \leqslant -1$. Hence, for any $\varepsilon > 0$, for all L large enough and R small enough, we have

$$\| g_n \| \leqslant CL^{-n(1-\varepsilon)} \| f_0 \|$$

where C depends on ε but not on n.
It follows that

$$\| f_n - Af^* \| \leqslant CL^{-n(1-\varepsilon)} \| f_0 \|. \tag{3.20}$$

Setting $t = L^{2\beta n}$, we see from (3.11) that

$$\| t^{1/2\beta} u(t^{1/2\beta} \cdot, t) - Af^*(\cdot) \| \leqslant C \| f_0 \| (t^{-1/2\beta})^{1-\varepsilon} \tag{3.21}$$

for t large enough and $\| f_0 \|$ small enough.
As we remarked in Section 2, this result can be extended to any $t \geqslant L^{2\beta}$.

Theorem 1′. *Let $\frac{1}{2} < \beta \leqslant 1$ and $p \geqslant 2$, then for $\| f_0 \|$ small enough and t_0 large enough, there exist constants A and $C > 0$ such that for all $t \geqslant t_0$*

$$\| u(\cdot t^{1/2\beta}, t) - \frac{A}{t^{1/2\beta}} f^*(\cdot) \| \leqslant C \| f_0 \| (t^{-1/\beta})^{1-\varepsilon}. \tag{3.22}$$

4. L_2- and L_∞-bounds

The error estimates in (3.22) also imply estimates in the L_2- and L_∞-norms on $\mathbb{R}$. For any function $f \in L^2(\mathbb{R})$, Plancherel's Theorem states that $\| f \|_2 = \| \hat{f} \|_{L^2} \leqslant C \| f \|$. Observing that

$$\| f(\gamma \cdot) \|_{L^2} = \gamma^{-1/2} \| f(\cdot) \|_{L^2},$$

(3.22) implies

$$t^{1/4\beta} \| u(\cdot, t) - \frac{A}{t^{1/2\beta}} f^*(\cdot / t^{1/2\beta}) \|_2 \leqslant C \| f_0 \| (t^{-1/2\beta})^{1-\varepsilon}. \tag{4.1}$$

Similarly $\| f \|_\infty \leqslant \| \hat{f} \|_{L^1} \leqslant C \| f \|$, and so we have

$$t^{1/2\beta} \| u(\cdot, t) - \frac{A}{t^{1/2\beta}} f^*(\cdot / t^{1/2\beta}) \|_\infty \leqslant C \| f_0 \| (t^{-1/2\beta})^{1-\varepsilon}. \tag{4.2}$$

These are precisely the estimates claimed in Theorem 1.

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