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ANALYTICITY OF SOLITARY-WAVE SOLUTIONS OF MODEL EQUATIONS FOR LONG WAVES*

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Abstract. It is shown that solitary-wave solutions of model equations for long waves have an analytic extension to a strip in the complex plane that is symmetric about the real axis. The classes of equations to which the analysis applies include equations of Korteweg–de Vries type, the regularized long-wave equations, and particular instances of nonlinear Schrödinger equations.

Key words. nonlinear dispersive wave equations, solitary waves, regularity, analyticity, Korteweg–de Vries-type equations, regularized long-wave-type equations, Schrödinger-type equations

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1. Introduction.

This note is concerned with solitary-wave solutions of model equations for long waves and aims to cast light on their regularity properties. The prototypical example in view is the well-known travelling-wave solution

$$u(x,t) = \phi_c(x-(c+1)t) = 3c\,\mathrm{sech}^2\left(\frac{c^{1/2}}{2}(x-(c+1)t)\right) \tag{1.1}$$

of the classical Korteweg–de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.2}$$

For any positive value of c, the function of x and t defined in (1.1) via the function ϕ_c of one real variable is an exact solution of (1.2) which is infinitely differentiable and which decays rapidly to zero at $\pm\infty$. These properties are possessed by solitary-wave solutions of a considerable range of evolution equations that feature a balance between nonlinearity and dispersion. As these special travelling-wave solutions of nonlinear, dispersive wave equations are known in many cases to play a significant role in the long-term asymptotics of general classes of solutions, they have come in for detailed study in the last couple of decades. Existence and regularity theory for solitary waves has been developed recently by Benjamin et al. [3] and Weinstein [12]. Their results apply to a broad class of model equations to be introduced presently. The outcome of these theories is that the relevant profiles ϕ_c of the solitary-wave solutions are often positive C^∞-functions having a single maximum and which decay monotonically to zero at infinity, just as does the sech^2 solutions of the Korteweg–de Vries equation displayed above. Moreover, ϕ_c and all its derivatives lie in $L_1 \cap L_\infty$.

In fact, the sech^2-solitary-wave solution of (1.2) has further regularity than just C^∞-smoothness. The function ϕ_c in (1.1) defined on the real axis $\mathbb{R}$ is real analytic and admits an analytic extension to the complex strip $\{z = x+iy : |y| < \pi/c^{1/2}\}$. It is this latter property on which attention will be focused in the present study. While the theory developed here seems to apply to a considerable range of equations, the ideas are most transparently presented in the context of the following relatively concrete

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classes of model equations for waves in nonlinear dispersive media:

(1.3) $$u_t + u_x + u^p u_x - (Mu)_x = 0 \quad \text{(Korteweg–de Vries type)},$$

(1.4) $$u_t + u_x + u^p u_x + (Mu)_t = 0 \quad \text{(regularized long-wave type)},$$

(1.5) $$iu_t - Mu + |u|^p u = 0 \quad \text{(Schrödinger type)}.$$

In the first two models, p is a positive integer, while p is a positive even integer in (1.5). The linear operator M is a Fourier multiplier operator defined by

(1.6) $$\widehat{(Mv)}(\xi) = \alpha(\xi)\hat{v}(\xi)$$

whose nonnegative symbol α satisfies certain growth conditions to be spelled out presently. The linear transformation M is called the dispersion operator and its symbol α is related to the linear dispersion relation for the model in question (see Benjamin [2] or Whitham [13]).

We intend to show that as a rule, solitary-wave solutions of these model equations possess the property of being extensible to an analytic function defined on a strip in the complex plane $\mathbb{C}$, which lies symmetrically about the real axis $\mathbb{R}$ on which the wave profile is ostensibly defined. This fact is interesting in its own right, but in addition, it has implications regarding uniqueness [9] and appears to be useful in assessing whether or not a particular solitary wave is actually a soliton (cf. [5], [6], [8]).

The plan of the paper is as follows. In the next section, a few convenient notational conventions are introduced. In §3, the main result for travelling-wave solutions of Korteweg–de Vries type and regularized long-wave type is enunciated and proved. Section 4 is concerned with the analogous result for nonlinear Schrödinger equations. The paper concludes with a few comments about regularity issues related to those discussed here.

2. Notation. By $L_p = L_p(\mathbb{R})$ for p in the range $1 \le p \le \infty$, we mean the standard class of pth-power Lebesgue-integrable functions on the real line $\mathbb{R}$ with the usual modification if $p = \infty$. The standard norm on L_p will be denoted by $\|\cdot\|_p$. The Fourier transform of a Lebesgue-measurable function ϕ defined on $\mathbb{R}$ is denoted by $\hat{\phi}$ and is defined to be

(2.1) $$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{i\xi x} dx.$$

The convolution of two functions f and g defined on $\mathbb{R}$ is written $f * g$. Multiple convolution of a function with itself will appear frequently, and it is therefore convenient to introduce notation for this operation. If ϕ is a measurable function defined on $\mathbb{R}$ and n is a positive integer, define the function $\mathcal{V}_n\phi$ by the recipe

$$\mathcal{V}_1\phi = \phi,$$

and for $n > 1$,

(2.2) $$\begin{aligned} \mathcal{V}_n\phi(x) &= (\phi * \mathcal{V}_{n-1}\phi)(x) \\ &= \int_{-\infty}^{\infty} \phi(x-y)\mathcal{V}_{n-1}\phi(y)dy. \end{aligned}$$

By a *solitary-wave solution* of (1.3) or (1.4) for a given positive integer p and dispersion symbol α, we shall mean a function $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ, ϕ', and $M\phi$ all lie in L_2 and such that for some positive constant c, $\phi(x - ct)$ defines an L_2-solution of (1.3) or (1.4). A similar definition will be adopted later for solutions of (1.5). As mentioned already, existence of such solutions for a wide range of symbols α has been dealt with in the recent works of Benjamin et al. [3] and Weinstein [12].

For any $x \in \mathbb{R}$, the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$.

3. Results for Korteweg–de Vries type and regularized long-wave models. After a preparatory lemma, the principal result for equations of the types depicted in (1.3) and (1.4) is stated and proved.

LEMMA 1. *Let $c > 1$ be given. Suppose $\phi = \phi(x - ct)$ defines a solitary-wave solution of* (1.3) *or* (1.4) *for a given value of p and symbol α of the dispersion operator M. Suppose also that for some positive constants A and r, $\alpha(\xi) \geq A|\xi|^r$ for all $\xi \in \mathbb{R}$. Then the function*

$$\hat{\psi}(\xi) = \frac{\hat{\phi}(\xi)}{\sqrt{2\pi}[(p+1)(c-1)]^{\frac{1}{p}}} \tag{3.1}$$

lies in $L_1 \cap L_2$ and solves the equation

$$(1 + \lambda\alpha(\xi))\,\hat{\psi}(\xi) = \mathcal{V}_{p+1}\hat{\psi}(\xi), \tag{3.2}$$

where $\lambda = 1/(c-1)$ if ϕ is a solution of (1.3) *and $\lambda = c/(c-1)$ if ϕ is a solution of* (1.4).

Proof. Suppose ϕ defines a solitary-wave solution of (1.3) as described in §2. Then

$$(-c+1)\phi' + \phi^p\phi' - M\phi' = 0,$$

from which it follows that, at least in the sense of tempered distributions,

$$[(c-1) + M]\phi - \frac{1}{p+1}\phi^{p+1} = \text{ constant}. \tag{3.3}$$

Since each term on the left-hand side is an L_2-function by assumption, the constant on the right-hand side must be zero. Applying the Fourier transform to (3.3) and using (1.6) leads directly to the desired result (3.2) with $\lambda = 1/(c-1)$.

Because $c > 1$ and M has a nonnegative symbol, it follows from (3.3) with the constant equal to zero that

$$\phi = \frac{1}{p+1}[(c-1) + M]^{-1}\phi^{p+1}. \tag{3.3'}$$

Since $\phi \in H^1$ by assumption, the product ϕ^{p+1} is also in H^1. For any $s \in \mathbb{R}$, the linear operator $(c - 1 + M)^{-1}$ maps H^s into H^{s+r}. Hence it transpires from (3.3′) that $\phi \in H^{1+r}$. In consequence, $\phi^{p+1} \in H^{1+r}$, whence $\phi \in H^{1+2r}$, and so on. It is thus inferred that $\phi \in H^\infty$, from which it is adduced at once that

$$\int_{-\infty}^{\infty} (1 + \xi^2)^m\, |\hat{\psi}(\xi)|^2 d\xi < \infty \tag{3.4}$$

for any m. An immediate consequence of (3.4) is that $\hat{\psi} \in L_1 \cap L_2$, as stated in the lemma.

The same considerations lead to the advertised result when ϕ defines instead a solitary-wave solution of (1.4). □

With this simple lemma in hand, the main issue may be confronted. The idea is to demonstrate that if ϕ defines a solitary-wave solution of (1.3) or (1.4), then its Fourier transform $\hat{\phi}$ has exponential decay at $\pm\infty$. In consequence, the Paley–Wiener theorem assures that ϕ itself is analytic in a complex strip centered about the real axis.

We begin with a special case of the main result, which will prove to be instructive and which contains the essence of the argument that applies to the more general situations.

THEOREM 2. *Let an integer $p \geq 1$ and a wave speed $c > 1$ be given. Suppose that ϕ as in Lemma 1 defines a solitary-wave solution of (1.3) or (1.4) corresponding to the dispersive symbol $\alpha(\xi) = |\xi|^m$ for some real number $m \geq 1$. Then there exists a constant $\sigma > 0$ such that for any μ with $0 < \mu < \sigma$,*

$$\sup_{\xi\in\mathbb{R}} e^{\mu|\xi|}|\hat{\phi}(\xi)| < \infty. \tag{3.5}$$

Proof. By Lemma 1, it suffices to prove (3.5) for the function $\hat{\psi}$ defined in (3.1) that satisfies equation (3.2).

For any k with $0 \leq k \leq m$ and $\lambda > 0$, define the nonnegative function f_k for $\xi \geq 0$ by

$$f_k(\xi) = \frac{\xi^k}{1 + \lambda\xi^m}.$$

It is straightforward to determine that for all $\xi \geq 0$,

$$f_k(\xi) \leq \frac{\delta_k}{\lambda^{\frac{k}{m}}}, \tag{3.6}$$

where $\delta_k = \left(\frac{k}{m}\right)^{\frac{k}{m}} \left(1 - \frac{k}{m}\right)^{1-\frac{k}{m}}$ if $0 < k < m$, and $\delta_m = \delta_0 = 1$.

Case I. $m \geq 1$ is an integer.

Suppose that $\hat{\psi}$ satisfies (3.2) and (3.4). When $0 \leq k \leq m-1$, (3.6) may be used to conclude that

$$\begin{aligned} \left|\xi^k \hat{\psi}(\xi)\right| &= \frac{|\xi|^k}{1+\lambda|\xi|^m} \left|\mathcal{V}_{p+1}\hat{\psi}(\xi)\right| \\ &\leq \frac{\delta_k}{\lambda^{\frac{k}{m}}} \left|\mathcal{V}_{p+1}\hat{\psi}(\xi)\right| \\ &\leq \frac{1}{\lambda^{\frac{k}{m}}} \mathcal{V}_{p+1}|\hat{\psi}|(\xi) \\ &\leq \frac{1}{\lambda^{\frac{k}{m}}} \left(\frac{kp}{m}+1\right)^{k-1} \mathcal{V}_{p+1}|\hat{\psi}|(\xi) \end{aligned} \tag{3.7}$$

for any $\xi \in \mathbb{R}$. On the other hand, for any $n \geq 0$ and any $\xi_1 \in \mathbb{R}$, we have

$$|\xi_1|^{m+n}|\hat{\psi}(\xi_1)| = \frac{|\xi_1|^{m+n}}{1+\lambda|\xi_1|^m}\left|\mathcal{V}_{p+1}\hat{\psi}(\xi_1)\right| \le \frac{|\xi_1|^n}{\lambda}\mathcal{V}_{p+1}|\hat{\psi}|(\xi_1)$$

$$= \frac{1}{\lambda}\int_{-\infty}^{\infty}\left|\hat{\psi}(\xi_1-\xi_2)\right|\int_{-\infty}^{\infty}\left|\hat{\psi}(\xi_2-\xi_3)\right|\int_{-\infty}^{\infty}$$

$$\cdots\int_{-\infty}^{\infty}\left|\left(\sum_{i=1}^{p}(\xi_i-\xi_{i+1})+\xi_{p+1}\right)^n\hat{\psi}(\xi_p-\xi_{p+1})\hat{\psi}(\xi_{p+1})\right|d\xi_{p+1}d\xi_p\cdots d\xi_2$$

(3.8)
$$\le \frac{1}{\lambda}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{\sum_{i=1}^{p+1}r_i=n}\frac{n!}{r_1!\cdots r_{p+1}!}$$

$$\cdot\left|\xi_{p+1}^{r_{p+1}}\hat{\psi}(\xi_{p+1})\prod_{i=1}^{p}(\xi_i-\xi_{i+1})^{r_i}\hat{\psi}(\xi_i-\xi_{i+1})\right|d\xi_{p+1}\cdots d\xi_2$$

$$\le \frac{1}{\lambda}\sum_{|r|=n}\binom{n}{r}\left|(\cdot)^{r_1}\hat{\psi}(\cdot)\right| * \left|(\cdot)^{r_2}\hat{\psi}(\cdot)\right| * \cdots * \left|(\cdot)^{r_{p+1}}\hat{\psi}(\cdot)\right|(\xi_1),$$

where $|(\cdot)^{r_i}\hat{\psi}(\cdot)|(\xi) = |\xi^{r_i}\hat{\psi}(\xi)|$ and we have introduced the standard multiindex notation $r=(r_1,\ldots,r_{p+1})$, $|r| = r_1+\cdots+r_{p+1}$, and $\binom{n}{r} = \frac{n!}{r_1!\cdots r_{p+1}!}$. If $0 \le l \le m-1$, then using (3.7) in (3.8) leads to the inequality

(3.9)
$$|\xi_1|^{m+l}|\hat{\psi}(\xi_1)| \le \frac{1}{\lambda}\sum_{|r|=l}\binom{l}{r}\left(\prod_{i=1}^{p+1}\frac{1}{\lambda^{\frac{r_i}{m}}}\right)\overbrace{\mathcal{V}_{p+1}|\hat{\psi}| * \cdots * \mathcal{V}_{p+1}|\hat{\psi}|}^{p+1 \text{ copies of } \mathcal{V}_{p+1}|\hat{\psi}|}(\xi_1)$$

$$= \frac{1}{\lambda^{1+\frac{l}{m}}}\sum_{|r|=l}\binom{l}{r}\mathcal{V}_{(p+1)^2}|\hat{\psi}|(\xi_1) = \frac{1}{\lambda^{\frac{m+l}{m}}}(p+1)^l\mathcal{V}_{(p+1)^2}|\hat{\psi}|(\xi_1).$$

It follows from (3.7) and (3.9) that for any $\xi \in \mathbb{R}$ and any k with $0 \le k \le 2m-1$, one has

(3.10)
$$\left|\xi^k\hat{\psi}(\xi)\right| \le \frac{1}{\lambda^{\frac{k}{m}}}\left(\frac{kp}{m}+1\right)^{k-1}\mathcal{V}_{(p+1)(\lfloor\frac{k}{m}\rfloor p+1)}|\hat{\psi}|(\xi),$$

where, as mentioned previously, $\lfloor\frac{k}{m}\rfloor$ denotes the greatest integer less than or equal to $\frac{k}{m}$.

It is intended to establish (3.10) for all values of k, and to this end we argue by induction, supposing that the inequality (3.10) is true for all k with $0 \le k \le nm-1$ for a fixed integer $n \ge 2$. Let $k = nm+l$ for some integer l in $[0, m-1]$. Then (3.8) and the induction hypothesis allow one to infer the following inequality:

(3.11)
$$\begin{aligned}\left|\xi^k\hat\psi(\xi)\right| &\le \frac{1}{\lambda}\sum_{|r|=(n-1)m+l}\binom{(n-1)m+l}{r}\left|(\cdot)^{r_1}\hat\psi\right|*\left|(\cdot)^{r_2}\hat\psi\right|*\cdots*\left|(\cdot)^{r_{p+1}}\hat\psi\right|(\xi)\\ &\le \frac{1}{\lambda}\sum_{|r|=(n-1)m+l}\binom{(n-1)m+l}{r}\left(\prod_{i=1}^{p+1}\frac{(\frac{r_ip}{m}+1)^{r_i-1}}{\lambda^{\frac{r_i}{m}}}\right)\mathcal{V}_{(p+1)(\lfloor\frac{r_1}{m}\rfloor p+1)}|\hat\psi|*\\ &\quad *\mathcal{V}_{(p+1)(\lfloor\frac{r_2}{m}\rfloor p+1)}|\hat\psi|*\cdots*\mathcal{V}_{(p+1)(\lfloor\frac{r_{p+1}}{m}\rfloor p+1)}|\hat\psi|(\xi)\\ &= \frac{1}{\lambda^{\frac{nm+l}{m}}}\sum_{|r|=(n-1)m+l}\binom{(n-1)m+l}{r}\left(\prod_{i=1}^{p+1}(\tfrac{r_ip}{m}+1)^{r_i-1}\right)\mathcal{V}_{\sum_{j=1}^{p+1}(p+1)(\lfloor\frac{r_j}{m}\rfloor p+1)}|\hat\psi|(\xi).\end{aligned}$$

If inequality (3.10) is specialized to the case $k=0$, one infers that $|\hat\psi|\le\mathcal{V}_{p+1}|\hat\psi|$. Using this fact and the elementary formula

$$\begin{aligned}&\sum_{i=1}^{p+1}(p+1)\left(\left\lfloor\frac{r_i}{m}\right\rfloor p+1\right)\\ &= (p+1)\left(p\sum_1^{p+1}\left\lfloor\frac{r_i}{m}\right\rfloor+\sum_1^{p+1}1\right)\le(p+1)\left(p\left\lfloor\sum_1^{p+1}\frac{r_i}{m}\right\rfloor+p+1\right)\\ &= (p+1)\left(p\left\lfloor\frac{(n-1)m+l}{m}\right\rfloor+p+1\right)=(p+1)\left(p\left\lfloor\frac{nm+l}{m}\right\rfloor+1\right)\\ &= (p+1)\left(\left\lfloor\frac{k}{m}\right\rfloor p+1\right),\end{aligned}$$

one obtains

$$\mathcal{V}_{\sum_1^{p+1}(p+1)(\lfloor r_i/m\rfloor p+1)}|\hat\psi|\le\mathcal{V}_{(p+1)(\lfloor\frac{k}{m}\rfloor p+1)}|\hat\psi|. \tag{3.12}$$

Using a specialization of the multinomial Abel identity (see [11, p. 26]), namely

$$\begin{aligned}&A_N(x_1,x_2,\dots,x_M)\\ &= \sum_{|k|=N}\binom{N}{k}\prod_{i=1}^{M}(x_i+k_i)^{k_i-1}\\ &= (x_1x_2\cdots x_M)^{-1}\left(\sum_1^M x_i\right)\left(\sum_1^M x_i+N\right)^{N-1},\end{aligned}$$

and the simple relation $\sum_1^{p+1}(r_i-1)=(n-1)m+l-p-1$, one obtains

$$\begin{aligned}&\sum_{|r|=(n-1)m+l}\binom{(n-1)m+l}{r}\prod_{i=1}^{p+1}\left(\frac{r_ip}{m}+1\right)^{r_i-1}\\ &= \sum_{|r|=(n-1)m+l}\binom{(n-1)m+l}{r}\left(\frac{p}{m}\right)^{\sum_1^{p+1}(r_i-1)}\prod_1^{p+1}\left(\frac{m}{p}+r_i\right)^{r_i-1}\end{aligned}$$

$$= \left(\frac{p}{m}\right)^{(n-1)m+l-p-1} A_{(n-1)m+l}\left(\frac{m}{p}, \frac{m}{p}, \ldots, \frac{m}{p}\right)$$

$$= \left(\frac{p}{m}\right)^{(n-1)m+l-p-1} \left(\frac{m}{p}\right)^{-(p+1)} \frac{(p+1)m}{p}\left(\frac{(p+1)m}{p} + (n-1)m + l\right)^{(n-1)m+l-1}$$

$$= (p+1)\left(\frac{nm+l}{m}p + 1\right)^{(n-1)m+l-1}$$

$$= (p+1)\left(\frac{kp}{m} + 1\right)^{k-1-m}$$

$$\leq \left(\frac{kp}{m} + 1\right)^{k-1}.$$

The latter inequality, when combined with (3.11), (3.12), and the induction hypothesis, yields

$$\left|\xi^k \hat{\psi}(\xi)\right| \leq \frac{1}{\lambda^{\frac{k}{m}}}\left(\frac{kp}{m} + 1\right)^{k-1} \mathcal{V}_{(p+1)(\lfloor \frac{k}{m} \rfloor p+1)}|\hat{\psi}|(\xi)$$

for any integer $k \geq 0$. It follows that (3.10) holds for any $\xi \in \mathbb{R}$ and any integer $k \geq 0$.

Using the fact that $\hat{\psi} \in L_1(\mathbb{R})$, we infer that

$$\mathcal{V}_{(p+1)(\lfloor \frac{k}{m} \rfloor p+1)}|\hat{\psi}| \leq \|\hat{\psi}\|_2 \left\|\mathcal{V}_{(p+1)(\lfloor \frac{k}{m} \rfloor p+1)-1}|\hat{\psi}|\right\|_2$$

$$\leq \|\hat{\psi}\|_2 \|\hat{\psi}\|_1 \left\|\mathcal{V}_{(p+1)(\lfloor \frac{k}{m} \rfloor p+1)-2}|\hat{\psi}|\right\|_2$$

$$\leq \cdots \leq \|\hat{\psi}\|_2^2 \|\hat{\psi}\|_1^{(p+1)(\lfloor \frac{k}{m} \rfloor p+1)-2},$$

and so

$$|\xi^k \hat{\psi}(\xi)| \leq \frac{\|\hat{\psi}\|_2^2}{\lambda^{k/m}}\left(\frac{kp}{m} + 1\right)^{k-1} \|\hat{\psi}\|_1^{(p+1)(\lfloor k/m \rfloor p+1)-2} \tag{3.13}$$

for any $\xi \in \mathbb{R}$.

To complete the proof for Case I, consider the sequence

$$a_k = \frac{1}{k!\,\lambda^{\frac{k}{m}}}\left(\frac{kp}{m} + 1\right)^{k-1} \|\hat{\psi}\|_1^{(p+1)(\frac{kp}{m}+1)} \tag{3.14}$$

for $k = 0, 1, 2, \ldots$. Because the ratio $\frac{a_{k+1}}{a_k}$ takes the form

$$\frac{a_{k+1}}{a_k} = \frac{k!\,\lambda^{\frac{k}{m}}\left(\frac{(k+1)p}{m} + 1\right)^{k+1-1} \|\hat{\psi}\|_1^{(p+1)\left(\frac{(k+1)p}{m}+1\right)}}{(k+1)!\,\lambda^{\frac{k+1}{m}}\left(\frac{kp}{m} + 1\right)^{k-1} \|\hat{\psi}\|_1^{(p+1)(\frac{kp}{m}+1)}}$$

$$= \frac{1}{\lambda^{1/m}} \|\hat{\psi}\|_1^{\frac{(p+1)p}{m}} \left(\frac{p}{m} + \frac{1}{k+1}\right)\left[\left(1 + \frac{p}{m+kp}\right)^{\frac{m+kp}{p}}\right]^{\frac{p(k-1)}{pk+m}},$$

it is readily seen that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{ep}{m\ \lambda^{1/m}} \|\hat{\psi}\|_1^{\frac{(p+1)p}{m}}.$$

Hence the power series $\sum_{k=0}^{\infty} a_k \mu^k$ converges for $|\mu| < \frac{m\lambda^{1/m}}{ep} \|\hat{\psi}\|_1^{\frac{-(p+1)p}{m}}$. In consequence of (3.13) and (3.14), it is seen that for any $\xi \in \mathbb{R}$,

$$\begin{aligned} e^{\mu|\xi|}|\hat{\psi}(\xi)| &= \sum_{k=0}^{\infty} \frac{\mu^k|\xi|^k}{k!}|\hat{\psi}(\xi)| \\ &\le \frac{\|\hat{\psi}\|_2^2}{\|\hat{\psi}\|_1^2} \sum_{k=0}^{\infty} \frac{\mu^k}{k!\ \lambda^{\frac{k}{m}}} \left(\frac{kp}{m}+1\right)^{k-1} \|\hat{\psi}\|_1^{(p+1)(\lfloor k/m \rfloor p+1)} \\ &\le \frac{\|\hat{\psi}\|_2^2}{\|\hat{\psi}\|_1^2} \sum_{k=0}^{\infty} a_k \mu^k < \infty, \end{aligned}$$

provided $|\mu| < \frac{m\lambda^{1/m}}{ep} \|\hat{\psi}\|_1^{\frac{-(p+1)p}{m}}$. Thus the function $e^{\mu|\xi|}|\hat{\psi}(\xi)|$ appears to be uniformly bounded for such choices of μ, and this is the desired conclusion in case m is a positive integer.

Case II. $m > 1$ is not an integer.

If $m_0 = \lfloor m \rfloor$ and $\varrho = \max_{0\le\xi<\infty} \frac{1+\lambda\xi^{m_0}}{1+\lambda\xi^m}$, then it follows from (3.2) that

$$|\hat{\psi}(\xi)| \le \frac{\varrho}{1+\lambda|\xi|^{m_0}} \mathcal{V}_{p+1}|\hat{\psi}|(\xi). \tag{3.15}$$

Now one may use (3.15) and induction as in the proof of Case I to prove that

$$\left|\xi^k\hat{\psi}(\xi)\right| \le \frac{\varrho^{(p+1)\lfloor k/m_0 \rfloor +1}}{\lambda^{k/m_0}} \left(\frac{kp}{m_0}+1\right)^{k-1} \mathcal{V}_{(p+1)\left(\lfloor \frac{k}{m_0} \rfloor p+1\right)} |\hat{\psi}|(\xi) \tag{3.16}$$

holds for any integer $k \ge 0$ and all $\xi \in \mathbb{R}$. In consequence, the following inequality is obtained for integers k and $\xi \in \mathbb{R}$:

$$\left|\xi^k\hat{\psi}(\xi)\right| \le \frac{\varrho^{(p+1)\lfloor k/m_0 \rfloor +1}}{\lambda^{k/m_0}} \left(\frac{kp}{m_0}+1\right)^{k-1} \|\hat{\psi}\|_2^2 \|\hat{\psi}\|_1^{(p+1)(\lfloor k/m_0 \rfloor p+1)-2}.$$

Thus it appears that for all $\xi \in \mathbb{R}$,

$$\begin{aligned} e^{\mu|\xi|}|\hat{\psi}(\xi)| &= \sum_{k=0}^{\infty} \frac{\mu^k|\xi|^k}{k!}|\hat{\psi}(\xi)| \\ &\le \frac{\|\hat{\psi}\|_2^2}{\|\hat{\psi}\|_1^2} \sum_{k=0}^{\infty} \frac{\mu^k \varrho^{(p+1)\lfloor k/m_0 \rfloor +1}}{k!\ \lambda^{k/m_0}} \left(\frac{kp}{m_0}+1\right)^{k-1} \|\hat{\psi}\|_1^{(p+1)(\lfloor k/m_0 \rfloor p+1)} < \infty \end{aligned}$$

for any μ satisfying $0 < \mu < \frac{m_0\lambda^{1/m_0}}{ep\varrho^{(p+1)/m_0}} \|\hat{\psi}\|_1^{-(p+1)p/m_0}$.

If the results just obtained for $\hat{\psi}$ are translated into results about $\hat{\phi}$, it appears that if $\phi(x-ct)$ defines a solitary-wave solution of (1.3), then

$$\sup_{\xi\in\mathbb{R}} e^{\mu|\xi|}|\hat{\phi}(\xi)| < \infty \tag{3.17}$$

for any μ satisfying

$$0 < \mu < \frac{m(c-1)^{p/m}(p+1)^{(p+1)/m}(2\pi)^{(p+1)p/2m}}{ep}\|\hat{\phi}\|_1^{\frac{-(p+1)p}{m}} = \rho_1(m,c,\phi)$$

when $1 \le \lfloor m \rfloor = m$, or for any μ satisfying

$$0 < \mu < \frac{m_0(c-1)^{p/m_0}(p+1)^{(p+1)/m_0}(2\pi)^{(p+1)p/2m_0}}{ep\varrho^{(p+1)/m_0}}\|\hat{\phi}\|_1^{\frac{-(p+1)p}{m}} = \rho_1(m,c,\phi)$$

when $1 \le m_0 = \lfloor m \rfloor < m$.

On the other hand, if $\phi(x-ct)$ defines a solitary-wave solution of (1.4), then (3.17) holds for this ϕ for any μ with

$$0 < \mu < \frac{mc^{1/m}(c-1)^{p/m}(p+1)^{(p+1)/m}(2\pi)^{(p+1)p/2m}}{ep}\|\hat{\phi}\|_1^{\frac{-(p+1)p}{m}} = \rho_2(m,c,\phi)$$

when $1 \le \lfloor m \rfloor = m$, or for any μ with

$$0 < \mu < \frac{m_0c^{1/m_0}(c-1)^{p/m_0}(p+1)^{(p+1)/m_0}(2\pi)^{(p+1)p/2m_0}}{ep\varrho^{(p+1)/m_0}}\|\hat{\phi}\|_1^{\frac{-(p+1)p}{m_0}} = \rho_2(m,c,\phi)$$

when $1 \le m_0 = \lfloor m \rfloor < m$.

The theorem is thus seen to be valid if one chooses $\sigma = \rho_1$ for solutions of (1.3) and $\sigma = \rho_2$ for solutions of (1.4). □

An inspection of the proof presented above shows that the specific assumption $\alpha(\xi) = |\xi|^m$ is not needed. Indeed, the presumption that there are positive constants $A > 0$ and $m \ge 1$ such that

$$A|\xi|^m \le \alpha(\xi) \tag{3.18}$$

for all $\xi \in \mathbb{R}$ suffices for our theory. The lower bound in (3.18) implies that the normalized Fourier transform $\hat{\psi}$ satisfies

$$|\hat{\psi}(\xi)| = \frac{1}{1+\lambda\alpha(\xi)}\left|\mathcal{V}_{p+1}\hat{\psi}(\xi)\right| \le \frac{1}{1+\lambda A|\xi|^m}\left|\mathcal{V}_{p+1}\hat{\psi}(\xi)\right|, \tag{3.19}$$

and it is this inequality that is the basis for the estimates appearing in the proof of Theorem 2. In consequence of these remarks, we can assert the following corollary to the proof of Theorem 2.

COROLLARY 3. *Let $u(x,t) = \phi(x-ct)$ be a solitary-wave solution of the equation*

$$u_t + u_x + u^p u_x - (Mu)_x = 0$$

or the equation

$$u_t + u_x + u^p u_x + (Mu)_t = 0,$$

where $p \ge 1$ is an integer and $\widehat{M\phi}(\xi) = \alpha(\xi)\hat{\phi}(\xi)$ with $\alpha(\xi)$ satisfying (3.18) *for some $m \ge 1$ and $A > 0$. Then there exists a constant $\sigma > 0$ such that*

$$\sup_{\xi\in\mathbb{R}} e^{\mu|\xi|}|\hat{\phi}(\xi)| < \infty$$

for any μ with $0 < \mu < \sigma$.

The result concerning analyticity of ϕ now follows immediately from Theorem 2 or Corollary 3 together with the Paley–Wiener theorem.

THEOREM 4. *Let ϕ satisfy the assumptions of Corollary 3 and let $\sigma > 0$ be as in the conclusion of this corollary. Then there is a function $\Phi(z)$ defined and holomorphic on the open strip $\{z \in \mathbb{C} : |\Im z| < \sigma\}$ such that $\Phi(x) = \phi(x)$ for all $x \in \mathbb{R}$.*

Proof. Let μ lie in the open interval $(0, \sigma)$. Choose a $\mu_1 > 0$ satisfying $0 < \mu < \mu_1 < \sigma$. Then it follows that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2\mu|\xi|}|\hat{\phi}(\xi)|^2 d\xi &= \int_{-\infty}^{\infty} e^{-2(\mu_1-\mu)|\xi|}e^{2\mu_1|\xi|}|\hat{\phi}(\xi)|^2 d\xi \\ &\le \sup_{\xi\in\mathbb{R}} \left(e^{\mu_1|\xi|}|\hat{\phi}(\xi)|\right)^2 \int_{-\infty}^{\infty} e^{-2(\mu_1-\mu)|\xi|} d\xi < \infty.\end{aligned} \tag{3.20}$$

Define the function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{-iz\xi} d\xi$$

for any $z = x + iy \in \Omega = \{z \in \mathbb{C};\ |\Im z| < \sigma\}$. Using (3.20) and the Paley–Wiener theorem [10], one may conclude that $\Phi(z)$ is a well-defined, analytic function on Ω. Of course, Plancherel's theorem implies that $\Phi(x) = \phi(x)$ for any $x \in \mathbb{R}$. $\square$

An immediate consequence of the analyticity expressed in Theorem 4 is the following interesting result.

COROLLARY 5. *Suppose the hypotheses of Corollary 3 to hold and let ϕ be a solitary-wave solution of (1.3) or (1.4). Then ϕ cannot have compact support, nor can it be the case that in any bounded set $S \subset \mathbb{R}$, there are more than a finite number of points $x_\nu \in S$ such that $\phi(x_\nu) = \nu$. In particular, ϕ has at most finitely many zeros in any bounded subset of $\mathbb{R}$.*

Remarks. It is worth contrasting the last result with that obtained for the evolution equation

$$u_t + u^p u_x + (u^q)_{xxx} = 0, \tag{3.21}$$

where $q > 1$ is an integer. In equation (3.21), the dispersive term is singular, and this fact accounts for the compactly supported travelling-wave solutions (compactons) discovered recently by Rosenau (see Hyman and Rosenau [7]). As Corollary 5 shows, such solutions are not possible when the dispersion is nonsingular.

In case the symbol $\alpha(\xi) = |\xi|^m$, where m is an even integer, one may establish the analyticity of ϕ by recourse to the local theory of ordinary differential equations. It is not immediately transparent even in this case that the real analyticity thereby established extends to analyticity in a complex strip. However, a little work in this context reveals the truth of this assertion. These methods make no impression in case the symbol α does not generate a local operator.

4. Further extensions. It is the purpose of this short section to expand the range of the discussion to include equations of Schrödinger type as depicted in (1.5). In (1.5), it is supposed that M is a dispersion operator with the symbol α as in (1.6) and that $p = 2r$ is an even natural number.

The travelling-wave solutions of (1.5) of interest here have the general form $e^{i\omega t}\psi_{\omega,\theta}(x - \theta t)$, where ω and θ are real numbers with $0 \le \omega \le 2\pi$, say, and with $\psi : \mathbb{R} \to \mathbb{C}$ a smooth function lying in $L_1 \cap L_\infty$. Of special interest are the so-called *bound states.* These are standing-wave solutions of (1.5) for which $\theta = 0$, $\omega = \Omega > 0$, and $|\psi|$ tends rapidly to zero at infinity. The function $\psi_{\omega,0} = \phi_\Omega$ defining a bound state $e^{i\Omega t}\phi_\Omega(x)$ satisfies the equation

$$\Omega\phi + M\phi - |\phi|^{2r}\phi = 0. \tag{4.1}$$

In the applications associated to Schrödinger equations, particular importance is attached to *ground states*, which are bound states that minimize energy subject to fixed charge. The associated waveforms ϕ_Ω are analogous to solitary-wave solutions of (1.3) and (1.4) in that they are real valued, even, and rapidly decreasing to zero at infinity. Such solutions fall under the auspices of our previous theory.

THEOREM 6. *Let $\Omega > 0$ and let ϕ_Ω be a ground-state solution of* (1.5) *that lies in $L_1 \cap L_2$. Suppose $p = 2r$, where r is a positive integer, and suppose the symbol α of M to satisfy* (3.18). *Then there exist a $\sigma > 0$ and a function Φ_Ω defined and analytic on the strip $\{z = x + iy : |y| < \sigma\}$ such that $\Phi_\Omega(x) = \phi_\Omega(x)$ for all $x \in \mathbb{R}$.*

The range of applicability of this result may be considerably broadened if the dispersion operator is suitably specialized. Consider, for example, the special case where $\alpha(k) = k^2$, corresponding to the one-dimensional equation

$$iu_t + u_{xx} + |u|^{2r}u = 0, \tag{4.2}$$

with $r = 1$ corresponding to the classical cubic Schrödinger equation. In this case, we have the following simple lemma (cf. Bona and Soyeur [4]) relating bound states to more general travelling-wave solutions. Define $T_\theta : H^1 \to H^1$ by

$$(T_\theta u)(x) = e^{i\frac{1}{2}\theta x}u(x). \tag{4.3}$$

LEMMA 7. *Let ϕ be an H^1-function and let $\psi = T_\theta\phi$ for some $\theta \in \mathbb{R}$. Then ϕ defines a bound state of* (4.2) *corresponding to the parameter $\Omega = \omega - \frac{1}{4}\theta^2 > 0$ if and only if $\psi = \psi_{\omega,\theta}$ defines a travelling-wave solution of* (4.2).

Suppose that $e^{i\omega t}\psi_{\omega,\theta}(x - \theta t)$ is a travelling-wave solution of (4.1) corresponding to a bound state $e^{i\Omega t}\phi_\Omega(x)$ under the transformation in (4.3). Suppose also that ϕ_Ω is actually a ground state. Then according to Theorem 6, ϕ_Ω is the restriction to the real axis of a function Φ_Ω that is analytic in a strip $\{z : |\Im(z)| < \sigma\}$. It follows that $\psi_{\omega,\theta}$ is likewise the restriction to the real axis of a function $\Psi_{\omega,\theta}$ analytic in the same strip, namely the function

$$\Psi_{\omega,\theta}(z) = e^{i\frac{1}{2}\theta z}\phi_\Omega(z).$$

While this result is a consequence of the general theory, such considerations are not required in this special case. Equation (4.1) for ϕ_Ω can be solved explicitly in case $M = -\partial_x^2$, and one readily finds that

$$\phi_\Omega(x) = A\,\mathrm{sech}^{1/r}(Bx),$$

where $A = \sqrt[2r]{(r+1)\Omega}$ and $B = r\sqrt{\Omega}$.

A more challenging situation arises when the symbol $\alpha(k)$ is a perturbation of the Laplacean. Suppose that $\psi_{\omega,\theta}$ defines a travelling-wave solution of (1.5) by the formula

$$u(x,t) = e^{i\omega t}\psi_{\omega,\theta}(x - \theta t).$$

Then $\psi_{\omega,\theta}$ satisfies the equation

$$-\omega\psi - i\theta\psi' - M\psi + |\psi|^{2r}\psi = 0.$$

Guided by the considerations that arose when $M = -\partial_x^2$ in (4.2) and (4.3), we write $\psi_{\omega,\theta}(x) = e^{i\frac{1}{2}\theta x}\phi(x)$. A computation shows ϕ to satisfy the equation

$$(-\omega + \theta^2/2)\phi - i\theta\phi' - \widetilde{M}\phi + |\phi|^{2r}\phi = 0, \tag{4.4}$$

where the symbol $\tilde{\alpha}$ of the operator $\widetilde{M}$ is given by

$$\tilde{\alpha}(\xi) = \alpha(\xi - \theta/2).$$

Assuming that ϕ is real valued, equation (4.4) takes the form

$$\widetilde{M}\phi + i\theta\phi' + (\omega - \theta^2/2)\phi = \phi^{2r+1}, \tag{4.5}$$

or, in Fourier-transformed variables,

$$\left[\alpha(\xi - \theta/2) + \xi\theta + \omega - \theta^2/2\right] \hat{\phi}(\xi) = \widehat{\phi^{2r+1}}(\xi). \tag{4.6}$$

Write $\alpha(\xi) = \xi^2 + \beta(\xi)$, where $\beta(\xi) \geq c|\xi|^m$ for some constants $m \geq 0$ and $c \geq 0$. Then the symbol on the left-hand side of (4.6) may be written as

$$\xi^2 + \beta(\xi - \theta/2) + \omega - \frac{1}{4}\theta^2 \geq \xi^2 + c|\xi - \theta/2|^m + \Omega,$$

where $\Omega = \omega - \frac{1}{4}\theta^2$ as before. If $\Omega > 0$, then obviously we have

$$\xi^2 + \beta(\xi - \theta/2) + \Omega \geq A_1 + A_2|\xi|^2 \tag{4.7}$$

for suitably chosen positive constants A_1 and A_2. Because of (4.7), the theory developed in §3 may be brought to bear, and we ascertain immediately that ϕ has an analytic extension Φ to a strip in $\mathbb{C}$ centered about the real axis. In consequence of the relationship between ϕ and ψ, the same conclusion is drawn about ψ. This result is summarized in our last proposition.

PROPOSITION 8. *Suppose the symbol α of the dispersion operator M to have the form $\alpha(\xi) = \xi^2 + \beta(\xi)$, where $\beta(\xi) \geq c|\xi|^m$ for some $c \geq 0$ and $m \geq 0$. Let $u(x,t) = e^{i\omega t}\psi_{\omega,\theta}(x - \theta t)$ be a travelling-wave solution of* (1.5), *where $\omega - \frac{1}{4}\theta^2 > 0$. Suppose $\psi_{\omega,\theta}(y) = e^{i\frac{1}{2}\theta y}\phi(y)$, where ϕ is real-valued. Then there is a $\sigma > 0$ and a function $\Psi_{\omega,\theta}$ analytic in the strip $\{z : |\Im(z)| < \sigma\}$ such that $\Psi_{\omega,\theta}(x) = \psi_{\omega,\theta}(x)$ for all $x \in \mathbb{R}$.*

Remark. Equation (4.1) arises in more than one space dimension in the form

$$iu_t - Mu + |u|^{2r}u = 0, \tag{4.8}$$

where $u = u(x_1, x_2, \ldots, x_n, t)$ and M is a Fourier multiplier operator defined by

$$\widehat{Mv}(\xi_1, \xi_2, \ldots, \xi_n) = \alpha(\xi_1, \xi_2, \ldots, \xi_n)\, \hat{v}(\xi_1, \xi_2, \ldots, \xi_n).$$

Travelling-wave solutions analogous to those considered in one dimension have the form $e^{i\omega t}\psi_{\omega,\theta}(x - \theta t)$ where $t, \omega \in \mathbb{R}$ and $x, \theta \in \mathbb{R}^n$. Bound states correspond to $\theta = 0$.

It follows readily from the techniques developed in §3 that a ground-state solution ϕ of (4.8) is the restriction to $\mathbb{R}^n$ of a function Φ which is defined in a "strip" $\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |\Im(z_j)| < \sigma, \text{ for } 1 \le j \le n\}$ and comprises an analytic function of n complex variables there.

In case $M = -\Delta$, then the n-dimensional analog of Lemma 7 allows bound states to be related to general travelling waves via the operator T_θ given by $T_\theta w(x) = e^{i\frac{1}{2}\theta \cdot x} w(x)$ and thereby to extend the results on analyticity to more general travelling-wave solutions.

5. Conclusion. Solitary-wave solutions of the classes (1.3), (1.4), and (1.5) of nonlinear, dispersive wave equations have been shown to possess an analytic extension into a complex strip around their original domain of definition. This further regularity property of such travelling-wave solutions lays the groundwork for a broader use of complex-variable methods in the study of these equations. Such techniques have already proven to be useful in discussing a number of thorny problems connected with uniqueness and soliton behavior (cf. [1], [5], [6], [8], [9]). Perhaps the door now stands ajar to further developments along these lines.

An interesting project for further study would be to determine the type of singularities that arise when a solitary wave is extended into the complex plane. The examples in hand indicate that these extensions will be meromorphic or fractional powers of meromorphic functions. We have conjectured this to be the case under fairly general conditions, but a proof has remained elusive.

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