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On the Role of Computation in Economic Theory*

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This essay is concerned with computation as a tool for the analysis of mathematical models in economics. It is our contention that the use of high-performance computers is likely to play the substantial role in providing understanding of economic models that it does with regard to models in the physical and biological sciences. The main thrust of our commentary is that numerical simulations of mathematical models are in certain respects like experiments performed in a laboratory, and that this view imposes conditions on the way they are carried out and reported. *Journal of Economic Literature* Classification Numbers: C63, C68. © 1997 Academic Press

1. INTRODUCTION

The principal aim of this article is to provide commentary on the use of high-performance computers combined with numerical algorithms in the investigation of mathematical models of economic activity. The use of computer simulation to provide insight into mathematical models is distinguished from the better developed use of computers in recording and processing economic data, and it is intended here to concentrate only on the former.

There are several points to be emphasized in the body of this essay. A central one is that computer simulation is destined to become more commonly used in attempting to understand detailed behavior of simple

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economic models, and for comprehending fundamental aspects of more complex ones. This lends importance to the primary item of discussion, which centers around the contention that computer simulations of mathematical models have much in common with laboratory or field studies in the natural sciences. All of these activities comprise sampling aspects of something that is usually too complicated to deal with as a whole, in the hope of finding patterns or other information that convey understanding of some of its essential features. In the case of a field or laboratory study, one obtains data about physical processes using various measuring or sensing devices. A numerical simulation gathers information about a mathematical model by using various approximation techniques and an associated computer program. In all these situations, there is an art to conducting the investigation in a telling way. But in the case of laboratory or field projects, there are also well-understood guidelines pertaining to setting up and conducting the study and reporting the outcome. If one agrees with the analogy we draw, then it becomes immediately attractive to carry these guidelines for laboratory and field work over to the realm of simulation of mathematical models. For example, it would never occur to a good experimentalist to report findings without the particulars of how they were secured. The same is true of field studies; data reported without a complete description of how and in what circumstances they were obtained is usually not going to be taken seriously.[1] It is our conviction that such attitudes should generally prevail regarding carrying out and reporting numerical simulations of economic models. For instance, to generate confidence in reported numerical results, it is necessary to set about these experiments in an environment of a firm analysis concerning possible differences in behavior between the computed and true solutions. A cursory remark to the effect that solutions were computed and the display of an associated computer-generated graph or table is at best inadequate reporting, and it may conceal a poor job of constructing and verifying a numerical approximation together with an associated computational algorithm for obtaining the putative solution. These sentiments will be amplified below. We believe that the themes discussed presently concerning computational methodology and reporting deserve to become common practice.

We begin the discussion with a brief commentary about the advent of computer simulation in the physical sciences to set the stage for remarks more specifically aimed at economic simulation. It must be acknowledged, however, that many of the general remarks made later in the Introduction and in Section 2 apply to computer simulations used to study issues in a wide variety of areas.

[1] Likewise, the editor of a good physics journal would not consider asking an author to delete such details, but would instead view them as an essential part of the research.

Through the 19th century, mathematical modeling was mainly confined to the physical sciences, though simple economic balance laws were certainly expressed in Walras' classical work and Mendel's work on genetics made use of elementary probability theory. In the physical sciences, much of the serious theoretical progress through this period was made on the basis of linear approximations, though there was no shortage of rather complex, nonlinear models in existence, some dating back to the 17th and 18th centuries. Analysis of nonlinear models in the physical sciences gained a little momentum toward the end of the 19th and in the early 20th century, e.g. in the works of Stokes, Rayleigh, Boussinesq, Poincaré, and later Birkhoff, Kellogg, Schauder, and Leray, to mention a few.

In the last few decades, remarkable progress has been made in developing tools and techniques for the analysis of nonlinear problems. In finite dimensions, one has the beautiful theory of dynamical systems. In infinite dimensions, a plethora of methods from functional analysis and differential geometry have been used on problems originating in mathematical models bearing on issues arising in the physical, biological, and social sciences. At the same time, stochastic analysis has reached new levels with discoveries such as the Itô calculus.

While the arsenal of techniques applicable to the analysis of nonlinear systems has recently grown enormously, it must be acknowledged that our collective hands are often tied when confronted with what appear to be simple nonlinear problems. Even in situations where modern analysis affords us with a helpful methodology, it frequently fails to provide the specific predictions needed to test a model. Moreover, if reliance is placed solely on analytical methods coupled with the occasional, small-scale numerical example carried out by hand, then the scope of inquiry is limited to only the most elementary of models.

In situations where our analytical resources fail to cast light, computational simulations of a model can provide much needed clues to what constitutes the true behavior of the system in question. This approach has had considerable recent success in many areas of the physical and biological sciences and in mathematics itself. Indeed, whole areas of inquiry owe their existence to the careful examination of well-conceived numerical computations. And some of the progress in nonlinear analysis referred to above was suggested initially by the outcome of computer simulations of particular, important models.

Economic modeling embraced a rigorous method of analysis with the general equilibrium theory of the members of the Carl Menger seminar in Wien, Abraham Wald being perhaps the most noteworthy personality of the group. It was not long before this activity reached North America, and found in John von Neumann a unique driving force. The mathematical approach to the study of market behavior came of age in the early 1950s

when the proof of existence of equilibria in a general competitive setting was provided by Arrow, Debreu, and McKenzie. Analysis of mathematical models was later extended to different economic frameworks, including the activity analysis approach pioneered by Koopmans and its development within the linear programming framework by Dorfman, Samuelson, Solow, and Gale, the mechanism design literature pioneered by Marschak and then Hurwicz, and the optimal growth or turnpike literature that merged the models of Ramsey and von Neumann and to which Gale, Morishima, McKenzie, Radner, and Uzawa were some of the crucial contributors. The optimal growth framework, together with the consumption-loan model of Samuelson, freed competitive equilibrium theory of its static straitjacket and set it on the road to a fully developed mathematical theory of time-dependent competitive equilibrium, realizing after some 30 years the research program outlined by John Hicks in his classic treatise *Value and Capital*.[2]

One troublesome aspect of a considerable portion of economic modeling is its essential nonlinearity, sometimes made even less tractable by being coupled with a stochastic element. Because of this, the analog in economics (linear programming, input–output analysis, and the like) of the very fruitful linear period that characterized much of 19th-century physics has had a shorter run.

Considering this state of affairs, it is not surprising that the use of numerical simulation in economic theory is increasing. Even linear or static models such as those found in the works of Leontief, Tinbergen, Johansen and Scarf (cf. [35, 60]) benefit from the use of high-speed computers, not so much because the behavior of solutions is mysterious as because the number of variables is quite large. Numerical methods show their power even more clearly on fully nonlinear problems such as the above-mentioned dynamic equilibrium frameworks of optimal growth and overlapping generations (cf. [38, 40, 54, 67]). While these tools are not yet commonplace in economic analysis the way they are in the physical and biological sciences and mathematics, it seems safe to predict that they will grow in importance in the same way as in these other disciplines. One should also expect that these methods will help to bridge the traditional gap between theoretical and applied economic analysis.

With the prospects presented by the relatively new computational tools at our disposal go certain responsibilities, and it is to these that the present work speaks. The crucial realization mentioned already is that usually numerical simulations of a model have many aspects of an experiment, and consequently they should be performed and evaluated with the same sort

[2] For further history and recent developments in these areas the reader is referred to the monographs [3, 4, 33, 59, 64].

of critical eye that is appropriate to laboratory or field studies. In the body of this paper, the reader will find discussed various technical aspects of numerical simulation as a tool in economic analysis, together with potential pitfalls associated thereto. The issues under discussion are illustrated with a couple of extended examples.

The plan of the paper is as follows. In Section 2 a framework is put forward for carrying out mathematical modeling in which our discussion of numerical simulation is to take place. Section 3 deals with the formulation of discrete models and algorithms for their solution, analysis of the algorithms, and procedures to follow when testing computer codes based on the algorithms. Procedures to follow when reporting numerical results are briefly discussed in Section 4, while Section 5 deals with the very important issue of interpreting the output of numerical simulations. We close our study in Section 6 with some suggestions for further reading.

Some of the important points that will be raised in the course of our discussion are the following.

- When the model is rendered fully discrete in preparation for generating a computer code to simulate it, does the discretization mirror well the general structure of the situation under consideration?
- Is the algorithm for the simulation of the model correctly coded (correctly realized as a set of computer instructions)? How do we know?
- How accurate is the algorithm? What tests were performed to determine its accuracy, stability and other characteristics?
- Is there any theory of convergence or provision of error bounds for the algorithm? If so, are the test simulations consistent with this theory?
- How efficient is the algorithm? Are there algorithms that produce the required accuracy at less cost?
- If the model's quantitative predictions are being taken seriously, how has the model been calibrated? Was the calibration independent of the predictions on which the model's success is being judged?
- When numerical results are reported, is enough information provided so that a skeptical reader can reproduce them?
- How well has the state space been sampled before general conjectures are formulated?
- Is there unwarranted identification between the numerical experiments and the mathematical model?
- Is there unwarranted confusion between the numerical experiment or the model and reality in forming interpretations of the results?

2. MATHEMATICAL MODELS

Our purpose here is to provide a general framework to which the more particular development of numerical simulation will refer. The structure presented is commonplace, and so our description need not be elaborate or exhaustive. Nevertheless, it will be useful presently to have these ideas specifically laid out. Not everyone will agree with all aspects of the conception put forward here, but most of our later commentary about numerical simulation is not strongly dependent upon the details of the discussion on which we now embark.

The framework we have in mind is that in which science is normally carried out. We intend to outline briefly the major components of the scientific method and some of their interrelations. The gist of the discussion is reflected in Fig. 1, which shows our view of the high points in scientific investigation.

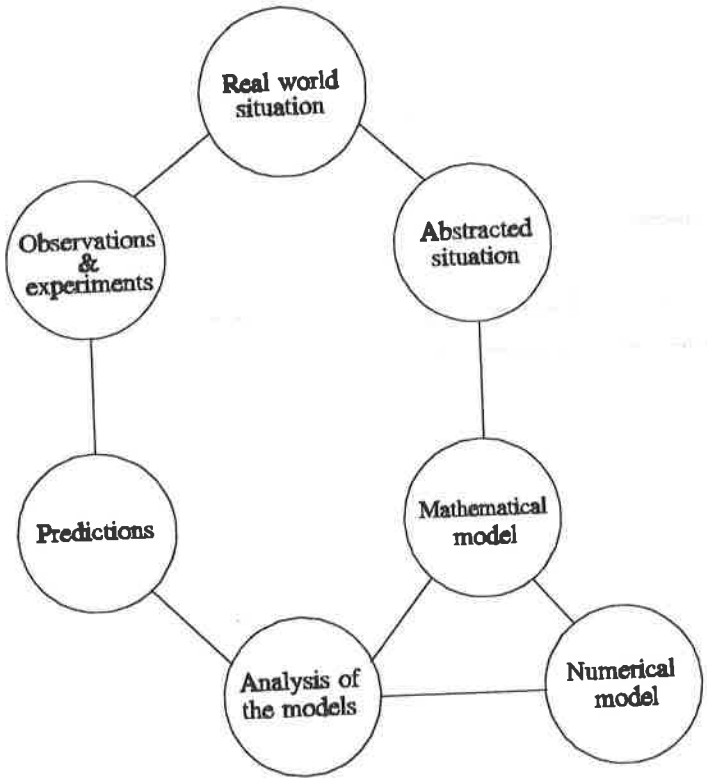

FIG. 1. Some aspects of scientific investigation.

Science begins with some phenomenon, activity, or situation that warrants investigation because of need or curiosity. A typical initial step is to make experiments or observations of the phenomenon. Often these are first undertaken in a mode of discovery, with more exacting scrutiny taking place at a later stage.

Armed with observations, one begins to theorize about the phenomenon. It is often useful to form an idealization of the real-world situation being studied. Constructs such as point masses, ideal gases, perfect fluids, consumers defined solely by preferences and income, or firms designated by simple production functions are sometimes very helpful in isolating the important mechanisms operating in the phenomenon or activity in question and in guiding one's further thinking. Indeed, carefully controlled observations are usually effected with some framework in mind to aid in the design of the study and in deciding exactly what to measure. At a certain point, need will be felt for the predictive power and precision of a quantitative description. At this stage, one builds a mathematical model wherein the crucial aspects of the phenomenon are represented by mathematical variables whose interrelations are meant to mirror the functioning of the phenomenon. The mathematical model is usually based on the idealized situation rather than on the real-world situation that is of underlying interest. The power of a quantitative formulation is twofold. First, and most obvious, it allows quantitative predictions to be made. This is helpful when one attempts to verify the model, and essential when one attempts to calibrate the model and use it to predict the future. Equally important, it allows one to discover general principles and new, sometimes unsuspected phenomena by using mathematical and computational analysis.

Naturally, predictions and discoveries made using the mathematical model must be referred back to the underlying real-world situation. Indeed, comparing model predictions with observations is the primary way in which a mathematical model is validated. Initially, observations and experiments should be made that correspond as well as possible to the idealization underlying the mathematical model. After one has some confidence in a model's predictive power in the center of the range in which it is expected to apply, further insights to the phenomenon under study may be gained by exploring a larger range of situations so as to determine the limits of the validity of the model.

When the performance of a model system is evaluated, different criteria will come to the fore depending on the purpose and stage of the modeling effort. If the theoretical construct has as its goal simply to understand better the interaction of the dependent and independent economic variables, then qualitative properties of the model may be all that one needs to ascertain. For example, if a money supply model is in question

Computation often plays a considerable role in the process just outlined. As mentioned in the Introduction, computational techniques may come to our aid both in controlling experiments and in gathering and processing data. Indeed, modern methods of robotic control and data analysis are becoming very sophisticated, but we eschew discussion of these matters. In the next sections, interest will be focused entirely on computation as it is related to the analysis of mathematical models.

3. FORMULATION, ANALYSIS AND TESTING OF NUMERICAL PROCEDURES

In this section, it is presumed that a mathematical model of an activity under consideration is in hand, and that it describes quantities that can be measured which are related to the phenomena. The discussion will center around general issues that arise when attempting to use computation to shed light on such a model.

Formulating a Numerical Model

Most mathematical models of economic activity are not initially fully discrete; that is, they are not formulated in such a way that they can be directly simulated/approximately solved via a finite list of computer instructions. Many have some discrete aspects, like discrete-time dynamical models, but there are typically exogenous, endogenous, and control or state quantities drawn from infinite-dimensional continua. Digital computers are not capable of directly coping with even domains in R^n, let alone open subsets of function spaces, and consequently most economic models cannot be realized fully intact as a computer code.

Thus, the first step one must take in order to involve the computer as an aid is the formulation of a fully discrete model of the activity in view. Here, one has at least two alternatives. One can resort to the idealized situation and derive directly a fully discrete mathematical model, or one can discretize an existing model which has continuum features using ideas from approximation theory. Some problems lend themselves to the first approach, and it is our feeling that this is preferable when it is convenient. The reason for this view is that a fully discrete model obtained directly from the idealized situation is more likely to preserve the mathematical structure inherent in the continuum model than one obtained by rote application of approximation theory. Experience gleaned from the numerical analysis of ordinary and partial differential equations indicates that a carefully chosen scheme preserving any special structure the equation may have tends to mimic the qualitative and quantitative aspects of solutions better than a scheme just posited without thought to the equation's structure.

For example, it is found that when integrating a Hamiltonian system, the use of a discrete scheme which has a discrete Hamiltonian describes solutions better than a scheme generated by finite differences or finite elements without thought to the Hamiltonian structure (cf. [66]).

It must be acknowledged, however, that most situations are more easily conceived with some of the quantities coming from infinite-dimensional spaces. Moreover, our techniques for analysis of continuum mathematical structures are far more powerful at present than are those for the analysis of discrete mathematical structures. Consequently, fully discrete models are most often and most easily derived by the use of discrete renditions of the continuous quantities already extant in the model. It is our view that even when following this path, it is worth looking for symmetries and other structure in the continuum model, and trying to maintain as much of this as possible in the discrete scheme.

Fortunately, there are well known methods from approximation theory that come to our aid when we are faced with discretizing a continuous or partially continuous problem. An advantage often gained by using well-studied techniques rather than some *ad hoc* invention that occurs on first inspection is that an analysis of the error embodied in the approximation is often readily available using standard theory. This point will figure in some of our later commentary. (It is worth note that if we agree that computation will continue to grow in importance, use, and acceptance, then it behooves us to suggest that our students make an effort to get a proper grounding in modern numerical analysis, the rudiments of approximation theory, and elementary structured programming, as well as becoming familiar with some of the more powerful software environments.)

EXAMPLE 1. Here is an illustrative example that focuses on some of the points under discussion. Consider the following optimization problem. Given $\delta > 0$ and k_0, find an (absolutely continuous) path $\{c(t), k(t)\}_{t \geq 0}$ as an optimal solution to

$$W(k_0) = \sup \int_0^{\infty} u(c(t))\, e^{-\delta t}\, dt$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t), \qquad \text{with} \quad k(0) = k_0,$$

for $t \geq 0$, where $c(t)$ and $k(t)$ are n-dimensional vectors, and $\dot{k}(t)$ denotes the time derivative of $k(t)$. This is a continuous-time version of the standard, neoclassical model of economic growth. From an analytical point of view, this problem is frequently approached in the following more abstract framework, which is convenient for our purposes:

$$W(k_0) = \sup \int_0^{\infty} v(k(t), \dot{k}(t))\, e^{-\delta t}\, dt$$

subject to (3.1)

$$(k(t), \dot{k}(t)) \in T \quad \text{with} \quad k(0) = k_0, \quad \text{for} \quad t \geqslant 0.$$

The following commonly used assumptions are easily derived from standard hypotheses on the primitive functions u and f.

Assumption A. The set $T \subset R^{2n}$ is convex with nonempty interior.

Assumption B. The mapping $v: T \to R$ is continuous and on the interior of its domain it is infinitely differentiable. Moreover, the function u has bounded second-order derivatives and there is some constant $\alpha > 0$ such that $v(k(t), \dot{k}(t)) + (\alpha/2)\, \|\dot{k}(t)\|^2$ is a concave mapping for all $(k(t), \dot{k}(t)) \in T$.

Assumption C. There exists an open set U in R^n such that for every k_0 in U there is a unique optimal solution $\{k(t)\}_{t \geqslant 0}$ to Problem (3.1) with $k(0) = k_0$ and $(k(t), \dot{k}(t)) \in \text{int}(T)$ for all $t \geqslant 0$.

The norm $\|\cdot\|$ is the usual Euclidean norm. The concavity requirement asserted in Assumption B is termed α_K-concavity. It follows from [56] that under the above assumptions the *value function* W is a C^2 mapping on U. Hence, this function must obey the functional equation of dynamic programming, the so-called *Bellman equation*,

$$\delta W(k_0) = \sup_{\dot{k}} v(k_0, \dot{k}) + DW(k_0) \cdot \dot{k}, \tag{3.2}$$

where $DW(k_0)$ denotes the derivative of W at k_0. Under the above assumptions, $\{k(t)\}_{t \geqslant 0}$ is an optimal path if and only if it satisfies at all times Eq. (3.2). Moreover, differentiating (3.2) with respect to $\dot{k}$, we obtain that the *optimal feedback control or policy* $\dot{k} = g(k)$ must fulfill at all times the first-order condition

$$D_2 v(k, \dot{k}) + DW(k) = 0. \tag{3.3}$$

Under the above postulates of concavity and differentiability, one readily sees from a simple application of the implicit-function theorem to (3.3) that g is a C^1-mapping on U.

In most applications, one faces the problem of computing the functions W and g. This is usually achieved by numerical methods, though if one is

satisfied to assume a quadratic objective, then the problem may be solved in closed form. (Incidentally, the solution to the quadratic problem can be very helpful when trying to test a numerical method that could be applicable to a whole class of problems of the form depicted in (3.1).) A thorough knowledge of the qualitative behavior and stability properties of the model is generally very useful in order to devise the most appropriate computational procedures. For example, if the model contains a unique, globally stable steady state, then it is generally possible to determine the law of motion of an optimal path at a desired level of accuracy by relatively fast methods (e.g., see [48]). Sometimes, however, the problem becomes less tractable, especially if the state space is multidimensional or if it comprises a stochastic component. In such circumstances, the behavior of an optimal model may be so complex that for the purposes of ascertaining properties of a solution one must rely for the most part on standard numerical techniques. Several issues are generally involved in an approximation of a model such as (3.1) in order to achieve a given level of accuracy.

Assume that a decision has been taken to discretize both the state space and the temporal variable if the model is not static. Then the following questions may arise.

(a) In what way and how finely must the state space be discretized?

(b) In what way and how finely must the temporal variable be discretized? How should these discretizations be related to each other?

(c) What is the most suitable length for the time horizon? Which terminal conditions are most appropriate if the horizon is to be truncated?

In many cases a discretization of the set of feasible strategies S, say, takes the form of replacing S by a suitable finite-dimensional space S_h whose elements are specified by a finite number of parameters, and then reformulating the model for S_h instead of S (e.g. a finite-element approximation of a function space; see [37]). In other cases, the full discretization is presented implicitly, as with a finite-difference scheme for solving an ordinary differential equation or a Newton method for computing the solution of a system of nonlinear equations.

With respect to the discretization of the temporal space, it should be observed that most dynamic idealizations in economics can be visualized as either continuous- or discrete-time processes. An issue that arises frequently is whether one should initially posit the discrete- or the continuous-time version of a given dynamic model. An answer to this question is heavily dependent on the qualitative properties of both frameworks and on the underlying computational requirements. Our subsequent analysis on computational requirements and loss of accuracy involved in the discretization of the temporal variable suggests that unless the model has some specific

structure it is worth posing from the start the discrete-time version of the model. This is certainly a good strategy if both versions share the same qualitative behavior. If, however, the problem in question is better conceived as a continuous-time process, then it is usually a good idea to consider the continuous-time version as the reference model and start with those approximations that most nearly resemble the fundamental aspects of the problem. On the other hand, regarding the issue of truncation of an infinite horizon and choice of an appropriate terminal condition, we shall illustrate below that sometimes it becomes fairly convenient to obtain numerical estimates of these terminal conditions from relatively fast computational procedures.

Following previous work on numerical analysis of problems such as (3.1) (cf. [12, 22]) the following discrete-time approximation is proposed:

$$W_m(k_0) = \sup \sum_{i=0}^{\infty} v\left(k_{mi}, \frac{k_{m(i+1)} - k_{mi}}{m}\right) m e^{-\delta m i}$$

$$\text{subject to} \tag{3.4}$$

$$\left(k_{mi}, \frac{k_{m(i+1)} - k_{mi}}{m}\right) \in T, \qquad i = 0, 1, 2, \ldots.$$

In this formulation it is assumed that controls and states are piecewise constant, and may jump at discrete times, $0, m, 2m, \ldots$.

For the discretization of the state space, let us assume that the set of feasible states k_0 lies in a compact polyhedron X for which Assumption C is still satisfied. This is not a restrictive condition for most economic applications. Let $\{S^j\}$ be a finite family of *simplices* which comprise a *triangulation* (i.e., $\bigcup S^j = X$ and $\text{int}(S^j) \cap \text{int}(S^k) = \varnothing$ for $i \neq j$). Let

$$h = \max_j \{\text{diam}(S^j)\}.$$

Denote by k^j a generic *vertex* of the triangulation. Consider then the space of continuous, piecewise affine functions

$$\mathcal{W}_m^h = \{W_m^h : X \to R \mid W_m^h \text{ is continuous, with constant derivative } DW_m^h \text{ in } \text{int}(S^j), \text{ for each } j\}$$

Observe that $\mathcal{W}_m^h$ is a complete metric space, as it is a closed subset of the space $C(X)$ of continuous functions on X equipped with the maximum norm $\|W_m^h\|_{C(X)} = \max_{k \in X} |W_m^h(k)|$.

The following equation will play a crucial role in our subsequent analysis:

$$W_m^h(k_0^j) = \max_{k_m} v\left(k_0^j, \frac{k_m - k_0^j}{m}\right) m + W_m^h(k_m)\, e^{-\delta m}$$

$$\text{subject to} \tag{3.5}$$

$$\left(k_0^j, \frac{k_m - k_0^j}{m}\right) \in T, \qquad \text{for all} \quad k_0^j.$$

This is the corresponding discretized version of Bellman's equation which must hold at each of the selected vertices, $k_0^j = k^j$. For each of these nodal points, define $g_m^h(k^j)$ as the optimal solution to (3.5) for $k_0^j = k^j$. If the solution is unique, this set of values uniquely defines a piecewise affine function g_m^h on X compatible with the given triangulation $\{S^j\}$. We shall refer to g_m^h as the policy function for W_m^h in (3.5). Our main concern now is with the error arising from these approximations.

Analysis of Error and Complexity

If the fully discrete model has been derived by approximation of a continuum model, it is generally very useful to analyze the difference between solutions of the fully discrete model and their continuum counterparts. Such an analysis gives an idea of how fine the approximation parameters such as mesh size, order of polynomial approximant, and so on need to be chosen so that the discrete model faithfully mimics the continuous model. This is also useful in helping to verify the code, as discussed below. In many situations, a rigorous error analysis can be performed using existing theory together with *a priori* derived qualitative information such as existence, uniqueness, and smoothness about solutions of the continuum model. In certain, novel situations, new theory must be conceived to effect an error analysis. Frequently, a formal error analysis can be carried out in which rigor is replaced with judicious guesses about certain aspects of the approximate and exact solutions. While not as satisfactory as a fully complete analysis, the information gleaned from a formal analysis of error may still be useful in the ways it is when there are rigorous error bounds established.

What often comes out of an error analysis is an order of accuracy estimate. For example, if N is the number of unknowns in a fully discrete model, then we may be able to infer that the error between the approximate solution and the exact solution is on the order of $1/N^r$, where r is a positive real number determined by the approximation scheme being used. This information lends credibility to the scheme, and it is handy at another stage to be discussed presently.

When the problem has been rendered fully discrete, one must face the issue of its implementation via a computer code. A working knowledge of modern structured programming is very helpful at this stage. In addition, one sometimes needs to think very carefully about the sequences of computational events to try to minimize the computational complexity of the code. If the computation to be performed is extensive, it is a good idea to carry out an approximate count to determine in terms of the input parameters how many of the more time-consuming operations like multiplications, divisions, exponentiations, comparisons and so on are to be performed. This sort of count is helpful to get an idea of running time, and it is also useful when trying to judge accuracy achieved for work expended.

EXAMPLE 2. Another example is introduced that shows several of the aspects just mentioned. The model in question is taken from Bona and Li [8] and follows from an earlier model of Grossman and Weiss [28] (see also the working paper [7]). It is an infinite-horizon model for the money supply that is especially designed to capture the economic response of prices, interest rates and so on to monetary injections. The principal issue in view is to determine how government intervention in the money supply can influence the economy in a positive way.

The model features consumers, firms, the public sector (government), and discrete temporal periods in which transactions are made. Firms are owned by the consumers and produce a perishable, exogenous good that is sold to other consumers at a profit. Because consumers need cash to purchase goods, they sell bonds to the firms and the government. The firms use the profit from selling goods to purchase bonds from consumers and immediately place them in the owners' interest-bearing accounts. The government issues cash to buy bonds to inject money into the market. The cash ends up in the hands of the consumers from the bond market (bank) via the sale of bonds. The transaction cost of making withdrawals is accounted for by requiring consumers to visit the bank only once every two periods, and these visits are staggered in that only half visit the bank in each period. The two types of consumers, type "a" and type "b," say, visit the bank at the end of the odd and even temporal periods, respectively, and withdraw enough cash to finance their consumption expenditures over the next two periods. The amount of withdrawal is determined by the possibility of intertemporal consumption substitution, and therefore is influenced by expected prices and future nominal interest rates. The model is completely deterministic and the consumers are taken to have perfect foresight.

The variables in the model are now defined. For $w = a$ or $w = b$, let M_t^w denote the cash that a consumer withdraws at the end of period t, C_t^w the amount of the single consumption good that consumer w consumes during

period t, and p_t the price level in period t. The quantity R_t will connote 1 plus the interest earned on deposits maintained at the bank during the entirety of period $t+1$ and $\alpha_t = (R_0)(R_1)(R_2)\cdots(R_{t-1})$ where $\alpha_0 = 1$. A consumer of type w with initial wealth W_0^w and initial money holdings M_0^w is confronted with choosing the appropriate bank withdrawal and consumption plan in order to maximize the total utility

$$\sum_{t=1}^{\infty} \beta^{t-1} U(C_t^w) \tag{3.6}$$

for $w = a, b$, where $0 < \beta < 1$ is a positive discount factor and $U = U^w$ is the consumer's utility function which is assumed to have the usual convexity properties. The following constraints are in force during this optimization,

$$\sum_t \frac{1}{\alpha_t} M_t^w = W_0^w, \tag{3.7}$$

where summation is over odd periods if $w = a$, and over even periods if $w = b$,

$$p_1 C_1^a = M_0^a, \tag{3.8}$$

$$p_{t+1} C_{t+1}^w + p_{t+2} C_{t+2}^w = M_t^w, \text{ where } w = \begin{cases} a & \text{if } t \in \{1, 3, 5, \ldots\} \\ b & \text{if } t \in \{0, 2, 4, \ldots\}, \end{cases} \tag{3.9}$$

and

$$M_t^w = M_{t-1}^w - p_t C_t^w, \quad \text{where } w = \begin{cases} a & \text{if } t \in \{2, 4, 6, \ldots\} \\ b & \text{if } t \in \{1, 3, 5, \ldots\}. \end{cases} \tag{3.10}$$

Equation (3.7) is a wealth constraint reflecting the fact that every agent must balance his budget. The initial, nonmonetary wealth W_0^w of agent w consists of three parts, a claim on a fraction of the firm's profits which is deposited into the agent's interest-bearing account every period, government bonds of one-period maturity, and, on the other side of the ledger, the taxes to finance the interest payments on government bonds. (The latter are assumed to be an equal lump-sum levy paid each period from each agent's interest-bearing account.) Equations (3.8)–(3.10) are cash-in-advance constraints. Without loss of generality, we assume that the interest rate is positive, so each optimizing consumer must spend all the money in hand by the end of the second period after having withdrawn it.

For the market to be in equilibrium, it is necessary that the flow of cash into the market at each period equals the consumers' desired withdrawals,

and that the goods produced equal the quantity of goods demanded by consumers. These conditions imply that for all $t>0$

$$C_t^a + C_t^b = C, \tag{3.11}$$

$$M_t^a + M_t^b = M_t, \tag{3.12}$$

where (C_t^w, M_t^w) is the solution of (3.6)–(3.10), $w=a, b$; C is the exogenously given output produced by the firms; and M_t is the exogenously given money supply, assumed to be equal to one. For consistency, in the initiating stage, it is required that $M_0^a + M_0^b = M_0$. The money supply changes only through government intervention.

The model is initiated in a steady-state configuration with a price level $p=p_1$, which is then disturbed by an unannounced, open market operation by the government at the end of the temporal period $t=1$ in which the money supply is increased by the fraction k, so that if M_0 was the initial money supply, then at the end of period 1 the money supply is $M_1=(1+k)\,M_0$. The goal is then to understand the economic consequences of this monetary injection. (As just described, the model appears in a predictive guise. A further goal is one of control of the consequences, but this aspect will not be dwelt upon here.)

Let us assume that the utility function $U(c)$ is identical for all consumers and that it is given by a power law $c^{1-\sigma}/(1-\sigma)$ (the case of constant elasticity of intertemporal substitution), where $\sigma>0$. The discussion in [8] does not require this stringent presumption, but it simplifies the exposition here and nothing is lost in regard to the issues about simulation under discussion. In this case, a straightforward analysis reduces solving the model outlined above to finding a sequence $q=\{q_t\}$ of normalized prices such that

$$q_t + \phi\left(\frac{q_t}{q_{t+1}}\right) q_{t-1} = 1, \tag{3.13}$$

for $t>1$, where $q_1=(p_1+k)/(1+k)$, $q_t=p_t/(1+k)$ for $t>1$, and

$$\phi(z) = \frac{\beta^{1/\sigma}}{\beta^{1/\sigma} + z^{(\sigma-1)/\sigma}}.$$

This problem, in turn, can be recast in another convenient form based on the *ansatz* that there is a solution $q=(q_1, q_2, \ldots)$ of (3.13) such that $q_{t+1}=f(q_t)$, $t=1, 2, \ldots$. If (3.13) can be solved in this form, then once the function f is determined, we will have at hand a solution for all initial

states. According to (3.13), the function f, which we will refer to below as the *price function*, must satisfy the relation

$$f(x) = 1 - x\phi\left(\frac{f(x)}{f(f(x))}\right) = F(f)(x). \tag{3.14}$$

It is shown in the aforementioned paper that (3.14) possesses a unique solution corresponding to any of the possible steady state configurations which are assumed to obtain prior to government intervention. Moreover, the solution is given by a price function f, and this function is smooth so that the problem (3.14) is robust in that small perturbations of the initial steady state lead to only small perturbations of the solution in a certain precise sense which need not concern us here.

The issue that is of concern here is the numerical approximation of solutions of (3.14). It is worth noting that an early attempt at approximating solutions of this equation (Bona and Grossman [7]) used a combinatorial algorithm and an approximate, finite-horizon problem. This method was poor in nearly all the ways we have outlined above: there was no theory for its accuracy, it was very difficult to check accuracy computationally, and the operation count grew exponentially with increasing accuracy. It did have the salutory property of being easy to program. Here we present a much better technique effected by a finite-element method. Let $C([0, 1])$ connote the continuous, real-valued functions defined on the closed interval $[0, 1]$ and let V_h connote the subspace

$$V_h = \{v \in C([0, 1]): v \text{ is linear on each interval } [x_j, x_{j+1}],$$
$$j = 0, \ldots, N-1\}$$

of piecewise linear functions on a uniform grid $0 = x_0 < x_1 < \cdots < x_N = 1$, where $h = 1/N$, $x_j = j/N$, $j = 0, 1, 2, 3, \ldots, N$. Any element v in the finite-dimensional space V_h is determined by its values $v(x_j)$, $j = 0, 1, \ldots, N$, and hence the space V_h is spanned by the tent functions $w_j \in V_h$, $0 \leqslant i \leqslant N$, such that

$$w_i(x_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta which is zero when $i \neq j$ and when $i = j$ it takes the value 1.

If v lies in $C([0, 1])$, define $I_h(v) \in V_h$ to be its nodal-value interpolant

$$I_h(v)(x) = \sum_{j=0}^{N} v(x_j)\, w_j(x).$$

Then the fully discretized version of (3.14) is the equation

$$f_h(x) = F_h(f_h)(x) = I_h(F(f_h))(x) \tag{3.15}$$

for f_h in V_h. A rigorous analysis of the approximate problem (3.15) is carried out in [8, 44]. This analysis provides several useful facts. First, for h small enough, it is inferred that the approximate problem possesses a unique solution for relatively small monetary shocks. Second, if f is the solution of the continuous problem and f_h is the solution of the discrete problem, then the difference between them is bounded above as follows,

$$\|f - f_h\|_{C([0,1])} \leqslant Mh^2 \|f''\|_{C([0,1])}, \tag{3.16}$$

where the constant M depends on the size of the shock, but not on h. Note, incidentally, that this estimate requires one to have ascertained that the solution f of the continuous problem (3.11)–(3.14) is at least twice continuously differentiable. This is the kind of information that is very often needed to obtain rigorous error estimates, and it provides impetus for understanding the smoothness properties of solutions of model problems. For the model under consideration here, one infers the solution to be infinitely differentiable (in fact, real analytic), and one can give bounds on the derivative f'' in terms of σ and β.

EXAMPLE 1 (Continuation). Discussion is now initiated of error estimates for the approximation sketched at the end of Example 1. This analysis will shed light on the important issue of how fine the discretization of the temporal intervals should be with respect to the discretization of the state space. An optimal choice of the two discretization parameters will have the error induced by either to be of the same order of magnitude. To put it another way, there is no point in refining the state space further if the total error is already dominated by the error in the discretization of the temporal variable.

Using similar discretization procedures, Falcone [22] has established approximation estimates for a related model. Sharper bounds will be derived here from smoothness properties of the value function. As shown in [55–57], under "regular" conditions the value function W in (3.1) is C^2, but fails to be higher-order differentiable.

Our main findings are summarized in the following theorem. Let g be the policy function for W in (3.1), and let g_m^h be the optimal policy for W_m^h in (3.5).

THEOREM 3.1. *Under the above assumptions, there are constants M and N such that $\|W - W_m^h\|_{C(X)} \leqslant M(m + h^2)$ and $\|g - g_m^h\|_{C(X)} \leqslant N(m^{1/2} + h)$.*

This theorem is proved in the Appendix. The result extends to the continuous-time model the analysis of [58] . It should be also observed that Lemma A.3 below illustrates that if in addition k_0 is such that $[DW(k_0) - DW_m(k_0)] \cdot g(k_0)/\|DW(k_0) - DW_m(k_0)\| \neq 0$, there is a constant N' such that $\|g(k_0) - g_m^h(k_0)\|_{C(X)} \leq N'(m+h)$. The bounds M, N and N' are independent of the values m and h, for m and h small enough. It thus becomes clear that generically mesh sizes m and h make similar contributions to the global estimate of the policy function, and consequently they should be specified with the same order of magnitude. Indeed, the exact, leading form for the error $\|g(k_0) - g_m^h(k_0)\|$ is $Am + Bh$, where A, B are constants. If an estimate for A and B is available, then for small values of m and h, the choice $h = m(A/B)$ balances the error generated by the two discretizations.

The above approximation exploits C^2-properties of W. Although higher-order approximations could be pursued, it should be realized that under standard conditions the value function may fail to be higher-order differentiable.

Testing the Computer Code

It is crucially important to test the code extensively once it has been constructed and debugged. Just because a computer program runs does not mean that it is computing what was intended. There are a number of ways in which codes can be tested to give an indication that they have been correctly entered into the machine and to try to determine their accuracy.

If exact solutions of the continuum model are known, then one can compute their fully discrete analogs and form the difference between the exact solution and the approximate solution. An idea of the absolute error is thereby obtained. In the same circumstance, it is good practice to compute the approximate solution for several values of the approximation parameters and to check that the error decreases at the rate predicted by the error analysis. If it does not, something is wrong, most likely with the code. In any case, no confidence should be placed in the output until this consistency is reached. If an exact solution is not available, then a standard procedure is to compute the solution with a very fine level of approximation and treat this as an exact solution. Then the same comparison of convergence rates can be made by computing the outcome at coarser levels of approximation. Again, one is looking for the rate of convergence predicted by the error analysis.

If an approximate operation count has been obtained, one is in a position to compute the accuracy achieved for work expended. In the final analysis, this is the most exacting test of a numerical method. If several approximation parameters appear in the problem, then by performing a series of computations featuring different values, one can determine optimal ratios or values of these parameters to achieve a given level of accuracy. This can be very helpful if many simulations of a similar type are contemplated.

The problem under consideration may have other consistency conditions that can be used to check the code. For example, the solution may necessarily be composed of nonnegative numbers, and the appearance of negative numbers in the computer output is a sure signal that something is wrong.

EXAMPLE 1 (Continuation). For illustrative purposes, we shall limit our attention to a simplified, discrete-time version of the above growth model. The optimization problem is written as

$$\begin{gathered} W(k_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} \\ c_t = f(k_t) - k_{t+1} \\ k_0 \text{ given, } 0 < \beta < 1, \, t = 0, 1, 2, \ldots, \end{gathered} \tag{3.17}$$

where all variables are positive real numbers. It is well known that for the functional forms $u(c) = \log c$ and $f(k) = Ak^{\alpha}$, with $A > 0$ and $0 < \alpha < 1$, the value function W has an analytical solution given by $W(k_0) = B + C \log k_0$, in which B and C are given constants. Moreover, the optimal policy is defined by the simple law of motion $k_{t+1} = \beta\alpha A k_t^{\alpha}$.

We now report some numerical results from [58]. The point to be stressed here is that before proceeding to a computational analysis of a given economy in the specified family (3.17), it is worthwhile to use the computer program based on the numerical model to attempt to compute the above simple analytical solutions as a check that the program has been correctly coded.

Following the iterative process outlined previously, we consider the operator $W_n^h = T^h(W_{n-1}^h)$, where the maximization

$$\begin{gathered} T^h(W_{n-1}^h(k_0^j)) = \max_{k_1} \log c_0 + \beta W_{n-1}^h(k_1) \\ \text{subject to} \\ c_0 = Ak_0^{\alpha} - k_1, \end{gathered} \tag{3.18}$$

is carried out at each vertex point k_0^j, for $n = 1, 2, \ldots$, and W_0 given. That is, at each iteration n the maximization is performed over a fixed grid of points $\{k_0^j\}$, with mesh size h, and these values define the piecewise affine function W_n^h. The iterative algorithm stops when the difference between two consecutive value functions W_{n-1}^h and W_n^h is such that $\| W_{n-1}^h - W_n^h \|_{C(X)} \leqslant \hat{N}h^2$, and $\hat{N}$ is selected in accordance with our estimates of the error involved in our discretization scheme along with further considerations

related to the computational cost of additional iterations. (Observe that if $\|W^h_{n-1} - W^h_n\|_{C(X)} \leq \hat{N}h^2$, then the contractive property of T^h implies that the fixed point W^h in (3.18) lies within a distanced $\beta\hat{N}h^2/(1-\beta)$ of W^h_n.)

We consider parameter values $\beta = 0.95$, $A = 5$, $\alpha = 0.34$. For such parameterization the policy function takes the simple form $k_{t+1} = 1.615k_t^{0.34}$. This function has a unique, globally stable steady state, $k^* = 2.0673$. For the purposes of our analysis the space of capital stocks X is restricted to $k \in [0.1, 10]$. This is a large enough domain so that it contains the steady state value k^* and has the property that all optimal solutions are interior, and so Assumption C is satisfied.

The numerical experiments, which were coded in standard FORTRAN 77 and run on a DEC 2000 workstation (300 ALPHA AXP, rated at 358.1 MFLOPS/150 Mhz) started with $h = 10^{-1}$ and an initial condition $W_0 \equiv 0$. In computing the approximate fixed point W^h in (3.18) for $h = 10^{-1}$, the program stops after 99 value-function iterations with a reported computer time of 3s. In a further computation of W^h, for $h = 10^{-2}$, a continuation method is implemented that takes as initial condition the previous piecewise linear function W^h obtained with $h = 10^{-1}$. The program stops now after 91 additional iterations with an extra reported time of 33 s. The same procedure is finally applied to the computation of W^h for $h = 10^{-3}$. Figures 2–4 depict the observed error $e^h(k) = |W(k) - W^h(k)|$ for the restricted domain of capital stocks k over the interval $[0.1, 10]$. Observe that the error function features the same

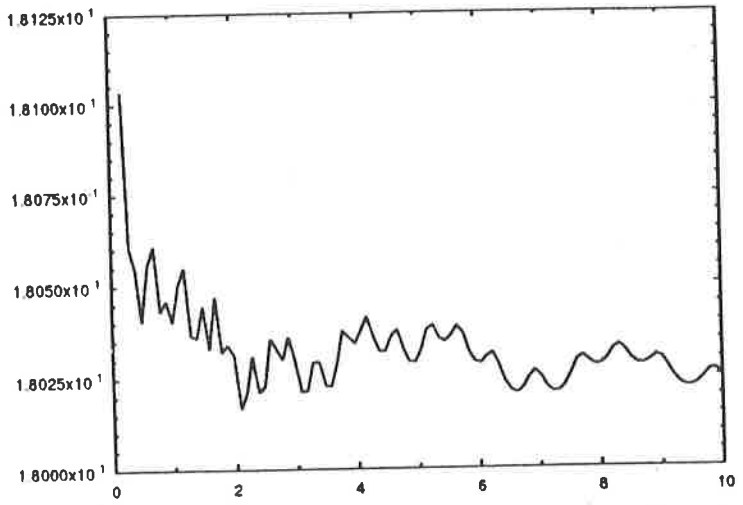

FIG. 2. Observed error $e^h_{\hat{n}}(k) = |W(k) - W^h_{\hat{n}}(k)|$, $h = 10^{-1}$, $\hat{n} = 99$, $\beta = 0.95$, for the deterministic growth model of Example 1.

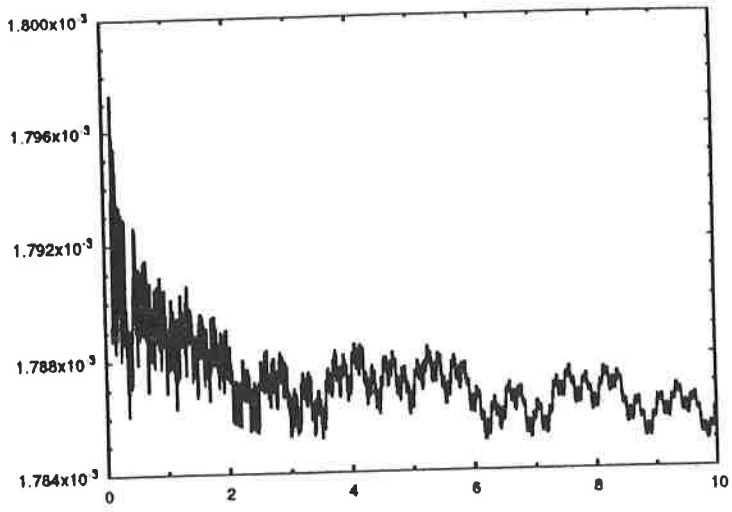

FIG. 3. Observed error $e_n^h(k) = |W(k) - W_n^h(k)|$, $h = 10^{-2}$, $\hat{n} = 91$, $\beta = 0.95$, for the deterministic growth model of Example 1.

asymptotic behavior predicted by the theoretical analysis. In other words, cutting the mesh size by a factor of 10 results in a decrease in the error on the order of 100, which is consistent with the predicted convergence being $O(h^2)$ as $h \to 0$. Therefore, the computed value function converges quadratically to the true value function. Additional information is provided in

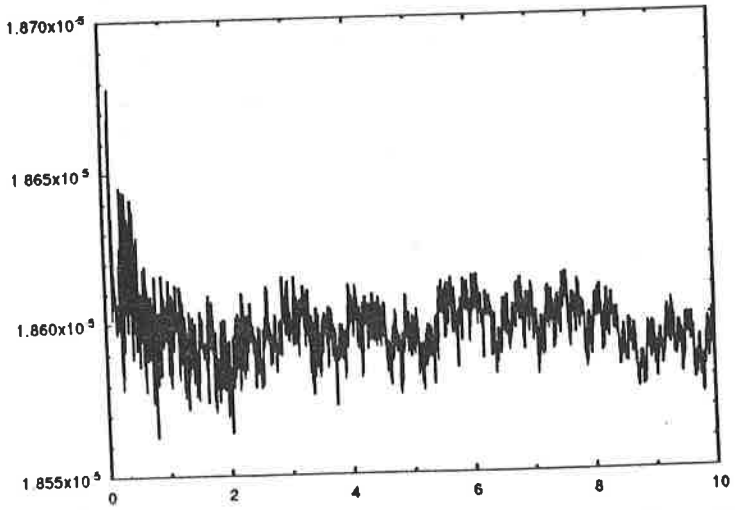

FIG. 4. Observed error $e_n^h(k) = |W(k) - W_n^h(k)|$, $h = 10^{-3}$, $\hat{n} = 90$, $\beta = 0.95$, for the deterministic growth model of Example 1.

TABLE I

A Numerical Experiment for the One-Sector Deterministic Growth Model with a Continuation Method

No. of vertex points	Mesh size	No. of additional iterations	CPU time (s)	Max. observed error in W
100	10^{-1}	99	3	1.8×10^{-1}
1000	10^{-2}	91	33	1.7×10^{-3}
10000	10^{-3}	90	298	1.8×10^{-5}

Table I, which summarizes some key points of this numerical experiment. For further details, the reader is referred to the aforementioned paper from which the example was taken.

Other theoretical properties of the model lend themselves to testing the code. For instance, in the above value-function iterated algorithm, one can instruct the program to verify at each consecutive iteration the contractive property of the operator T^h in (3.18). Likewise, one can solve for the steady states of the true and computed solutions, and analyze the possible sources of the observed difference. Many properties of the mathematical model have a corresponding counterpart in the numerical algorithm, and such properties provide basic information for testing the code.

In problems involving a sequence of computational steps, it is also very important to design a modular testing procedure, effecting separate checks of each individual unit of the calculations. This procedure is usually very effective in detecting errors, and allows one to assess the performance of the different parts composing the program. (In devising and interpreting a numerical experiment, one should always be aware of how the inaccuracies of different maximizations, integrations, and related operations unfold into the numerical algorithm.)

The above growth model offers an appropriate illustration of the points just made. It is well known (e.g., see [40, 45]) that in such a model one could compute analytically the sequence of functions $\{W_n\}_{n\geq 0}$, where $W_n(k_0) = \max_{k_1} \log[Ak_0^\alpha - k_1] + \beta W_{n-1}(k_1)$ and $W_0 \equiv 0$. This sequence of analytical forms, along with the known fixed point $W(k_0) = \max_{k_1} \log[Ak_0^\alpha - k_1] + \beta W(k_1)$, provides us with very useful information within which to evaluate the performance of the maximization subroutines or related operations. In addition, if the model contains stochastic variables, then proposed integration procedures and related operations should be thoroughly tested with the aid of analytically simple forms before incorporating such sets of instructions to the body of the computer program.

It is also important to realize that several approximation schemes are generally available in a given problem. For instance, in the formulation of our discrete model in (3.5) we confined our attention to piecewise linear functions over triangular subdivisions. In some cases, however, it seems more natural to consider alternative (possibly higher-order) interpolations over rectangular subdivisions. When several choices are at our disposal a theoretical analysis of the problem along with some illustrative computations should give an idea of the best way to proceed in order to attain a given level of accuracy.

Again, the above numerical calculations are useful for this purpose. As can be seen from Table I, it takes roughly 0.03 s to effect an iteration for $h = 10^{-1}$ (first row of Table I), it takes roughly 0.3 s to effect an iteration for $h = 10^{-2}$ (second row of Table I), and it takes roughly 3 s to effect an iteration for $h = 10^{-3}$. Hence, the computer time spent in each iteration increases linearly with the number of grid points. Thus, if we were to consider mesh size $h = 10^{-4}$, then it would be expected that each single iteration would take roughly 30 s. In this case, in order to attain such a fine level of accuracy, it would likely be profitable to resort to higher-order interpolations.

One should also be aware that some approximation methods fare better in small-scale models, whereas other methods are more suitable for large-scale models. Some numerical methods are subject to what is known as the "curse of dimensionality," in which the computational requirements grow exponentially with the number of exogenous and endogenous variables. These methods then become unfeasible for the computation of solutions of large-scale models. It should be remarked, however, that, excepting cases having a special structure, rigorously analyzed schemes that allow for arbitrary levels of accuracy are generally subject to the "curse of dimensionality".

EXAMPLE 2 (Continuation). Lastly we verify the order of magnitude of the approximation error in our monetary model. This model does not feature a closed form solution. In spite of this fact, we can still inspect numerically that for sufficiently small h,

$$e(h) \approx Ch^p \tag{3.19}$$

for some constant C and $p = 2$, and where

$$e(h) = \|f - f_h\|_{C[0,1]}.$$

Since the values C and p are expected to be independent of h, we obtain from (3.19) that for sufficiently small h,

$$e\left(\frac{h}{2}\right) \approx C\frac{h^p}{2^p}. \tag{3.20}$$

TABLE II

Error Analysis of Numerical Computations

j	$c(j)$	$p(j)$
2	0.0043	1.5650
3	0.0045	1.9449
4	0.0045	1.9923
5	0.0047	1.9423
6	0.0049	1.9570
7	0.0050	1.9679
8	0.0049	2.0221

Combining (3.19) and (3.20) yields

$$p = \log_2 \frac{e(h)}{e(h/2)}$$

We first let $h = 2^{-12}$ and compute the approximate solution for f. Such an approximation is fairly accurate and will play the role of the "exact" solution in all of our computations. Let $h_j = 2^{-j}$ and define

$$c(j) = e(h_j)/h_j^2,\ p(j) = \log_2 \frac{e(h_j)}{e(h_{j+1})}$$

For j ranging between 2 and 8, the computed results are listed in Table II. We observe that $c(j)$ converges to a constant 0.0049 and $p(j)$ hovers around 2. This array of numerical outcomes conforms with our previous theoretical analysis; moreover, although not implied by our theory, the approximating constant C settles down to a given number. This type of regularity is observed in many other similar numerical experiments.

4. REPORTING NUMERICAL SIMULATIONS

The important point here is that a numerical simulation of a mathematical model constitutes an experiment in much the same way as does a laboratory study or field observation. There is a well defined culture about reporting laboratory studies, and in our view these guidelines apply equally to reporting numerical studies.

First, the formulas used to generate the code should be written explicitly. The general structure and any numerical shortcuts used in the actual coding should be mentioned. If standard software packages are used in the

code, these should be identified explicitly and the conditions of their use specified. It is even a good idea to report the machine and operating system with which the code was executed. The overall goal is that enough information be provided for a skeptical reader to reproduce your simulation.

It is worth pointing out to journal editors that deleting the sort of material mentioned and substituting a reference to the author(s) for further details does not lead to a satisfactory state of affairs. If the relevant details are not in the permanent record, it is nearly certain they will be lost in due course, and perhaps before the article's useful lifetime is exceeded.

Second, whatever error analysis was deemed possible to carry out and useful should be reported, along with at least an indication of why it was expected to be valid. Best is a rigorous error analysis and a compilation of accuracy achieved for work expended based on a careful operation count together with a series of simulations. If rates of convergence or other approximation characteristics are relevant, tabulate some outcomes to show that they are consistent with the error analysis. (Such material can be relegated to an appendix if it interrupts the flow of the exposition unduly.) Keep in mind that you are expecting the reader to believe the numbers and the trends you report. Unlike the proof of a theorem or other logically based piece of analysis, the reader cannot verify your results by detailed inspection without good reporting from you on the conditions under which you performed the numerical experiment. It is incumbent on you to provide an indication of the care with which you constructed and tested your code and carried out the calculations to generate a reasonable level of trust in the claims put forth based on your computations.

5. INTERPRETING NUMERICAL EXPERIMENTS

Perhaps the most important aspect of the entire exercise is embodied in the interpretation put on the results. There are several pitfalls that appear regularly in reports of numerical simulation of mathematical models.

First, one must take care not to confuse the original mathematical model unduly with the numerical model. If care was taken to ensure that the program actually computes what is desired, and if one has a convincing case for the level of accuracy achieved by the program, obviously one is on a firmer ground at this stage. Even with this level of care, if the model is time dependent or has recursive features, the small errors can easily accumulate in ways that make for even quantitatively different behaviors between solutions of the mathematical and the numerical models.

It is similarly wrong to impute overall behavior to the numerical model based on a small sampling of the state space. Again, one will reach a healthy conception of this aspect if it is kept in mind that the simulations

are akin to experiments. It is not expected that a physicist will put forth a new law of nature on the basis of a single set of experiments. Instead, a conjecture will be formulated which is then amenable to further testing. A well conceived conjecture is tremendously valuable in science because it focuses activity on a defined and, it is hoped, an important objective. Numerical simulations of a mathematical model should be treated similarly as a guide to formulating conjectures about the model which one can then investigate by further simulation or by mathematical analysis, aided by the particular view provided by the conjecture.

Another confusion that sometimes occurs is to identify the abstracted or the real situation too closely with the model and with the outcome of the numerical simulations. This is an overall objective, of course, but care must be exercised not to establish this circularly by making the identification too early in the investigation. A nice example of this sort of confusion occurs in an interesting discussion of [38, Chap. 13] concerning the computation of a simple rational expectations model, namely an infinite-horizon growth model of the form

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t = f(k_t) - k_{t+1}, \qquad 0 < \beta < 1, \qquad t = 0, 1, \ldots.$$

In discussing the approach to this problem in which one attempts to solve the associated Euler equation, the issue of accuracy of a certain approximation is addressed. One method, discussed above and in [38], is to compute a solution by other, slower but more reliable means, and compute the difference between the approximate solution thereby obtained with that obtained via the much faster method based on the Euler equations. This certainly gives a direct measure of the accuracy achieved. However, another route is suggested in cases where it is awkward or infeasible to carry out the method just defined. Namely, one checks by how much the Euler equation is violated at an approximate solution. In a particular calculation, it is determined that the Euler equation is very nearly satisfied, and that, indeed, the error is less than two dollars in every hundred thousand dollars, say. The author then goes on to argue that "...that approximate policy function is as compelling a description of behavior as the equilibrium policy function since it is unclear why individuals would bother making the nontrivial effort to find the "true" policy function if the gain is so small." Of course, this argument ignores the well known mathematical problem that there may be multiple solutions of the Euler equations associated with an optimization problem, and that some of these solutions have nothing to do with an optimum policy, and it likewise ignores the general problem that a function E such that $F(E)$ is nearly zero may be

a long way from a function H such that $F(H) = 0$. If, indeed, E and H are not close, then there is no reason to expect the utility associated with them to be close. The author is aware of both of these pitfalls as one can see from other parts of his text. However, invoking an argument from the point of view of an individual about effort expended to reach a better solution when the model itself takes no account of this aspect is exactly the kind of confusion between the model and the idealized economic situation that should be avoided. To put the matter another way, heuristic arguments of the sort just reported may be helpful at the level of the idealized model or even with regard to the original situation under consideration, but once one has reduced the issue to a mathematical problem, it is only mathematical argument that can win the day.[3]

6. SUGGESTIONS FOR FURTHER READING

It is the purpose of this brief section to offer a few suggestions aimed at someone who is interested in incorporating numerical simulation into their research program, but who is a relative neophyte. Scholars more experienced with numerical simulation will probably not feel the need to study the following remarks.

One of the first, practical issues that arises is in what language the computer code is going to be assembled. There are many good programming languages available. Generally speaking, if a scientist already has command of one of the major languages, then for most purposes learning a new language is not necessary. Good computer codes can be constructed in all the major, all-purpose languages. If one has to choose a language, then it makes sense to select one of the modern ones such as C or one of its descendants, or the most recent update of FORTRAN.

Perhaps more important than the choice of language is the way in which one programs. It is extremely helpful in this respect to acquire some command of structured programming. This is probably best done in a computer science course, but it can be learned on one's own. Some good texts in this area are [17, 29, 41, 69]. The advantages of structured programming over just assembling the necessary computer instructions willy-nilly is that the resulting program is more readable by others, it is more portable, and is far easier to debug, update, and extend.

[3] A related example from the macroeconomics literature is to formulate a rational expectations model (with perfectly rational agents) and interpret the outcome of an associated numerical simulation as the solution of a different model with boundedly rational agents. The point here is that if one is interested in the analysis of a model with the possibly more realistic assumption of boundedly rational agents, such an assumption and the corresponding equilibrium concept should be specified from the outset of the formulation of the model.

As the examples offered in Section 3 should attest, it is very useful to have a working knowledge of some of the standard ideas and methods from numerical analysis. Contrary to what one might think at first, this is far from a new subject. Indeed, Newton, Gauss, and many other scientists from earlier centuries made substantial contributions to this area that are still in use today, even though their analyses pre-date the modern high-performance computer. Most modern economic theorists have a sufficient knowledge of mathematical analysis so that the texts mentioned below are technically accessible. Perhaps the most fruitful strategy is to learn one's way around one of the more comprehensive of the advanced undergraduate/early graduate-level texts. It is also useful to have at hand one or two of the many compendia of formulas for approximation of certain functions (e.g., [1]).

Numerical analysis is a large subject worth lifetimes of work by its many practitioners. As mentioned above, it is an old subject, but one that has come to the fore because of the development of computing machines. The modern version of the subject comprises the design and analysis of methods for the approximation of quantities using digital computers. It is distinguished from abstract approximation theory, which is really a branch of functional analysis, and which generally does not deal with deriving algorithms that can be implemented as computer programs. The novice will find a bewildering array of methods, many of which have more than one name. In fact, most of the overall ideas in modern numerical analysis are relatively simple and easy to grasp. More daunting are the technical aspects of the subject where these simple ideas are turned into powerful, multipurpose methods whose analysis can be anything but transparent. It is not necessary for a user of the theory to master every detail of the sometimes difficult analyses, but it is important to understand the implications of these as they often bear on practicalities.

Another aspect of practical concern is the dramatic technological development that the field has experienced in the last decade, especially in issues related to computation, algorithm design, and software. Such progress has fostered the numerical analysis of a wide range of problems in various areas to limits beyond what one could possibly imagine even in the very recent past. Further advances in computing and data processing will certainly extend the boundaries of numerical analysis, and allow for a better effected use of computational methods.

At the present stage, there are a large number of professionally designed subroutines that can tackle a variety of standard numerical problems. Some of these are bound together in packages designed for particular classes of computational tasks, such as LINPACK, QUADPACK, MINPACK and SLATEC, to mention a few. There are also some all-purpose libraries such as NAG and IMSL. There are several advantages in using software devised

by recognized experts in the field. First, the algorithms and the coding have been extensively tested and often come equipped with various safeguards that alert the user to possible anomalies or misuse. Also, these algorithms and their implementation are generally familiar to a large community of researchers, and hence the outcome of their use is understood with relative ease and confidence.

One should have a good idea of the nature of the algorithm and the performance characteristics of this kind of software before using it in an essential way as part of a code for the simulation of an economic model. For example, it is a good idea to test the software in a familiar situation to be sure one understands its use and the form of its output. In general, it probably is not efficient to attempt to redesign such software for special purposes. Often the code has been optimized in a way that makes tampering with it an uncertain business. If the software does not fit the job, look for more suitable material or seek advice at a friendly mathematics or computer science department.

We now offer some suggestions for further reading. For a first approach to the subject, we recommend a start with the introductory textbook by Kahaner, Moler, and Nash [39]. This book provides a good background on several topics and features some nice historical excursions together with discussion of modern software. For additional theoretical material we recommend two classic monographs in the area: Conte and de Boor [16] at an intermediate level, and Stoer and Bulrisch [63] at a more comprehensive and advanced level. Regarding more practical issues about scientific computing, the well known source *Numerical Recipes* [50] is a most appropriate reference. All these monographs analyze at different levels various subjects such as systems of linear equations, interpolation, numerical integration, ordinary differential equations, solving nonlinear systems, optimization, simulation and random numbers, and partial differential equations. These topics are also covered in a host of specialized treatises. The following is a partial list of some more specialized references.

1. *Matrices and Systems of Linear Equations.* They frequently arise in linearizations of economic models. The classical monographs [27, 34, 68] cover a wide range of topics in this area, and should be suitable for most applications.

2. *Ordinary Differential Equations.* This is a field that has expanded considerably in the last 15 years with extensive work on computational methods. Gear [25], Henrici [32], and Lambert [43] are classical treatises in this area. The two-volume monograph by Harier *et al.* [30, 31] provides a more updated and encyclopedic treatment.

3. *Approximation.* A good grounding of these methods is often necessary for the efficient formulation of numerical models. In addition to

the above-mentioned treatments [16, 63], Davis [19], Powell [51], and Rivlin [53] discuss issues on optimal approximation and further topics on the theory and approximation methods. Schumaker [61] is devoted to spline functions. Modern ideas about multigrid, spectral approximation, and projection methods can be found in the recent monographs [9, 11]. Also, wavelets have proven most effective for the analysis of time series, and an excellent introduction to this topic is provided in [18].

4. *Optimization.* This is a basic ingredient of many static and dynamic economic models. Techniques for solving optimization problems generally involve finding zeroes of systems of equations. Ortega and Rheinboldt [49] is a recommended source for basic methods for solving equations. Moreover, for static optimization, Bazaraa *et al.* [5], Dennis and Schnabel [21], Fletcher [23], and Gill *et al.* [26] cover several theoretical aspects and algorithms for the solution of linear and nonlinear models. Regarding dynamical models, two recent texts [24, 42] survey important theoretical work in this area.

5. *Partial Differential Equations.* These equations often arise in the finance literature and in other optimization problems. There are several standard procedures for computation of the solutions, such as finite-difference methods, finite-element methods, multigrid methods, mixed methods, and the like. Ames [2] is still a recommended reference for some basic theory. Ciarlet and Lions [13], Richtmyer and Morton [52], and Smith [62] offer an introduction to finite-difference methods, whereas Brenner and Scott [10] , Ciarlet and Lions [14], and Johnson [36] cover extensively the topic of finite-element methods as applied to different types of partial differential equations. In addition, and as mentioned previously, Canuto *et al.* [11] present an excellent modern treatment of spectral approximation and projection methods.

6. *Numerical Integration.* This is a subject with numerous applications in economics, and deserves to be treated rigorously. The corresponding chapters in [50, 63] contain a good introduction to some basic procedures. Also, [20, 65] are useful general references.

APPENDIX

Our goal here is to prove Theorem 3.1. The proof follows from th following series of lemmas.

LEMMA A.1. *Assume that W is the value functions defined in* (3.1), a *W_m is the value function defined in* (3.4). *Then under Assumptions A to there is a constant $M > 0$ such that $\|W - W_m\|_{C(X)} \leqslant Mm$.*

Proof. For given k_0, let $\dot{k} = g(k_0)$ be the optimal solution to (3.2), and let $\dot{k}_m = g_m(k_0) = (k_m - k_0)/m$ be the optimal solution to (3.4). Then we have

$$\delta W(k_0) = v(k_0, \dot{k}_0) + DW(k_0) \cdot \dot{k} \tag{7.1}$$

and

$$W_m(k_0) = v(k_0, \dot{k}_m)\, m + W(k_0 + m\dot{k}_m)\, e^{-\delta m} \tag{7.2}$$

It is well known [e.g., 12] that on compact sets, the functions W_m and g_m converge uniformly to W and g, respectively. Hence, for m small enough, $\dot{k}_m = g_m(k_0)$ lies in the interior, and so W_m is a C^1-function [cf. 6]. Then an application of the mean-value theorem to (7.2) yields that

$$W_m(k_0) = v(k_0, \dot{k}_m)\, m + [\, W_m(k_0) + DW_m(k_s) \cdot m\dot{k}_m]\, e^{-\delta m} \tag{7.3}$$

for some k_s in the segment $(k_0, k_0 + m\dot{k}_m)$. Rearranging terms, we obtain

$$\frac{W_m(k_0)[1 - e^{-\delta m}]}{m} = v(k_0, \dot{k}_m) + [DW_m(k_s) \cdot \dot{k}_m]\, e^{-\delta m}. \tag{7.4}$$

It follows that

$$\delta W_m(k_0) = v(k_0, \dot{k}_m) + DW_m(k_0) \cdot \dot{k}_m + F(m). \tag{7.5}$$

In this case, it is easy to show that DW_m is a Lipschitz function and the Lipschitz constant can be defined independently of m, for m small enough [cf. 47]. Hence, the residual term $F(m)$ is a Lipschitz function of m such that $F(0) = 0$. Moreover, from (7.4) and (7.5) and the definition of $\dot{k}_m$ we deduce the existence of a Lipschitz function $G_{k_0}(m)$ with $G_{k_0}(0) = 0$ such that

$$\delta W_m(k_0) + G_{k_0}(m) \geqslant \max_{\dot{k}} v(k_0, \dot{k}) + DW_m(k_0) \cdot \dot{k}. \tag{7.6}$$

Consider now an arbitrary solution to (3.1), $\{k'(t)\}_{t \geqslant 0}$ with $k'(0) = k_0$. Then, we have

$$\begin{aligned} W_m(k_0) - e^{-\delta t} W_m(k'(T)) &= \int_0^T [\delta W_m(k'(s)) - DW_m(k'(s)) \cdot \dot{k}'(s)]\, e^{-\delta s}\, ds \\ &\geqslant \int_0^T [v(k'(s), \dot{k}'(s)) - G_{k'_s}(m)]\, e^{-\delta s}\, ds \\ &\geqslant \int_0^T [v(k'(s), \dot{k}'(s)) - G(m)]\, e^{-\delta s}\, ds, \end{aligned}$$

where the first equality comes from the definition of the integral and the other two inequalities come from (7.6) and the fact that on a compact domain X, the Lipschitz constant G can be selected independently of k.

Letting T go to infinity we then see that

$$W_m(k_0) \geqslant \int_0^{\infty} v(k'(s), \dot{k}'(s))\, e^{-\delta t}\, dt + K(m),$$

where $K(m)$ is a Lipschitz function. Hence, for all k_0 in a compact domain X,

$$W_m(k_0) + Mm \geqslant W(k_0)$$

for some constant M. The reverse inequality

$$W(k_0) \geqslant W_m(k_0) - Mm$$

follows easily from the basic observation that $W(k_0) \geqslant e^{\delta m} W_m(k_0)$, as piecewise constant controls are feasible solutions to (3.1).

LEMMA A.2. *Assume that g is the policy function for W defined in* (3.1) *and g_m is the policy function for W_m defined in* (3.4). *Then under Assumptions A to C there is a constant $N > 0$ such that $\| g - g_m \|_{C(X)} \leqslant N m^{1/2}$.*

Proof. The proof becomes more transparent in the one-dimensional case, and so we assume temporarily that k and $\dot{k}$ are simply real numbers. We shall prove that the established bound in the previous theorem imposes an additional restriction on the first-order derivatives DW and DW_m. This is because such derivatives are Lipschitz functions.

To establish the lemma, consider for some arbitrary k in X the extreme case

$$W(k) = W_m(k) + Mm, \qquad M > 0,$$

and

$$DW(k) = DW_m(k) - G, \qquad G > 0.$$

In this situation, G cannot be an arbitrary constant. Indeed, the worst possible case is to assume the existence of a point $k' > k$ such that

$$W(k') + Mm = W_m(k')$$

and

$$DW(k') = DW_m(k').$$

Then $k' - k \geq G/K$, where K is a Lipschitz constant for both W and W_m. Hence

$$[W_m(k') - W_m(k)] - [W(k') - W(k)] = 2Mm$$

$$= \int_k^{k'} [DW_m(s) - DW(s)]\, ds$$

$$\geq \int_0^{G/K} (G - Ks)\, ds = \frac{1}{2}\frac{G^2}{K}.$$

Therefore, $2Mm \geq \frac{1}{2}G^2/K$. Thus, $G \leq (4KM)^{1/2} m^{1/2}$. Since k was an arbitrarily chosen point, we have proved the existence of a constant K' such that

$$\|DW - DW_m\|_{C(X)} \leq K' m^{1/2}. \tag{7.7}$$

Moreover, the same sort of result holds in the multidimensional case, as the same argument can be applied separately to each coordinate, leaving constant the remaining components. Lemma A.2 is now an easy consequence of the definitions of g and g_m [cf. Eqs. (7.1)–(7.3)], since by Assumption B the function v is α_K-concave.

We now illustrate that under additional assumptions it is possible to obtain higher orders of convergence.

LEMMA A.3. *Assume that the sequence of values*

$$\frac{[DW(k_0) - DW_m(k_0)] \cdot g(k_0)}{\|DW(k_0) - DW_m(k_0)\|}$$

is uniformly separated from zero for all m small enough. Then under the conditions of the previous lemma, there is a constant N' such that $\|g(k_0) - g_m(k_0)\| \leq N'm$. In particular, for k_0 a real number, $\|g(k_0) - g_m(k_0)\| \leq N'm$ if $g(k_0) \neq 0$.

Proof. It follows from Lemma A.1 and Eqs. (7.1) and (7.5) that there exists a constant M' such that

$$-M'm \leq v(k_0, \dot{k}) + DW(k_0) \cdot \dot{k} - v(k_0, \dot{k}_m) - DW_m(k_0) \cdot \dot{k}_m \leq M'm. \tag{7.8}$$

Under the maintained assumptions, we now claim the existence of $K>0$ such that $\|DW(k_0)-DW_m(k_0)\| \leqslant Km$. Let $\varepsilon = DW(k_0)-DW_m(k_0)$. Consider the optimization problem

$$V(\varepsilon) = \max_{\dot{k}} v(k_0, \dot{k}) + (DW(k_0)+\varepsilon)\cdot \dot{k}.$$

Let $\dot{k}_\varepsilon$ be the optimal solution. Then it follows from (7.1), (7.5), and (7.8) that for some ε' in the interval $(0, \varepsilon)$,

$$|V(\varepsilon)-V(0)| = |DV(\varepsilon')\cdot\varepsilon| = |\dot{k}_{\varepsilon'}\cdot\varepsilon| \leqslant Lm$$

for some constant L. As $\dot{k}_{\varepsilon'}$, converges to $\dot{k}$, and $|\dot{k}\cdot\varepsilon| \neq 0$, we must have in view of (7.4) and (7.5) that under the asserted concavity of v, $\|DW(k_0)-DW_m(k_0)\| \leqslant Km$ for some $K>0$. The proof now proceeds as in the previous lemma.

We now provide an error bound for the discretization of the state space X. Let us define

$$\mathcal{W}_m = \{W_m : X \to R \mid W_m \text{ is continuous}\}$$

$$\mathcal{W}^h_m = \{W^h_m : X \to R \mid W^h_m \text{ is continuous and the derivative } DW^h_m \text{ is constant in int}(S^j), \text{ for each } S^j\}.$$

Observe that both $\mathcal{W}_m$ and $\mathcal{W}^h_m$ are metric spaces with the distance induced by the norm $\|W\|_{C(X)} = \max_{k\in X}|W(k)|$. Also, define the functional operators, $T_m : \mathcal{W}_m \to \mathcal{W}_m$ and $T^h_m : \mathcal{W}_m \to \mathcal{W}^h_m$, by

$T_m(V(k)) = \max_{\dot{k}} v(k, \dot{k})\, m + V(k+m\dot{k})\, e^{-\delta m}$ subject to $(k, \dot{k}) \in T$, for all k in X, and V in $\mathcal{W}_m$; and

$T^h_m(V(k^j)) = \max_{\dot{k}} v(k^j, \dot{k})\, m + V(k^j + m\dot{k})\, e^{\delta} m$ subject to $(k^j, \dot{k}) \in T$, for each vertex k^j in X, and V in $\mathcal{W}_m$.

LEMMA A.4. *Under Assumptions A and B, Eq.* (3.4) *has a unique fixed point* W_m *in* $\mathcal{W}_m$, *and Eq.* (3.5) *has a unique fixed point* W^h_m *in* $\mathcal{W}^h_m$.

Proof. The proof is the standard one [cf. 22]. One immediately sees that T^h and T^h_m are well defined and that both are contraction mappings with modulus $0 < e^{-\delta m} < 1$. By a well known fixed-point theorem, Eq. (3.4) has a unique fixed point W_m in $\mathcal{W}_m$, and Eq. (3.5) has a unique fixed point W^h_m in $\mathcal{W}^h_m$.

LEMMA A.5. *Assume that W_m^h is a fixed point for Eq.* (3.5). *Under Assumptions A to C, there is a constant $K>0$ such that for sufficiently small m and h, it must be the case that $\|T_m W_m^h - T_m^h W_m^h\|_{C(X)} \leqslant Kh^2 m$.*

Proof. Pick an arbitrary point $k \in X$ in a simplex S^j. Then $k = \sum_{i=1}^{n+1} a_i(k) k^i$, where k^i is a generic vertex point in S^j and $0 \leqslant a_i(k) \leqslant 1$. Assume that $\dot{k}'$ is an optimal solution to

$$\max v(k, \dot{k})\, m + W_m^h(k + m\dot{k})\, e^{-\delta m}$$
$$\text{subject to} \qquad (k, \dot{k}) \in T.$$

Also, for $i = 1, \ldots, n+1$ assume that $\dot{k}^i$ is an optimal solution to

$$\max v(k^i, \dot{k})\, m + W_m^h(k^i + m\dot{k})\, e^{-\delta m}$$
$$\text{subject to} \qquad (k^i, \dot{k}) \in T.$$

Then for sufficiently small m and h it must hold that

$$\begin{aligned} &|v(k, \dot{k}')\, m + W_m^h(k + m\dot{k}')\, e^{-\delta m} \\ &\qquad - \sum_{i=1}^{n+1} [a(k^i)[(v(k^i, \dot{k}^i)\, m + W_m^h(k^i + m\dot{k}^i)\, e^{-\delta m})]| \\ &\leqslant \sum_{i=1}^{n+1} [a(k^i)\, |v(k, \dot{k}') - v(k^i, \dot{k}')|\, m] \\ &\qquad + \sum_{i=1}^{n+1} [a(k^i)\, |v(k, \dot{k}^i) - v(k^i, \dot{k}^i)|\, m] \leqslant Kh^2 m, \end{aligned}$$

where the first inequality is true if the optimal controls $\dot{k}'$ and $\dot{k}^i$ $(i = 1, \ldots, n+1)$ are all feasible for both k and every vertex point k^i (by the convergence properties of W_m and W_m^h, this requirement is always satisfied if h is small enough, or else if the domain of controls $\dot{k}$ is a given fixed set). The last step follows from [58, Lemma 3.4] for some constant K.

LEMMA A.6. *Assume that W_m is a fixed point of Eq.* (3.4) *and that W_m^h is a fixed point of Eq.* (3.5). *Then under Assumptions A to C, there is a constant $M>0$ such that $\|W_m - W_m^h\|_{C(X)} \leqslant Mh^2$.*

Proof. Let T_m and T_m^h be the functional operators defined previously from equations (3.4) and (3.5), respectively. Then

$$\begin{aligned} \|W_m - W_m^h\|_{C(X)} &= \|T_m W_m - T_m^h W_m^h\|_{C(X)} \\ &\leqslant \|T_m W_m - T_m W_m^h\|_{C(X)} + \|T_m W_m^h - T_m^h W_m^h\|_{C(X)}. \end{aligned}$$

Hence, by Lemma A.4,

$$\|W_m - W_m^h\|_{C(X)} \leq e^{-\delta m}\|W_m - W_m^h\|_{C(X)} + \|T_m W_m^h - T_m^h W_m^h\|_{C(X)}.$$

Moreover, by Lemma A.5,

$$\|W_m - W_m^h\|_{C(X)} \leq \frac{1}{1-e^{-\delta m}}\|T_m W_m - T_m^h W_m^h\|_{C(X)} \leq \frac{Kh^2 m}{1-e^{-\delta m}}.$$

Lemma A.6 is now easily established from the observation that $m/(1-e^{-\delta m})$ converges to $1/\delta$ as m converges to 0.

LEMMA A.7. *Let k^j be a generic vertex point. Let $g_m(k^j)$ be the policy function for optimization problem* (3.4) *at k^j, and let $g_m^h(k^j)$ be a point of the optimal policy for optimization problem* (3.5) *at k^j. Then under Assumptions A to C, there is an $N>0$ such that*

$$\|g_m(k^j) - g_m^h(k^j)\|_{C(X)} \leq Nh.$$

Proof. Let $k^j \in X$ be a vertex point. Assume $\dot{k}_m = g_m(k^j)$ and $\dot{k}_m^h \in g_m^h(k^j)$. Observe that

$$D_2 v(k^j, \dot{k}_m) + DW_m(k^j + m\dot{k}_m) = 0 \tag{7.9}$$

and

$$-D_2 v(k^j, \dot{k}_m^h) \in \partial W_m^h(k^j + m\dot{k}_m^h), \tag{7.10}$$

where ∂W_m^h denotes the generalized gradient of the Lipschitz function W_m^h (cf. [15]). Moreover, for every point k in the interior of a simplex S^j for which k^j is a vertex point we must have

$$\begin{aligned}&|[W_m(k) - W_m(k^j)] - [W_m^h(k) - W_m^h(k^j)]| \\ &\quad = |[DW_m(k') - DW_m^h(k')] \cdot (k - k^j)| \leq 2Mh^2,\end{aligned}$$

where for some k' in (k^j, k) the first equality comes from the mean-value theorem and the fact that the derivative DW_m^h is constant over the interior of each simplex S^j, and the upper bound $2Mh^2$ follows from Lemma A.6. Since k is an arbitrary interior point, it must hold that there is a constant N' such that for every vertex point k^j and every d in $\partial W_m^h(k^j)$ one has

$$\|DW_m(k^j) - d\| \leq N'h. \tag{7.11}$$

From (7.9) and (7.10), Lemma A.7 is now a simple consequence of (7.11) and the α_K-concavity of v, for every vertex point k^j.

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Corrections to the Galley Proofs "On the Role of Computation in Economic Theory"

J. Bona M. S. Santos

- Page 1, line 9: Insert "1994"

  Received December 23, 1994, Revised March 1, 1996

- Page 3, line 14: change "beautified" to "beautiful"

- Page 4, end: Footnote 3 should be moved to page 29. (It is the last remark in Section 5). Observe that in this footnote there is an "e" missing off one of the "the".

- Page 7, line -9: Delete capital W and close up.

- Page 8, line 8: Delete comma; at most use a "comma" superscript for possession. (See the original manuscript).

- Page 9, line -16: We are missing a period and a capital D.

  ... continua. Digital computers

- Page 9, line -6: Delete comma

- Page 10, line -14: write "proper", not "propel".

- Page 11, equation (3.2): $\dot{k}$ below sup (not k). Please see original manuscript.

- Page 11, line -3: dash between C^1 and mapping.

  C^1-mapping

- Page 11, line -2: comma after applications

  In most applications, one faces...

- Page 12, line 8: comma after state

  ...stable steady state, then it is...

- Page 12, line 19: replace colon by period

  ... may arise.

- Page 13, line 6: "nearly" instead of "neatly"

- Page 13, formula (3.4): It needs a period at the end of it.

  $i = 0, 1, 2, ...$

- Page 13, line -5: close up in the definition of $\mathcal{W}_m^h$ that is marked.

- Page 14, line 8: Insert "If the solution is unique, this..."

  - line -20: Insert "comma", delete "and" to make it read

    such as mesh size, order of...

- Page 15, line 11: "to" instead of "too". Delete one "o"

- Page 16, line -12: non-monetary, insert a dash.

- Page 17, line 7: Insert " , assumed to be equal to one."

- Page 17, line -13: change "analysis" to "discussion" (we use analysis in the next sentence)

  The discussion in [8]...

- Page 17, line -10: take out dash

- Page 18, line 10: take out dash

- Page 18, second display : close up between "v" and "is linear on..."

- Page 18, line -9: x_N instead of x_n

- Page 18, line -8: We left out zero here

  j=0,1,2,3...

- Page 19, line -7: insert "under"

  in [55-57], under "regular conditions"...

- Page 19, line -4: Change "policy function" to "optimal policy"

- Page 20, line 1: Change "this" to "the"

  The result extends...

  - line 13: Dash between C^2 and properties

    C^2-properties of W. ...

- Page 21, line -11: "attempt" , add a "t".

  - line -7: something wrong with the first W^h_{n-1}, it should be the same as the following ones.

- Page 22, line 2: Delete comma, insert "and"

  value functions W^h_{n-1} and W^h_n is such...

  - line 10: It should read $k^* = 2.0673$
  - line 15: Insert the sentence (note that we must change "experiment" to "experiments"):

    The numerical experiments which were coded in standard FORTRAN 77 and run on a DEC 2000 workstation (300 ALPHA AXP, rated at 358.1 MFLOPS/150 Mhz.), started with h=...

- Page 22, line 21: change "460" to "91"

  - line 22: change " 2 min 51 s." to "33 s."

- Page 23, legend of figure 3: change "$\hat{n} = 559$" to "$\hat{n} = 91$"

- Page 23, legend of figure 4: change "$\hat{n} = 1478$" to "$\hat{n} = 90$"

- Table 1: The Table should be as folows

No. of vertex points	Mesh size	No. of additional iterations	CPU time in seconds	Max. observed error in W
100	10^{-1}	99	3	1.8×10^{-1}
1000	10^{-2}	91	33	1.7×10^{-3}
10000	10^{-3}	90	298	1.8×10^{-5}

- Page 25, lines 13-15: A dot after s (for seconds) seems to be missing.
- Page 25, line -18: Change "may" to "would likely". A stronger statement is warranted here. Also insert a comma after "accuracy".

  ...accuracy, it would likely be profitable

  - last sentence before the example: Change [not viable for full-fleshed models involving four or more variables] to [subject to the "curse of dimensionality".]
  - last line: period missing at the end.

- Page 25 and 26, formulas (3.19) and (3.20): delete the "=" sign

  - line -9, -8: Change "gets very close to" to "hovers around".

    ...p(j) hovers around 2. This array...

- Page 29, line 18: Insert footnonte 3 after "day"

  ...argument that can win the day.[3]

- Page 29, line -3: Delete the "r" sitting there by itself.

  - line -2: It is more portable; not "its portable".

- Page 30, line 6: "modern", lower case m, not Modern

  - Last line: change "ISML" o "IMLS"

- page 31, line 26: Change "there" to "these"

- line -6: change "considerable" to "considerably"

- Page 33, line 7: Insert "the"

  ...on compact sets, the functions...

  - line 9: insert dash; "C^1-function"
  - line 10: insert a dash; "mean-value theorem".
  - line 12: insert a comma after "terms"

    ...Rearranging terms, we obtain...

  - line -10: "of" instead of "on"

    Lipschitz function of m such...

  - Equation (7.6): $\dot{k}$ below max, (not k). See manuscript.
  - line -5: "an" instead of "any"; delete the "y".

    Consider now an arbitrary...

  - line -5: it should say $\{k'(t)\}_{t\geq 0}$ (See the manusript)
  - line -4: make it read

    Then, we have

- Page 34, line 2: transpose "two" and "other"

  other two inequalities come...

  - End of the statement of lemma A.2: needs a period on line -14
  - line -13: change "unidimensional" to "one-dimensional"
  - line -10: Insert a dash

    first-order derivatives...

  - line -3: start it as "In this situation,"

    In this situation, G cannot be...

- Page 35, line 3: no indentation

  - line 4: make it read

Hence, it follows that

- line 6: the subscript is "m" not "s"
- line 11: insert a dash; multi-dimensional
- line -7: take out the dash between "for" and "all".

- Page 36, line 10: It is capital W (see manuscript)
  - line 11: leave a blank line between this one and the next to set off the end of the proof
  - line -14: no indentation
  - line -13: delete "let us"; it should read

    ... Also, define the functional...
  - line -12: delete a comma and delete "given"

    ...$\mathcal{W}_m^h$ by...
  - line -11: The phrase "subject to " could go on the next line
  - line -9: The phrase "subject to " could go on the next line

Thus both of these would have the form (3.1)-(3.5).

- line -7: insert comma after B

  ...A and B, Eq. 3.4
- line -4: delete a comma between "defined" and "and"

  are well defined and that...

- Page 37, line 3: Replace "hold" by "be the case"; comma after "h"

  m and h, it must be the case that ...
  - line 5: delete comma after S^j

    vertex point in $S^j$and $0 \leq$...
  - line -11: space between k^i and $(i = 1, ..., n + 1)$
  - line -7: delete comma after (3.4)

...fixed point of Eq. (3.4) and that...

- line -6: insert "a constant" after "is"

  ...C, there is a constant M>0 such...

- last line: period missing at the end.

- Page 38, line 8-9: change "the policy function" to "a point of the optimal policy"

  - line 10: insert "an" after "is"

    there is an N>0 such that

  - line 13: Change "=" to "$\in$" (change the 'equal' sign to the 'belongs to' sign)

  - line -8: insert dash;

    ...the mean-value theorem...

  - last line: dot should go just above k (see the manuscript)

- In the references:

[6]: In the title, change "continuous case" to "continuous-time case"
[8]: The new title of the paper is:

Stabilizing Monetary Injection Policies

[15]: In Frank Clarke's paper change "274" to "247"
[42]: there is the word "for" missing from the title

"Numerical Methods for Stochastic...