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INSTABILITY OF SOLITARY-WAVE SOLUTIONS OF THE 3-DIMENSIONAL KADOMTSEV-PETVIASHVILI EQUATION

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Abstract. The generalized Kadomtsev-Petviashvili system of equations in three space dimensions,

$$\begin{cases} u_t + u^p u_x + u_{xxx} - v_y - w_z = 0, \\ v_x = u_y, \\ w_x = u_z, \end{cases} \tag{$*$}$$

has been shown by de Bouard and Saut to possess solitary-wave solutions if and only if $1 \le p < 4/3$. It is demonstrated here that these localized traveling-waves, when considered as solutions of the initial-value problem for $(*)$, are dynamically unstable to perturbations.

1. INTRODUCTION

Considered herein is the 3-dimensional generalization of the Kadomtsev-Petviashvili equation (KP-equation henceforth)

$$\begin{cases} u_t + u^p u_x + u_{xxx} - v_y - w_z = 0, \\ v_x = u_y, \\ w_x = u_z, \end{cases} \tag{KP-3D}$$

where $(x, y, z) \in \mathbf{R}^3$ and $t \ge 0$, say. The KP-type-equations are universal models for the propagation of weakly nonlinear dispersive long waves that are essentially uni-directional, but which allow for weak transverse effects (see Kadomtsev and Petviashvili 1970, Petviashvili and Yan'kov 1989, the monograph of Enfeld and Rowland 1990, and, for commentary in a plasma physics context on the cylindrical solitary waves considered here, Kuznetsov

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and Turitsyn 1982 and Mikhailovskaya and Erokhin 1987). In certain technical senses, (KP-3D) is a natural extension of the classical one-dimensional Korteweg-de Vries equation (KdV-equation henceforth) and the Benjamin-Ono (BO) equation to three dimensions, as pointed out by Spector and Miloh (1985), for example.

The initial-value problem for (KP-3D) consists in posing a suitable starting point $u_0(x, y, z)$ for the dependent variable u. This problem has been the subject of a number of studies, and a satisfactory local (in time) existence theory in both two and three space dimensions with data posed in all of $\mathbf{R}^n, n = 2, 3$, or with periodic boundary conditions is now in hand (see Ukai 1989, Saut 1993, 1995, Bourgain 1993). The issue of whether or not particular initial data generate a global solution is more subtle. For example, certain classes of initial data posed on all of $\mathbf{R}^3$ for (KP-3D) with $p \geq 2$ are known to lead to solutions that form singularities in finite time. The essence of this result is a virial identity written by Turytsyn and Falkovich (1985), but a rigorous proof waited for the paper of Saut (1993). Moreover, it was recently shown by Liu (2001) that solutions of (KP-3D) may form singularities in finite time even for p in the range $1 \leq p \leq \frac{4}{3}$. This latter theory makes use of some invariant sets for the flow which allows one to optimize the use of the virial identity. Global existence in the range $\frac{4}{3} < p \leq 2$ is an open question.

An issue that is often related to whether or not global existence obtains for arbitrary classes of data is the stability of the solitary-wave solutions. Solitary waves are special traveling-wave solutions of (KP-3D) that are localized in space. Often, when nonlinear dispersive wave equations have solutions that lose regularity in finite time, the transition to singularity formation is associated with a solitary wave going unstable. For (KP-3D), de Bouard and Saut (1995a, 1995b) have shown existence of solitary waves $u(x, y, z, t) = \varphi_c(x, -ct, y, z)$ for p in the range $1 \leq p < 4/3$ (and non-existence for p outside this range).

It is our purpose here to show that these solitary-wave solutions are indeed unstable when considered as solutions of the initial-value problem for (KP-3D). Thus there are perturbations arbitrarily close to $\varphi_c(x, y, z)$ which, when posed as initial data for (KP-3D), lead to solutions that move away from the solitary wave in a way to be made precise presently. These results complement those of de Bouard and Saut (1996) which dealt with the same issue in two space dimensions.

To establish the result in view, we follow the general approach to stability and instability pioneered by Grillakis, Shatah and Strauss (1987) and Shatah and Strauss (1985), but using the detailed analysis of solitary waves provided by de Bouard and Saut. Employed in our development are E and Q, two of the five known invariants

$$E(u,v,w) = \frac{1}{2}\int_{\mathbf{R}^3} u_x^2 + v^2 + w^2 - \frac{1}{(p+1)(p+2)}\int_{\mathbf{R}^3} u^{p+2} \tag{1.1}$$

$$Q(u) = \frac{1}{2}\int_{\mathbf{R}^3} u^2 \tag{1.2}$$

$$I(u) = \int_{\mathbf{R}^3} u \tag{1.3}$$

$$P_1(u,v,w) = \int_{\mathbf{R}^3} uv \qquad \text{and} \tag{1.4}$$

$$P_2(u,v,w) = \int_{\mathbf{R}^3} uw. \tag{1.5}$$

of (KP-3D). By modifying slightly the ideas in Shatah and Strauss (1985), Bona, Souganidis and Strauss (1987) and Souganidis and Strauss (1990), it will be shown that a sharp instability criterion involving the so-called moment of instability applies in the present circumstance. Detailed properties of the solitary wave φ_c come naturally to the fore when analyzing the moment of instability.

The plan of the paper is as follows. In Section 2, notation is introduced and the principal results about the solitary-wave solution of (KP-3D) are described. The main result of the paper is also stated to give focus to the technical developments in Section 3, where instability is established. The paper concludes with a short summary and further commentary.

2. Preliminaries and the principal result

Notation. Throughout, p will be a rational number $m/n, m$ and n relatively prime, and n odd so that, by choosing the proper branch of the mapping $z \mapsto z^{1/n}$, u^p is real-valued whenever u is real.

If $f \in L_q(\mathbf{R}^3)$, its norm is written $|f|_q$, whereas if $f \in W_q^s(\mathbf{R}^3)$, its norm is $\|f\|_{s,q}$. If $q = 2$, we abbreviate the norm of $H^s(\mathbf{R}^3) = W_2^s(\mathbf{R}^3)$ to simply $\|f\|_s$. The notation $< f, g >$ connotes the $L_2(\mathbf{R}^3)$–inner product of the two measurable functions f and g. The linear space $C_0^\infty(\mathbf{R}^3)$ is the usual collection of real-valued, C^∞–functions having compact support in $\mathbf{R}^3$. Use will also be made of a couple of non-standard spaces. For $s \geq 0$, define the asymmetric Sobolev space

$$X_s = \{f \in H^s(\mathbf{R}^3) : D_x^{-1} f \in H^s(\mathbf{R}^3)\}$$

with the obvious norm

$$\|f\|_{X_s} = \|f\|_s + \|D_x^{-1} f\|_s.$$

Here and below, $D_x^{-1} f$ is defined via the Fourier transform as

$$\widehat{D_x^{-1} f}\,(\xi_1, \xi_2, \xi_3) = (i\xi_1)^{-1} \hat{f}(\xi_1, \xi_2, \xi_3).$$

As above, the dual variables to (x, y, z) with regard to the Fourier transform are denoted (ξ_1, ξ_2, ξ_3). In a similar vein, the operator D_x^{-k}, for $k = 1, 2, \ldots$, is the Fourier multiplier operator with symbol $(1/i\xi_1)^k$. Let Y be the closure of the linear space

$$\{g : \mathbf{R}^3 \to \mathbf{R} : \ g = \partial_x f \quad \text{for some } f \in C_0^\infty(\mathbf{R}^3)\}$$

with the norm

$$\|g\|_Y = \|\partial_x f\|_Y = (|\partial_x f|_2^2 + |\partial_y f|_2^2 + |\partial_z f|_2^2 + |\partial_x^2 f|_2^2)^{\frac{1}{2}}.$$

Notice that $D_x^{-1}\partial_y$ and $D_x^{-1}\partial_z$ are well defined on Y. In fact, $D_x^{-1} u_y$ is the unique element $v \in L_2(\mathbf{R}^3)$ such that $v_x = u_y$, and similarly for $D_x^{-1} u_z$. Of course, $X_s \subset Y$ provided that $s \geq 2$. If $u \in Y$, then by abuse of notation, we will write

$$E(u) = E(u, D_x^{-1}\partial_y u, D_x^{-1}\partial_z u).$$

This abbreviation, which will be used throughout, simplifies the appearance of formulas in Section 3. If $u \in Y$, a distinguished role will be played by the *action* $S = S_c$ defined for a given phase speed $c > 0$ to be

$$S_c(u) = E(u) + cQ(u). \tag{2.1}$$

Note that if u is a solution of (KP-3D) that, for each time t lies in Y, then the action $S_c(u(\cdot, t))$ is independent of t.

The solitary waves. The detailed notation in de Bouard and Saut (1995a, 1995b) will be incorporated into the present description. By a solitary wave of (KP-3D), we mean simply a solution (u, v, w) whose first component has the form $u(x, y, z, t) = \varphi_c(x - ct, y, z)$ where $\varphi_c \in Y$. Such a solution satisfies the system of equations

$$\begin{cases} -c\partial_x \varphi_c + \varphi_c^p \partial_x \varphi_c + \partial_x^3 \varphi_c - v_y - w_z = 0, \\ v_x = \partial_y \varphi_c, \\ w_x = \partial_z \varphi_c, \end{cases} \tag{2.2}$$

in all of $\mathbf{R}^3$. It follows readily that φ_c must satisfy the nonlinear elliptic equation

$$-\Delta_c\varphi_c + \partial_x^4\varphi_c + \frac{1}{p+1}\partial_x^2\varphi_c^{p+1} = 0 \tag{2.3}$$

in $\mathbf{R}^3$, where

$$\Delta_c = c\partial_x^2 + \partial_y^2 + \partial_z^2.$$

Conversely, if $\varphi_c \in Y$ is a non-trivial solution of (2.3) and we define

$$u(x,y,z,t) = \varphi_c(x-ct,y,z), \quad v = D_x^{-1}u_y, \quad \text{and} \quad w = D_x^{-1}u_z,$$

then (u,v,w) is a solitary-wave solution of (KP-3D). Notice also that the phase speed c can be normalized to 1 by the transformation

$$\varphi(x,y,z) = c^{-\frac{1}{p}}\varphi_c(\frac{x}{\sqrt{c}},\frac{y}{c},\frac{z}{c}).$$

Thus, if $\Phi = \varphi_1 \in Y$ is any non-trivial solution of (2.3) with $c = 1$, we automatically obtain an associated smooth branch of solitary-wave solutions of (KP-3D) by setting

$$\varphi_c(x,y,z) = c^{\frac{1}{p}}\Phi(\sqrt{c}x, cy, cz) \tag{2.4a}$$

and

$$\begin{cases} u(x,y,z,t) = \varphi_c(x-ct,y,z), \\ v(x,y,z,t) = (D_x^{-1}\partial_y\varphi_c)(x-ct,y,z), \\ w(x,y,z,t) = (D_x^{-1}\partial_z\varphi_c)(x-ct,y,z). \end{cases} \tag{2.4b}$$

If φ_c is a solitary wave in Y, then the *moment of instability* $d(c)$ associated to the branch (2.4a) emanating from φ_c is

$$\begin{aligned} d(c) &= E(\varphi_c) + cQ(\varphi_c) = S_c(\varphi_c) \\ &= E(\varphi_c, D_x^{-1}\partial_y\varphi_c, D_x^{-1}\partial_z\varphi_c) + cQ(\varphi_c). \end{aligned} \tag{2.5}$$

As mentioned already, de Bouard and Saut have given a sharp set of results about solutions of (2.3) lying in Y. Their theory is summarized now for the reader's convenience. De Bouard and Saut obtain their solitary-wave solutions via the constrained minimization problem

$$I_\lambda = \inf\left\{I_c(u) : \ u \in Y \text{ and } \int_{\mathbf{R}^3} u^{p+2} = \lambda\right\} \tag{2.6}$$

for a particular choice $\lambda = \lambda^* > 0$, where

$$I_c(u) = \int_{\mathbf{R}^3} u_x^2 + cu^2 + (D_x^{-1}\partial_y u)^2 + (D_x^{-1}\partial_z u)^2.$$

Indeed, they show existence of solitary-wave solutions which are ground states, so minimizing the action $S_c(u)$ among all solutions of equation (2.2). Such solutions u^*, say, are proven to lie in Y and have the property that

$$K(u^*) = 0 = \inf\Big\{K(u) : u \in Y, \int_{\mathbf{R}^3} u_x^2 = \int_{\mathbf{R}^3} (u_x^*)^2\Big\}, \tag{2.7a}$$

where

$$\begin{aligned} K(u) = \frac{1}{2}\int_{\mathbf{R}^3} cu^2 + (D_x^{-1}\partial_y u)^2 + (D_x^{-1}\partial_z u)^2 \\ + \frac{1}{6}\int_{\mathbf{R}^3} u_x^2 - \frac{1}{(p+1)(p+2)}\int_{\mathbf{R}^3} u^{p+2} \end{aligned} \tag{2.7b}$$

(see de Bouard and Saut 1995b, Lemma 2.1). Notice that if $u^* = \varphi_c$ is a non-trivial de Bouard–Saut solitary wave, then

$$d(c) = S_c(\varphi_c) = K(\varphi_c) + \frac{1}{3}\int_{\mathbf{R}^3} (\partial_x \varphi_c)^2 > 0. \tag{2.8}$$

Theorem 2.1. (de Bouard – Saut) *For any $c > 0$ and $1 \le p < 4/3$, the system* (KP-3D) *has non-trivial solitary-waves φ_c which are solutions of the minimization expressed in* (2.6) *for a suitable choice of $\lambda > 0$. These solitary waves tend to zero at infinity, and are cylindrically symmetric in the transverse variables (y, z), which is to say, $\varphi_c(x, y, z) = \varphi_c(x, |x'|)$ where $x' = (y, z)$ and $|x'| = \sqrt{y^2 + z^2}$. They also have the properties that $r^\delta \varphi_c, r^{1+\delta}\nabla\varphi_c \in L_2(\mathbf{R}^3)$ for any δ with $0 \le \delta < 3/2$, where $r^2 = x^2 + y^2 + z^2$.*

Remark. The issue of uniqueness up to translations, even of a ground state (minimizing solution of (2.6)), is open for this system. The restriction $p < 4/3$ is sharp, as de Bouard and Saut show there are no solitary waves for $p \ge 4/3$. In the theory put forward in Section 3, some of the auxiliary results require for their proof properties established by de Bouard and Saut of minimizers of (2.6). When these properties are needed, we so indicate by stating the relevant result for de Bouard-Saut solitary waves.

Local existence and the main result. The last ingredient needed in our development is a local existence theory for the initial-value problem. This has been provided by Saut (1995) (see also Ukai 1989). In the absence of at least a local existence result in a function class that includes the solitary waves, the question of stability or its absence has no clear significance. The following lemma takes account of Molinet's (1999) commentary on Saut's basic result.

Theorem 2.2. (Saut, Molinet) *Suppose* $u_0 \in X_s$, *for* $s \geq 3$. *Then there exists* $T > 0$ *and a unique solution* (u, v, w) *of* (KP-3D) *with*

$$u \in C([0,T); H^s(\mathbf{R}^3)) \cap C^1([0,T); H^{s-3}(\mathbf{R}^3))$$

and

$$v, w \in C([0,T); H^{s-1}(\mathbf{R}^3)).$$

Moreover, the functionals $E(u(\cdot,t)), Q(u(\cdot,t))$ *and* $I(u(\cdot,t))$ *take values independent of* t *when evaluated on the solution* (u, v, w).

We say that a solitary-wave solution φ_c of (KP-3D) is *stable for the space* Y if for any $\epsilon > 0$, there is a $\delta > 0$ such that if $u_0 \in X_s$ for some $s \geq 3$ and $\|u_0 - \varphi_c\|_Y < \delta$, the solution u of the equation (KP-3D) with initial value u_0 satisfies

$$\inf_{\vec{r} \in \mathbf{R}^3} \|u(\cdot,t) - \varphi_c(\cdot - \vec{r})\|_Y < \epsilon$$

for all $t \geq 0$. Otherwise, φ_c is considered to be *unstable*, at least with regard to the space Y.

The principal result of the present paper may now be enunciated.

Theorem 2.3. (Main Result) *Let* p *lie in the range* $1 \leq p < 4/3$ *and corresponding to this value of* p, *let* φ_c *be a de Bouard-Saut cylindrically symmetric solitary-wave solution of* (KP-3D) *with phase speed* $c > 0$. *Then* φ_c *is unstable in* Y.

3. Proof of instability

In this, the primary section of the paper, a proof of the Main Result is completed. Despite the extra complexity of KP-type equations, we are able to adapt the development put forward for KdV-type equations in Bona *et al.* (1987).

Let φ_1 be a solitary-wave solution of (2.3) with phase speed equal to 1, say, and let φ_c be the associated branch of solitary waves parameterized by the phase speed c as depicted in (2.4a). Suppose $\varphi \in X_s$ for some $s \geq 3$ as well. Then the function d defined in (2.5) is differentiable with respect to c and

$$d'(c) = \langle E'(\varphi_c) + cQ'(\varphi_c), \frac{\partial \varphi_c}{\partial c} \rangle + Q(\varphi_c) = Q(\varphi_c) = c^{\frac{4-5p}{2p}} Q(\varphi_1). \qquad (3.1)$$

In this latter calculation, use has been made of the fact that the Fréchet derivative $S_c' = E' + cQ'$ at φ_c, evaluated in the direction h may be written in the form

$$E'(\varphi_c)h + cQ'(\varphi_c)h = \int_{\mathbf{R}^3} \big[-\Delta_c\varphi_c + \partial_x^4\varphi_c + \frac{1}{p+1}\partial_x^2\varphi_c^{p+1}\big]h \tag{3.2}$$

after suitable integrations by parts. In consequence of (2.3), this quantity vanishes identically, independently of the choice of h. Differentiating d' and evaluating at c_0 yields

$$d''(c_0) = \frac{4-5p}{2p} c_0^{\frac{4-7p}{2p}} Q(\varphi_1) < 0$$

since $1 \le p < 4/3$. The proof of the main result is approached via a series of lemmas.

Lemma 3.1. *Let d be as defined in (2.5) relative to the branch $\{\varphi_c\}_{c>0}$ of solitary-wave solutions defined in (2.4a), where $\varphi = \varphi_1$ is a de Bouard-Saut solitary wave. For $c > 0$,*

$$d(c) = \inf\Big\{ S_c(u) :\ u \in Y,\ |\partial_x u|_2 = |\partial_x\varphi_c|_2 \Big\}, \tag{3.3}$$

where $S_c(u) = E(u) + cQ(u)$ as in (2.6).

Proof. Notice that

$$K(u) = S_c(u) - \frac{1}{3}\int_{\mathbf{R}^3} u_x^2,$$

where K is defined in (2.7b). Since $K(\varphi_c) = 0$ by the theory of de Bouard and Saut, so

$$S_c(\varphi_c) = \frac{1}{3}\int_{\mathbf{R}^3} (\partial_x\varphi_c)^2$$

and it follows from (2.7a) that

$$\begin{aligned} &\inf\Big\{ S_c(u) : u \in Y,\ |\partial_x u|_2 = |\partial_x\varphi_c|_2 \Big\} \\ &= \inf\Big\{ K(u) :\ u \in Y,\ |\partial_x u|_2 = |\partial_x\varphi_c|_2 \Big\} + \frac{1}{3}\int_{\mathbf{R}^3} (\partial_x\varphi_c)^2 \\ &= \frac{1}{3}\int_{\mathbf{R}^3} (\partial_x\varphi_c)^2 = S_c(\varphi_c) = d(c). \quad \square \end{aligned} \tag{3.4}$$

Lemma 3.2. *Fix $c = c_0 > 0$ and let $\varphi_0 = \varphi_{c_0}$ be a solitary wave with speed c_0 of (KP-3D). Then for $\alpha > 0$ and any C^2-curve $u : (-\alpha, \alpha) \to Y$ such that $u(0) = \varphi_0$ and $Q(u(\lambda)) = Q(\varphi_0)$ for $\lambda \in (-\alpha, \alpha)$, it follows that*

$$\frac{d^2}{d\lambda^2} E(u(\lambda))\Big|_{\lambda=0} = \big\langle (E''(\varphi_0) + c_0 Q''(\varphi_0)) y_0, y_0 \big\rangle \tag{3.5}$$

where $y_0 = u'(0)$.

Proof. Differentiating E along the curve $u(\lambda)$ yields

$$\frac{d}{d\lambda}E(u(\lambda)) = \langle E'(u(\lambda)), \frac{du}{d\lambda}\rangle, \tag{3.6}$$

and therefore

$$\frac{d^2}{d\lambda^2}E(u(\lambda)) = \langle E''(u(\lambda))\frac{du}{d\lambda}, \frac{du}{d\lambda}\rangle + \langle E'(u), \frac{d^2u}{d\lambda^2}\rangle. \tag{3.7}$$

Since $Q(u(\lambda)) = Q(\varphi_0)$, it must be the case that

$$0 = \frac{d^2}{d\lambda^2}Q(u(\lambda)) = \langle Q''(u)\frac{du}{d\lambda}, \frac{du}{d\lambda}\rangle + \langle Q'(u), \frac{d^2u}{d\lambda^2}\rangle. \tag{3.8}$$

Adding c_0 times (3.8) to (3.7) and evaluating the result at $\lambda = 0$, there obtains the advertised result

$$\frac{d^2}{d\lambda^2}E(u(\lambda))\Big|_{\lambda=0} = \langle (E''(\varphi_0) + c_0Q''(\varphi_0))y_0, y_0\rangle \tag{3.9}$$

because of (3.2). □

View φ_0 as a critical point of the energy E subject to constant values of Q. The next lemma states that if $d''(c_0) < 0$, then φ_0 is a saddle point of the energy E under this constraint.

Lemma 3.3. *Let $c_0 > 0$ be given and $\varphi_0 = \varphi_{c_0}$ a solitary wave with speed c_0. Let φ_c connote the branch of solitary waves passing through φ_0 defined in (2.4a) and let*

$$\chi_c(x,y,z) = \varphi_c\Big(\frac{x}{\sigma(c)}, \frac{y}{\sigma(c)^2}, \frac{z}{\sigma(c)^2}\Big), \tag{3.10a}$$

where

$$\sigma(c)^5 = \frac{Q(\varphi_0)}{Q(\varphi_c)}. \tag{3.10b}$$

Assume $d''(c_0) \neq 0$. Then it transpires that

a) $\frac{d^2}{dc^2}E(\chi_c)\Big|_{c=c_0} \le d''(c_0)$.

Moreover, if $d''(c_0) < 0$, then

b) $E(\chi_c) < E(\varphi_0)$ *for c near c_0, $c \neq c_0$.*

Proof. A direct calculation reveals that

$$E(\chi_c) + cQ(\chi_c) = \frac{1}{2}\int_{\mathbf{R}^3}(D_x^{-1}\partial_y\chi_c)^2 + (D_x^{-1}\partial_z\chi_c)^2 + (\partial_x\chi_c)^2$$
$$- \frac{1}{(p+1)(p+2)}\int_{\mathbf{R}^3}\chi_c^{p+2} + \frac{c}{2}\int_{\mathbf{R}^3}\chi_c^2$$

$$
\begin{aligned}
&= \frac{1}{2}\sigma(c)^3 \int_{\mathbf{R}^3} (D_x^{-1}\partial_y\varphi_c)^2 + (D_x^{-1}\partial_z\varphi_c)^2 + (\partial_x\varphi_c)^2 \qquad (3.11)\\
&\quad - \frac{1}{(p+1)(p+2)}\sigma(c)^5 \int_{\mathbf{R}^3} \varphi_c^{p+2} + \frac{c}{2}\sigma(c)^5 \int_{\mathbf{R}^3} \varphi_c^2\\
&= \frac{1}{2}\sigma(c)^3(1-\sigma(c)^2)\int_{\mathbf{R}^3} \left((D_x^{-1}\partial_y\varphi_c)^2 + (D_x^{-1}\partial_z\varphi_c)^2 + (\partial_x\varphi_c)^2\right) + \sigma(c)^5 d(c).
\end{aligned}
$$

Differentiating the last equation with respect to c and evaluating at $c = c_0$ leads to the formula

$$
\begin{aligned}
Q(\varphi_0) = &\frac{3}{2}\sigma'(c_0)\int_{\mathbf{R}^3} \left((D_x^{-1}\partial_y\varphi_0)^2 + (D_x^{-1}\partial_z\varphi_0)^2 + (\partial_x\varphi_0)^2\right)\\
&- \frac{5}{(p+1)(p+2)}\sigma'(c_0)\int_{\mathbf{R}^3} \varphi_0^{p+2} + \frac{5c_0}{2}\sigma'(c_0)\int_{\mathbf{R}^3} \varphi_0^2 + d'(c_0).
\end{aligned}
$$

Since $d'(c_0) = Q(\varphi_0)$, it follows that

$$
\begin{aligned}
&\frac{3\sigma'(c_0)}{2}\int_{\mathbf{R}^3} \left((D_x^{-1}\partial_y\varphi_0)^2 + (D_x^{-1}\partial_z\varphi_0)^2 + (\partial_x\varphi_0)^2\right)\\
&- \frac{5\sigma'(c_0)}{(p+1)(p+2)}\int_{\mathbf{R}^3} \varphi_0^{p+2} + \frac{5c_0\sigma'(c_0)}{2}\int_{\mathbf{R}^3} \varphi_0^2 = 0,
\end{aligned}
$$

or, what is the same,

$$
5\sigma'(c_0)\left(d(c_0) - \frac{1}{5}\int_{\mathbf{R}^3} (D_x^{-1}\partial_y\varphi_0)^2 + (D_x^{-1}\partial_z\varphi_0)^2 + (\partial_x\varphi_0)^2\right) = 0. \qquad (3.12)
$$

On the other hand, it is easy to see that

$$
\sigma'(c_0) = -\frac{1}{5}\frac{d''(c_0)}{d'(c_0)} > 0, \qquad (3.13)
$$

and hence that

$$
d(c_0) = \frac{1}{5}\int_{\mathbf{R}^3} (D_x^{-1}\partial_y\varphi_0)^2 + (D_x^{-1}\partial_z\varphi_0)^2 + (\partial_x\varphi_0)^2.
$$

As c_0 was arbitrary in this computation, we see that for any $c > 0$,

$$
d(c) = \frac{1}{5}\int_{\mathbf{R}^3} (D_x^{-1}\partial_y\varphi_c)^2 + (D_x^{-1}\partial_z\varphi_c)^2 + (\partial_x\varphi_c)^2. \qquad (3.14)
$$

It now follows from formula (3.11) that

$$
\begin{aligned}
E(\chi_c) + cQ(\chi_c) &= \frac{5}{2}\sigma(c)^3(1-\sigma(c)^2)d(c) + \sigma(c)^5 d(c)\\
&= \frac{1}{2}\left(5\sigma(c)^3 - 3\sigma(c)^5\right)d(c). \qquad (3.15)
\end{aligned}
$$

Differentiating (3.15) with respect to c yields

$$\frac{d}{dc}(E(\chi_c)+cQ(\chi_c)) = \frac{1}{2}(5\sigma(c)^3-3\sigma(c)^5)d'(c) + \frac{15}{2}(\sigma(c)^2-\sigma(c)^4)\sigma'(c)d(c); \tag{3.16}$$

differentiating a second time gives

$$\frac{d^2}{dc^2}(E(\chi_c)+cQ(\chi_c)) = 15(\sigma(c)^2-\sigma(c)^4)\sigma'(c)d'(c) + \frac{1}{2}(5\sigma(c)^3-3\sigma(c)^5)d''(c) + \Big(15(\sigma(c)-2\sigma(c)^3)\sigma'(c)^2+\frac{15}{2}(\sigma(c)^2-\sigma(c)^4)\sigma''(c)\Big)d(c). \tag{3.17}$$

At $c=c_0$, where $\sigma(c)=1$, it is found that

$$\frac{d^2}{dc^2}\Big(E(\chi_c)+cQ(\chi_c)\Big)\Big|_{c=c_0} = d''(c_0)-15\sigma'(c_0)^2d(c_0)\le d''(c_0).$$

This proves (a) since $Q(\chi_c)\equiv Q(\varphi_0)$. Part (b) follows from Part (a) because, as in (3.6),

$$\frac{d}{dc}E(\chi_c)\Big|_{c=c_0} = \frac{d}{dc}\Big(E(\chi_c)+c_0Q(\chi_c)\Big)\Big|_{c=c_0} = \Big\langle E'(\varphi_0)+c_0Q'(\varphi_0),\ \frac{d\chi_c}{dc}\Big|_{c=c_0}\Big\rangle = 0 \tag{3.18}$$

on account of (3.2) and (2.3). □

Lemma 3.4. *With the notation in the last lemma, assume again that* $d''(c_0)\neq 0$. *It follows that*

a) $\langle (E''(\varphi_0)+c_0Q''(\varphi_0))y_0,\, y_0\rangle \le d''(c_0)$,

b) $\langle Q'(\varphi_0), y_0\rangle = \int_{\mathbf{R}^3}\varphi_0 y_0 = 0$ *and*

c) if $d''(c_0)<0$, $\int_{\mathbf{R}^3}\partial_x\varphi_0\partial_x y_0 > 0$ *where* $y_0 = \frac{\partial\chi_c}{\partial c}|_{c=c_0}$.

Proof. Let $u(\lambda)=\chi_{c_0+\lambda}$ as in (3.10a). Applying Lemma 3.2 and Lemma 3.3 then yields Part (a) directly. Differentiating $Q(\chi_c)$ with respect to c and evaluating at $c=c_0$ gives (b). To establish (c), argue as follows. Differentiate the formula

$$|\partial_x\chi_c|_2^2 = \sigma(c)^3|\partial_x\varphi_c|_2^2$$

with respect to c and evaluate at $c=c_0$ to obtain

$$2\int_{\mathbf{R}^3}\partial_x\varphi_0\partial_x y_0 = 3\sigma(c_0)^2\sigma'(c_0)\int_{\mathbf{R}^3}|\partial_x\varphi_0|^2 + \sigma(c_0)^3\,\frac{d}{dc}\int_{\mathbf{R}^3}|\partial_x\varphi_c|^2\Big|_{c=c_0}$$

$$= 3\sigma'(c_0)|\partial_x\varphi_0|_2^2 + 3d'(c_0). \tag{3.19}$$

Part (c) follows from (3.19), (3.13) and the fact that $d'(c_0) = Q(\varphi_0) > 0$. $\square$

An important role will be played by the tubular neighborhoods of the orbit of a solitary wave. For $\epsilon > 0$, these are defined to be

$$U_\epsilon = \Big\{ u \in Y : \inf_{\vec{r}\in\mathbf{R}^3} \|u - \varphi_c(\cdot - \vec{r})\|_Y < \epsilon \Big\}.$$

Lemma 3.5. *Fix $c > 0$ and a non-trivial de Bouard-Saut solitary wave solution φ_c of* (KP–3D). *There is an $\epsilon > 0$ and a C^1−map $\alpha : U_\epsilon \to \mathbf{R}^3$ such that for all $u \in U_\epsilon$ and $r \in \mathbf{R}^3$,*

i) $\langle u(\cdot + \alpha(u)), \partial_{x_i}\varphi_c\rangle = 0 \qquad i = 1, 2, 3,$ and

ii) $\alpha(u(\cdot + r)) = \alpha(u) - r$.

iii) Moreover, if u is cylindrically symmetric, i.e., if $u(x, y, z) = u(x, |x'|)$ with $x' = (y, z)$ and $|x'| = \sqrt{y^2 + z^2}$, then $\alpha(u) = (\alpha_0(u), 0, 0)$, where

$$\alpha_0'(u) = \frac{\partial_x\varphi_c(\cdot - \alpha(u))}{\langle u, \partial_x^2\varphi_c(\cdot - \alpha(u))\rangle}. \tag{3.20}$$

Proof. Define $F : Y \times \mathbf{R}^3 \to \mathbf{R}^3$ by

$$F(u, \alpha) = \int_{\mathbf{R}^3} u(\vec{x} + \alpha)\nabla\varphi_c(\vec{x})d\vec{x} \tag{3.21}$$

with $\vec{x} = (x, y, z)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Since φ_c is cylindrically symmetric in (y, z) (see Theorem 2.1), it follows from the decay properties of φ_c together with Fubini's Theorem that

$$\langle\partial_x\varphi_c, \partial_y\varphi_c\rangle = \langle\partial_x\varphi_c, \partial_z\varphi_c\rangle = \langle\partial_y\varphi_c, \partial_z\varphi_c\rangle = 0. \tag{3.22}$$

It follows from (3.22) that the Jacobian matrix of F at $\alpha = 0$ and $u = \varphi_c$ is diagonal with positive diagonal entries, so invertible. Therefore, by the Implicit-Function Theorem, there is a unique C^1−functional $\alpha(u)$ satisfying (i) in a neighborhood of φ_c. By translation invariance, $\alpha(u)$ can be uniquely extended to U_ϵ for ϵ small enough. By (i) $u(\cdot + \alpha(u)) = u(\cdot + r + (\alpha(u) - r))$ is orthogonal to $\nabla\varphi_c$. Hence by the uniqueness of $\alpha(u)$, (ii) holds. For (iii), argue as follows. Again by the Implicit–Function Theorem, making use of the fact that

$$\int_{\mathbf{R}^3} (\partial_x\varphi_c(x, x'))^2 dx dx' \neq 0,$$

it is adduced that there is a mapping α_0 defined for u in a neighborhood of φ_c, which is such that

$$\int_{\mathbf{R}^3} u(x+\alpha_0(u),x')\partial_x\varphi_c(x,x')dxdx' = 0. \tag{3.23}$$

On the other hand, if u happens to be cylindrically symmetric, then

$$\begin{aligned}&\int_{\mathbf{R}^3} u(x+\alpha_0(u),x')\partial_y\varphi_c(x,x')dxdx'\\ &=\int_{\mathbf{R}^3} u(x+\alpha_0(u),x')\partial_z\varphi_c(x,x')dxdx' = 0\end{aligned} \tag{3.24}$$

since φ_c is also cylindrically symmetric. Hence, by the uniqueness provided by the Implicit-Function Theorem, it is inferred that $\alpha(u) = (\alpha_0(u),0,0)$ for u in U_ϵ provided ϵ is small enough. The proof of Part (iii) is completed simply by forming the Fréchet derivative of the relation

$$\int_{\mathbf{R}^3} u(x,x')\partial_x\varphi_c(x-\alpha_0(u),x')dxdx' = 0 \tag{3.25}$$

in Y with respect to u. □

Define another mapping B by

$$\begin{aligned}B(u) &= y_0(\cdot-\beta(t),\cdot,\cdot) - \langle y_0(\cdot-\beta(t),\cdot,\cdot),\, u\rangle\, \partial_x\alpha_0'(u)\\ &= y_0(\cdot-\beta(t),\cdot,\cdot) - \frac{\langle y_0(\cdot-\beta(t),\cdot,\cdot),\, u\rangle}{\langle u, \partial_x^2\varphi_0(\cdot-\beta(t),\cdot,\cdot)\rangle}\partial_x^2\varphi_0(\cdot-\beta,\cdot,\cdot)\end{aligned} \tag{3.26}$$

for $u \in U_\epsilon^s$, where $U_\epsilon^s = \{u \in U_\epsilon : u \text{ is cylindrically symmetric}\}$, $y_0 = \frac{d\chi_c}{dc}|_{c=c_0}, \varphi_0 = \varphi_{c_0}$ is a de Bouard-Saut solitary wave and $\beta(t) = \alpha_0(u(\cdot,t))$. The important properties of B are expressed in the following auxiliary result.

Lemma 3.6. *The mapping B is a C^1–function from U_ϵ^s to Y. Moreover, B commutes with translations, $B(\varphi_0) = y_0$ and $\langle B(u),u\rangle = 0$ for all $u \in U_\epsilon^s$.*

Proof. It is shown first that $y_0 \in Y$. Indeed, we may write $\varphi_c(x,y,z) = c^{\frac{1}{p}}\varphi(\sqrt{c}x,cy,cz)$ where $\varphi = \varphi_1$ is independent of c and satisfies equation (2.3) with $c = 1$. A calculation shows that

$$\begin{aligned}y_0 = \frac{d\chi_c}{dc}\Big|_{c=c_0} &= \frac{d}{dc}\Big(c^{\frac{1}{p}}\varphi(\frac{\sqrt{c}}{\sigma(c)}x, \frac{c}{\sigma(c)^2}y, \frac{c}{\sigma(c)^2}z)\Big)\Big|_{c=c_0}\\ &= \frac{1}{p}c_0^{\frac{1-p}{p}}\varphi\Big(\frac{\sqrt{c_0}}{\sigma(c_0)}x, \frac{c_0}{\sigma(c_0)^2}y, \frac{c_0}{\sigma(c_0)^2}z\Big)\\ &+(c_0^{\frac{1}{p}}x\partial_x\varphi)\Big(\frac{\sqrt{c}}{\sigma(c)}\Big)'\Big|_{c=c_0} + (c_0^{\frac{1}{p}}y\partial_y\varphi + c_0^{\frac{1}{p}}z\partial_z\varphi)\Big(\frac{c}{\sigma(c)^2}\Big)'\Big|_{c=c_0}\end{aligned}$$

$$= c_1\varphi + c_2x\partial_x\varphi + c_3y\partial_y\varphi + c_4z\partial_z\varphi \tag{3.27}$$

where c_1, c_2, c_3 and c_4 are constants depending only on $c_0, p, Q(\varphi_0)$ and $Q'(\varphi_0)$, and, in the last line, $\varphi, \partial_x\varphi, \partial_y\varphi, \partial_z\varphi$ are all evaluated at the point

$$\Big(\frac{\sqrt{c_0}}{\sigma(c_0)}x, \frac{c_0}{\sigma(c_0)}y, \frac{c_0}{\sigma(c_0)}z\Big).$$

To show $y_0 \in Y$, it thus suffices by (3.27) to show that $y\partial_x\partial_y\varphi$, $yD_x^{-1}\partial_y^2\varphi$, $yD_x^{-1}\partial_y\partial_z\varphi$, $z\partial_x\partial_z\varphi$, $zD_x^{-1}\partial_z^2\varphi$ and $zD_x^{-1}\partial_y\partial_z\varphi$ lie in $L_2(\mathbf{R}^3)$. Rewrite equation (2.3) in the form

$$-D_x^{-1}\varphi_{yy} - D_x^{-1}\varphi_{zz} - \varphi_x + \varphi_{xxx} + \frac{1}{p+1}(\varphi^{p+1})_x = 0 \tag{3.28}$$

with $c = 1$. This is possible since $\varphi \in Y$, being a de Bouard–Saut solitary wave. Multiplying equation (3.28) by $-y^2D_x^{-1}\varphi_{yy}$ and integrating by parts leads to the relation

$$\begin{aligned}|yD_x^{-1}\varphi_{yy}|_2^2 + |yD_x^{-1}\varphi_{yz}|_2^2 + \langle 2yD_x^{-1}\varphi_{yz},\, D_x^{-1}\varphi_z\rangle + |y\varphi_y|_2^2 + 2\,\langle y\varphi, \varphi_y\rangle \\ + |y\varphi_{xy}|_2^2 + 2\,\langle y\varphi_x, \varphi_{xy}\rangle + \frac{1}{p+1}\,\langle y^2\varphi^{p+1}, \varphi_{yy}\rangle = 0.\end{aligned} \tag{3.29}$$

This in turn implies that

$$\begin{aligned}&|yD_x^{-1}\varphi_{yy}|_2^2 + |yD_x^{-1}\varphi_{yz}|_2^2 + |y\varphi_{xy}|_2^2 + |y\varphi_y|_2^2 \\ &\le c_1|D_x^{-1}\varphi_z|_2^2 + c_2|\varphi_y|_2^2 + |y\varphi_x|_2^2 + |\varphi_{xy}|_2^2 + |\varphi|_2^2 + c_3|\varphi_{yy}|_\infty|\varphi|_\infty^{p-1}|y\varphi|_2^2 < \infty.\end{aligned} \tag{3.30}$$

Thus, $yD_x^{-1}\varphi_{yy}$, $yD_x^{-1}\varphi_{yz}$ and $y\varphi_{xy} \in L_2(\mathbf{R}^3)$. One shows similarly that $z\varphi_{xz}$, $zD_x^{-1}\varphi_{zz}$, $zD_x^{-1}\varphi_{yz} \in L_2(\mathbf{R}^3)$. It is thus concluded that $y_0 \in Y$. Since $y_0 \in Y$ and $\partial_x^2\varphi_0 \in Y$, it is clear that $B(u) \in Y$. To show that B is C^1, it is sufficient to show $\partial_x y_0 \in Y$ which follows if $x\varphi_{xxx}$, $x\varphi_{xy}$, $x\varphi_{xz} \in L_2(\mathbf{R}^3)$, $y\varphi_{xxy}$, $y\varphi_{yy}$, $y\varphi_{yz} \in L_2(\mathbf{R}^3)$ and $z\varphi_{xxz}$, $z\varphi_{zz}$, $z\varphi_{yz} \in L_2(\mathbf{R}^3)$. Multiplying equation (3.28) by $\partial_x(x^2\varphi_{xx})$ and integrating the result by parts leads to

$$\begin{aligned}&|x\varphi_{xxx}|_2^2 + \langle 2x\varphi_{xxx}, \varphi_{xx}\rangle + |x\varphi_{xx}|_2^2 + \int_{\mathbf{R}^3} x^2\varphi_{xx}\varphi_{yy} \\ &+ \int_{\mathbf{R}^3} x^2\varphi_{xx}\varphi_{zz} - \frac{1}{p+1}\int_{\mathbf{R}^3} x^2\varphi_{xx}(\varphi^{p+1})_{xx} = 0.\end{aligned} \tag{3.31}$$

It follows from (3.31) that

$$|x\varphi_{xxx}|_2^2 + |x\varphi_{xx}|_2^2 + |x\varphi_{xy}|_2^2 + |x\varphi_{xz}|_2^2 \le 2\int_{\mathbf{R}^3} |x\varphi_{xxx}\varphi_{xx}| \tag{3.32}$$

$$+2\int_{\mathbf{R}^3}|x\varphi_{xy}\varphi_y|+2\int_{\mathbf{R}^3}|x\varphi_{xz}\varphi_z|+2\int_{\mathbf{R}^3}|x\varphi^p\varphi_{xx}\varphi_x|+\int_{\mathbf{R}^3}|x^2\varphi^p\varphi_{xxx}\varphi_x|$$

and, therefore,

$$|x\varphi_{xxx}|_2^2+|x\varphi_{xx}|_2^2+|x\varphi_{xy}|_2^2+|x\varphi_{xz}|_2^2\le C, \tag{3.33}$$

where C is a constant depending only on $\|\varphi\|_2$ and $|x\varphi_x|_2$. To prove $y\varphi_{xxy}$, $y\varphi_{yy}$ and $y\varphi_{yz}$ lie in $L_2(\mathbf{R}^3)$, multiply equation (3.28) by $\partial_x(y^2\varphi_{yy})$. Upon integrating the result over $\mathbf{R}^3$ and after integrating by parts, one obtains

$$\begin{aligned}&|y\varphi_{yy}|_2^2+|y\varphi_{xy}|_2^2+|y\varphi_{yz}|_2^2+2\int_{\mathbf{R}^3}y\varphi_{xy}\varphi_x+2\int_{\mathbf{R}^3}y\varphi_{yz}\varphi_z\\&+|y\varphi_{xxy}|_2^2+2\int_{\mathbf{R}^3}y\varphi_{xxy}\varphi_{xx}-\frac{1}{p+1}\int_{\mathbf{R}^3}y^2\varphi^{p+1}\varphi_{xxyy}=0.\end{aligned} \tag{3.34}$$

This implies that

$$|y\varphi_{yy}|_2^2+|y\varphi_{xy}|_2^2+|y\varphi_{yz}|_2^2+|y\varphi_{xxy}|_2^2\le C, \tag{3.35}$$

where C depends only on $|y\varphi|_2$ and $\|\varphi\|_2$. Hence, $y\varphi_{yy}, y\varphi_{yz}$ and $y\varphi_{xxy}\in L_2(\mathbf{R}^3)$. Similarly, one shows $z\varphi_{xxz}, z\varphi_{zz}, z\varphi_{zz}$ and $z\varphi_{yz}\in L_2(\mathbf{R}^3)$. By Lemma 3.4 (b), $\langle\varphi_0, y_0\rangle=0$, hence,

$$B(\varphi_0)=y_0-\langle y_0,\varphi_0\rangle\,\partial_x\alpha_0'(u)=y_0$$

and

$$\langle B(u),u\rangle=\langle y_0(\cdot-\beta(t),\cdot,\cdot),u\rangle-\langle y_0(\cdot-\beta(t),\cdot,\cdot),u\rangle\,\langle\partial_x\alpha_0'(u),u\rangle\,. \tag{3.36}$$

Since

$$\langle\partial_x\alpha_0'(u),u\rangle=\frac{-\langle\partial_x u,\partial_x\varphi_0(\cdot-\alpha)\rangle}{-\langle\partial_x u,\partial_x\varphi_0(\cdot-\alpha)\rangle}=1 \tag{3.37}$$

for $u\in U_\epsilon^s$, it thus transpires that

$$\langle B(u),u\rangle=0. \tag{3.38}$$

Finally, note that for $u\in U_\epsilon^s$ and $\alpha(u)=(\alpha_0(u),0,0)$ as in Lemma 3.5, we have

$$\begin{aligned}B(u(\cdot+r))&=y_0(\cdot-\alpha(u)+r)-\frac{\langle u(\cdot+r),y_0(\cdot-\alpha(u)+r)\rangle}{\langle u(\cdot+r),\partial_x^2\varphi_0(\cdot-\alpha+r)\rangle}\partial_x^2\varphi_0(\cdot-\alpha+r)\\&=y_0(\cdot-\alpha(u)+r)-\frac{\langle u,y_0(\cdot-\alpha)\rangle}{\langle u,\partial_x^2\varphi_0(\cdot-\alpha)\rangle}\partial_x^2\varphi_0(\cdot-\alpha+r)=(Bu)(\cdot+r).\end{aligned} \tag{3.39}$$

This completes the proof of Lemma 3.6. □

Remark. It follows from the above ruminations that $B'(u)$ is bounded on bounded subsets (see (3.33) and (3.35)). Hence, B is locally Lipschitz continuous.

Lemma 3.7. *Let B be the operator defined in* (3.26) *relative to a de Bouard-Saut solitary wave $\varphi_0 = \varphi_{c_0}$. Let $\epsilon > 0$ be such that B is a C^1–mapping of U_ϵ^s into Y. Corresponding to any $u \in U_\epsilon^s$ there exists a solution $u_\lambda = R(\lambda, u)$ of the initial-value problem*

$$\frac{du_\lambda}{d\lambda} = B(u_\lambda), \qquad u_0 = u,$$

and a positive number $\lambda_0(u)$ for which
i) R is C^2 as a function of λ for $|\lambda| < \lambda_0(u)$,
ii) for each fixed λ, R commutes with translations,
iii) $Q(R(\lambda, u))$ is independent of λ, and
iv) $\frac{\partial}{\partial\lambda}R(\lambda, \varphi_0)\Big|_{\lambda=0} = y_0$.
Moreover, $\lambda_0(u)$ is bounded below on bounded subsets.

Proof. Let $u \in U_\epsilon^s$ and consider the initial-value problem

$$\frac{du_\lambda}{d\lambda} = B(u_\lambda), \quad u_\lambda\big|_{\lambda=0} = u_0 = u. \tag{3.40}$$

Here, $\epsilon > 0$ is chosen so that $B: U_\epsilon^s \to Y$ is a C^1–map. Since B is C^1, there exists $\lambda_0(u) > 0$ for which (3.40) can be solved at least in the internal $[-\lambda_0(u), \lambda_0(u)]$ and u_λ is a C^2–function of λ there. The fact that B is locally Lipschitz allows one to infer a non-zero value of λ_0 corresponding to any bounded set in U_ϵ^s that applies uniformly there. That is, for any bounded set S of data, there is a $\lambda_0 > 0$ such that the differential equation with initial value drawn from S can be solved in $[-\lambda_0, \lambda_0]$. Conclusions (i), (ii) and (iv) are obvious from the properties of B delineated in Lemma 3.6. For (iii), simply note that

$$\frac{dQ(u_\lambda)}{d\lambda} = \langle u_\lambda, B(u_\lambda)\rangle = 0. \tag{3.41}$$

Lemma 3.8. *Fix $c_0 > 0$ and suppose $d''(c_0) < 0$. Let $\varphi_0 = \varphi_{c_0}$ be a de Bouard-Saut solitary-wave solution of* (KP-3D). *Then there is an $\epsilon > 0$ such that for any $u \in U_\epsilon^s$ which is not a translate of φ_0, but which satisfies $Q(u) = Q(\varphi_0)$, there is a $\lambda = \lambda(u) \in (-\epsilon, \epsilon)$ such that*

$$E(\varphi_0) < E(u) + \lambda \left\langle E'(u), B(u)\right\rangle. \tag{3.42}$$

Proof. Let u_λ be the curve defined in Lemma 3.7. Straightforward calculations show

$$\frac{\partial}{\partial_\lambda}E(u_\lambda)\Big|_{\lambda=0} = \Big\langle E'(u), \frac{du_\lambda}{d\lambda}\Big|_{\lambda=0}\Big\rangle = \Big\langle E'(u), B(u)\Big\rangle, \tag{3.43}$$

$$\frac{\partial^2}{\partial\lambda^2}E(u_\lambda)\Big|_{\lambda=0} = \Big\langle E''(u)\frac{du_\lambda}{d\lambda}, \frac{du_\lambda}{d\lambda}\Big\rangle\Big|_{\lambda=0} + \Big\langle E'(u), \frac{d^2u_\lambda}{d\lambda^2}\Big\rangle\Big|_{\lambda=0} \tag{3.44}$$

and

$$0 = \frac{\partial^2}{\partial\lambda^2}Q(u_\lambda)\Big|_{\lambda=0} = \Big\langle Q''(u)\frac{du_\lambda}{d\lambda}, \frac{du_\lambda}{d\lambda}\Big\rangle\Big|_{\lambda=0} + \Big\langle Q'(u), \frac{d^2u_\lambda}{d\lambda^2}\Big\rangle\Big|_{\lambda=0}. \tag{3.45}$$

Combining (3.44) with (3.45) yields

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}E(u_\lambda)\Big|_{\lambda=0} &= \Big\langle (E''(u) + c_0Q''(u))\frac{du_\lambda}{d\lambda}, \frac{du_\lambda}{d\lambda}\Big\rangle\Big|_{\lambda=0} \\ &\quad + \Big\langle E'(u) + cQ'(u), \frac{d^2u_\lambda}{d\lambda^2}\Big\rangle\Big|_{\lambda=0}.\end{aligned} \tag{3.46}$$

Apply this calculation to the curve u_λ starting at $u = \varphi_0$ (see (3.41)) to obtain

$$\frac{\partial^2}{\partial\lambda^2}E(\varphi_0) = \langle (E''(\varphi_0) + c_0Q''(\varphi_0))y_0, y_0\rangle \tag{3.47}$$

since

$$E'(\varphi_0) + c_0Q'(\varphi_0) = 0.$$

By Lemma 3.4 (a), the quantity on the right-hand side of (3.47) is negative. If u is near φ_0, $B(u)$ is near $B(\varphi_0)$ and hence the solutions of the differential equation in Lemma 3.7 starting at φ_0 and u are close, along with their first two derivatives. Hence, for u near φ_0 in U_ϵ^s,

$$\frac{\partial^2}{\partial\lambda^2}E(u_\lambda)\Big|_{\lambda=0} < 0 \tag{3.48}$$

when $u_\lambda|_{\lambda=0} = u$. The Taylor expansion thus implies

$$E(u_\lambda) < E(u) + \lambda\langle E'(u), B(u)\rangle \tag{3.49}$$

for λ near 0 and u near φ_0. On the other hand, if we consider again the curve u_λ starting at φ_0, then by Lemma 3.4 (c), since $d''(c_0) < 0$,

$$\frac{d}{d\lambda}Q(\partial_x u_\lambda)\Big|_{\lambda=0} = 2\int_{\mathbf{R}^3} \partial_x\varphi_0\partial_x y_0 > 0. \tag{3.50}$$

Consider the function G defined as follows. For u near φ_0 and λ near zero, solve the differential equation of Lemma 3.7 with initial value u. As remarked previously, if attention is restricted to a bounded neighborhood S of φ_0, say, then $u_\lambda = R(\lambda, u)$ exists on $[-\lambda_0, \lambda_0] \times S$ for some positive value λ_0. In particular, we may take S to be a small enough ball around φ_0 and λ_0 small enough that $R(\lambda, u)$ exists and (3.50) holds throughout. Then, for $(\lambda, u) \in [-\lambda_0, \lambda_0] \times S$, set $G(\lambda, u) = Q(\partial_x u_\lambda)$. The mapping G is plainly C^1 and the transversality condition in (3.50) allows us to apply the Implicit-Function Theorem. As $G(0, \varphi_0) = Q(\partial_x \varphi_0)$, it is concluded there is a neighborhood N of φ_0 and a C^1–mapping $\lambda : N \to \mathbf{R}^+$ such that

$$Q(\partial_x \varphi_0) = G(0, \varphi_0) = G(\lambda(u), u) = Q(\partial_x u_{\lambda(u)}). \tag{3.51}$$

Because of Lemma 3.1, there obtains

$$S_{c_0}(u_\lambda) = E(u_\lambda) + c_0 Q(u_\lambda) \geq E(\varphi_0) + c_0 Q(\varphi_0) = d(c_0). \tag{3.52}$$

This implies that $E(u_\lambda) \geq E(\varphi_0)$ and hence it is adduced that

$$E(\varphi_0) < E(u) + \lambda(u) \langle E'(u), B(u) \rangle. \qquad \square \tag{3.53}$$

Lemma 3.9. *Let $c_0 > 0$ be given along with a de Bouard-Saut solitary wave $\varphi_0 = \varphi_{c_0}$. Let φ_c given as in (2.4a) be a branch of solitary waves passing through φ_0 at c_0 and let χ_c be as in (3.10a). Assume $d''(c_0) < 0$. Then the curve χ_c satisfies*

i) $E(\chi_c) < E(\varphi_0)$ for $c \neq c_0$ and c near c_0,
ii) $Q(\chi_c) = Q(\varphi_0)$ and
iii) $\langle E'(\chi_c), B(\chi_c) \rangle$ changes sign as c passes through c_0.

Proof. Part (i) was noted in Lemma 3.3, while (ii) is obvious from the definition of χ_c. Applying Lemma 3.8 with $u = \chi_c$ and c near c_0, there obtains the inequality

$$\lambda(\chi_c) \langle E'(\chi_c), B(\chi_c) \rangle > 0 \qquad \text{for } c \neq c_0. \tag{3.54}$$

Thus, it suffices to show that $\lambda(\chi_c)$ changes sign as c passes through c_0. Let

$$w(c) = R(\lambda(\chi_c), \chi_c)$$

and remember that because of (3.51),

$$|\partial_x w(c)|_2^2 = |\partial_x \varphi_0|_2^2.$$

Differentiate the last relation with respect to c to derive the relation

$$2 \int_{\mathbf{R}^3} \partial_x u_\lambda \partial_x \Big(\frac{\partial R}{\partial \lambda} \frac{\partial \lambda}{\partial c} + \frac{\partial R}{\partial \chi_c} \frac{\partial \chi_c}{\partial c} \Big) = 0. \tag{3.55}$$

Evaluate this at $c = c_0$, $u_0 = \chi_{c_0}$ and $\lambda = 0$ to reach the conclusion

$$\int_{\mathbf{R}^3} \partial_x \varphi_0 \Big(\partial_x y_0 \frac{d\lambda}{dc}\Big|_{c=c_0} + \partial_x y_0 \Big) = 0. \tag{3.56}$$

Since

$$\int \partial_x\varphi_0\partial_x y_0 > 0,$$

it must be the case that

$$\frac{d\lambda}{dc}\Big|_{c=c_0} = -1 \neq 0 \tag{3.57}$$

almost everywhere. As $\lambda(c_0) = 0$, it is ascertained that λ changes sign as c passes through c_0, thereby concluding the proof of Lemma 3.9. □

Lemma 3.10. *Let $\varphi_0 = \varphi_{c_0}$ be a de Bouard-Saut solitary wave with speed $c_0 > 0$. Let φ_c be the solitary-wave branch defined in (2.4a). Let*

$$v = D_x^{-1}y_0,$$

where $y_0 = \frac{\partial\chi_c}{\partial c}|_{c=c_0}$, $\chi_c(x,y,z) = \varphi_c\big(\frac{x}{\sigma(c)}, \frac{y}{\sigma(c)^2}, \frac{z}{\sigma(c)^2}\big)$ and $\sigma(c) = \frac{Q(\varphi_0)}{Q(\varphi_c)}$ as before. Then $v \in L_2(\mathbf{R}^3)$.

Proof. Write $\varphi_c = c\varphi(\sqrt{c}x, cy, cz)$ where $\varphi = \varphi_1$ is independent of c and satisfies equation (2.3) with $c = 1$. As before, this amounts to taking $c_0 = 1$ or to rescaling φ_0. A calculation shows that

$$v = D_x^{-1}y_0 == C_1D_x^{-1}\varphi + C_2x\varphi + C_3yD_x^{-1}\varphi_y + C_4zD_x^{-1}\varphi_z, \tag{3.58}$$

where C_1, C_2, C_3 and C_4 are constants. Thus $v \in L_2(\mathbf{R}^3)$, if $x\varphi$, $D_x^{-1}\varphi$, $yD_x^{-1}\varphi_y$ and $zD_x^{-1}\varphi_z$ all lie in $L_2(\mathbf{R}^3)$. In fact, by Theorem 2.1, we have $x\varphi \in L_2(\mathbf{R}^3)$. To be convinced that $D_x^{-1}\varphi \in L_2(\mathbf{R}^3)$, consider the following argument. Since φ satisfies (3.28), its Fourier transform satisfies the equation

$$\xi_1^{-1}\hat{\varphi}(\xi) = \frac{\xi_1}{|\xi|^2 + |\xi_1|^4}\Big(\frac{1}{p+1}\widehat{\varphi^{p+1}}\Big) \tag{3.59}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$. Thus it transpires that

$$D_x^{-1}\varphi = h \star \Big(\frac{1}{p+1}\varphi^{p+1}\Big), \tag{3.60}$$

where $\hat{h}(\xi) = \frac{\xi_1}{|\xi|^2+|\xi_1|^4}$. Applying Young's inequality and Sobolev embedding gives

$$|D_x^{-1}\varphi|_2 \le |h|_2\Big|\Big(\frac{1}{p+1}\Big)\varphi^{p+1}\Big|_1 \le \frac{1}{p+1}|\hat{h}|_2\|\varphi\|_2^{p+1}. \tag{3.61}$$

It follows from (3.61) that $D_x^{-1}\varphi \in L_2(\mathbf{R}^3)$ because of the following estimate of h :

$$
\begin{aligned}
|\hat{h}|_2^2 &= \int_{\mathbf{R}^3} \frac{\xi_1^2}{(|\xi_2|^2+|\xi_3|^3+|\xi_1|^2(1+|\xi_1|^2))^2} d\xi_1 d\xi_2 d\xi_3 \\
&= \int_{\mathbf{R}} \frac{\xi_1^2}{|\xi_1|^4(1+|\xi_1|^2)^2} \Big(\int_{\mathbf{R}^2} \frac{d\xi_2 d\xi_3}{\left(1+\frac{\xi_2^2+\xi_3^2}{\xi_1^2(1+\xi_1^2)}\right)^2} \Big) d\xi_1 \\
&= \int_{\mathbf{R}} \frac{\xi_1^2 \cdot \xi_1^2(1+|\xi_1|^2)}{|\xi_1|^4(1+|\xi_1|^2)^2} d\xi_1 \int_{\mathbf{R}^2} \frac{d\eta_1 d\eta_2}{(1+\eta_1^2+\eta_2^2)^2} \\
&= \int_{\mathbf{R}} \frac{d\xi_1}{(1+\xi_1^2)} \int_{\mathbf{R}^2} \frac{d\eta_1 d\eta_2}{(1+\eta_1^2+\eta_2^2)^2} < \infty.
\end{aligned}
\tag{3.62}
$$

Next it is established that $yD_x^{-1}\varphi_y$ and $zD_x^{-1}\varphi_z \in L_2(\mathbf{R}^3)$. Multiplying equation (3.28) by $r_1^2\varphi D_x^{-1}$ where $r_1^2 = y^2+z^2$ and integrating over $\mathbf{R}^3$ yields

$$
\begin{aligned}
&|r_1\varphi|_2^2 + |r_1 D_x^{-1}\varphi_y|_2^2 + |r_1 D_x^{-1}\varphi_z|_2^2 + 2\int_{\mathbf{R}^3} yD_x^{-1}\varphi \cdot D_x^{-1}\varphi_y \\
&+ 2\int_{\mathbf{R}^3} zD_x^{-1}\varphi D_x^{-1}\varphi_z + |r_1\varphi_x|_2^2 = \frac{1}{p+1}\int_{\mathbf{R}^3} r_1^2\varphi^{p+2}.
\end{aligned}
\tag{3.63}
$$

The Cauchy-Schwarz inequality then implies

$$
|r_1 D_x^{-1}\varphi_y|_2^2 + |r_1 D_x^{-1}\varphi_z|_2^2 + |r_1\varphi_x|_2^2 \le C|D_x^{-1}\varphi|_2^2 + C(|r\varphi|_2, \|\varphi\|_2) \tag{3.64}
$$

for some constant C. This means that $r_1 D_x^{-1}\varphi_y$ and $r_1 D_x^{-1}\varphi_z \in L_2(\mathbf{R}^3)$. Hence, the proof of Lemma 3.10 is completed. □

The preceding lemmas lead to a proof of the Main Result, which is the instability of the solitary wave φ_c.

Proof of Theorem 2.3. Let $\epsilon > 0$ be given in Lemma 3.8 and U_ϵ^s the cylindrically symmetric subset of the associated tubular neighborhood of the orbit of φ_0. Since $\partial_x C_0^\infty(\mathbf{R}^3)$ is dense in Y, we may choose $u_0 \in \partial_x C_0^\infty$ with u_0 close in Y-norm to χ_c for c near c_0 and such that $Q(u_0) = Q(\chi_c)$. Moreover, by Lemma 3.9, for c sufficiently close to c_0, it must also be the case that $E(u_0) < E(\varphi_0)$ and $\langle E'(u_0), B(u_0)\rangle > 0$. Since $u_0 \in X_s$ with $s \ge 3$, it follows that the solution u of (2.6) with initial value u_0 lies in $C([0,T^*), X_s)$ for some $T^* > 0$. Let T^* be the maximum time for which $u \in C([0,T^*), Y)$. We may assume $T^* = +\infty$, for otherwise $\limsup_{t\to T^*<\infty} \|u(t)\|_Y = \infty$ and φ_0 is strongly unstable in Y. Assume now the solution $u(\cdot,t) \in U_\epsilon^s$ for $t \in [0,T]$. We intend to show that $T < +\infty$ which means that $u(\cdot,t)$ eventually exits the tube U_ϵ^c. This will complete the proof of instability. Define a Liapunov function

$$A(t) = \int_{\mathbf{R}^3} v(x-\beta(t), y, z)u(x, y, z, t)dxdydz, \tag{3.65}$$

where $\beta(t) = \alpha_0(u(\cdot, t))$, $v = D_x^{-1}y_0$ where $y_0 = \frac{\partial}{\partial c}\chi|_{c=c_0}$ and u is as above, the solution of the equation (KP-3D) with initial value u_0. By the Cauchy-Schwarz inequality and Lemma 3.10,

$$|A(t)| \leq |v|_2|u(\cdot, t)|_2 = |v|_2|u_0|_2 < +\infty. \tag{3.66}$$

On the other hand, using the Hamiltonian formulation

$$\frac{du}{dt} = \partial_x E'(u)$$

of (KP-3D), one computes as follows:

$$\begin{aligned} \frac{dA}{dt} &= -\beta'(t) \int_{\mathbf{R}^3} y_0(x-\beta(t), y, z)u(\cdot, t) + \int_{\mathbf{R}^3} v(x-\beta(t), y, z)\frac{du}{dt} \\ &= -\Big\langle \alpha_0'(u(\cdot, t)), \frac{du}{dt}\Big\rangle\Big\langle y_0(\cdot - \alpha(u(\cdot, t)), u\Big\rangle + \Big\langle v(\cdot - \alpha(u(\cdot, t)), \frac{du}{dt}\Big\rangle \\ &= -\Big\langle E'(u(\cdot, t)), B(u(\cdot, t))\Big\rangle. \end{aligned} \tag{3.67}$$

Since

$$0 < E(\varphi_0) - E(u_0) = E(\varphi_0) - E(u(\cdot, t)),$$

Lemma 3.8 implies that

$$\lambda(u(\cdot, t)) \big\langle E'(u(\cdot, t)), B(u(\cdot, t))\big\rangle \geq E(\varphi_0) - E(u_0) > 0.$$

Because $u(\cdot, t) \in U_\epsilon^s$, for $0 \leq t \leq T$, we know that $\langle E'(u_0), B(u_0)\rangle > 0$. Hence, $\lambda(u(\cdot, t)) > 0$ and since $\lambda(\varphi_0) = 0$, it may be assumed that $0 < \lambda(u(\cdot, t)) < 1$ by choosing ϵ smaller if necessary. Therefore for all $t \in [0, T]$,

$$\big\langle E'(u(\cdot, t)), B(u(\cdot, t))\big\rangle \geq E(\varphi_0) - E(u_0) > 0. \tag{3.68}$$

Hence (3.67) yields the lower bound

$$-\frac{dA}{dt} \geq E(\varphi_0) - E(u_0) > 0. \tag{3.69}$$

Comparing (3.66) and (3.69), it is concluded that $T < \infty$. This completes the proof of Theorem 2.3. □

4. Conclusion

Consideration has been given to a natural three-dimensional version of the Katomtsev-Petviashvili equation. The KP-equation has a certain universality as a model for nonlinear dispersive wave motion that propagates in essentially one direction, but with allowance made for weak effects of dispersion in the transverse direction. This model has been discussed in the context of issues in plasma physics. De Bouard and Saut have discussed this system of equations from the perspective of existence of lump-type solitary waves, showing that in a certain range of the power of the nonlinearity, there are such solutions and that in the complementary range, there are not.

As a rule, we expect solitary-wave solutions of nonlinear dispersive wave equations to play a distinguished role in the evolution of certain classes of initial disturbances. On the other hand, there are at least two different paradigms that one observes. The first is when the solitary waves are stable, in which case we expect initial disturbances to go over asymptotically to solitary waves and other more dispersive structures. In this case, the underlying evolution equation is usually globally well posed in reasonable function classes. Another situation is that obtaining when the solitary waves are unstable. In this case, we sometimes (see *e.g.* Bona, Dougalis, McKinney and Karakashian 1995), though not always (see Bona, McKinney and Restrepo 2000), see singularity formation in finite time associated with the instabiity.

For the system investigated here, it is shown uniquivocally that all the solitary waves are unstable. Thus we would tentatively predict, and in fact it is known to be true (see Liu 2001) that the initial-value problem is not globally well posed. In particular, this shows clearly the system cannot serve as a model for smooth processes in situations where other than small initial data is contemplated.

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