On the Stability of KdV Multi-Solitons

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Abstract

Stability results for multi-soliton solutions of the Korteweg-de Vries equation are stated and proved. The theory developed here contributes to earlier discussions of this issue by Maddocks and Sachs, Martel and Merle and Tsai and Schuur.

1 Introduction

Multi-soliton solutions of the Korteweg-de Vries equation are solutions which represent the interactions of multiple solitary waves. In general, the term "solitary wave" is used to refer to a localized disturbance which propagates without change in form. In the context of the Korteweg-de Vries equation,

$$u_t + uu_x + u_{xxx} = 0, \tag{1.1}$$

posed for $-\infty < x < \infty$ and $t \geq 0$, a solitary wave is represented by a function $u(x,t)$ of the real variables x and t which takes the form

$$u(x,t) = \phi(x - ct),$$

where c is a constant and $\phi(\xi)$ is a function of one variable whose values are small when $|\xi|$ is large. It is easy to see that the only non-singular solutions of (1.1) of this form are those given by

$$u_{c,\theta}(x,t) = \phi_c(x - ct + \theta),$$

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where $c > 0$, $\theta \in \mathbf{R}$, and

$$\phi_c(\xi) = 3c\ \mathrm{sech}^2(\sqrt{c}\ \xi/2).$$

These Korteweg-de Vries solitary waves are not run-of-the-mill solitary waves, however. They go by the special appellation of "soliton" because they exhibit strong stability properties, like those of particles. Suppose for example that a solution $u(x,t)$ of (1.1) contains a soliton $u_{c,\theta}$ at time $t = 0$, in the sense that $|u(x,0) - u_{c,\theta}(x,0)|$ is small at least for x in a large interval centered at $x = \theta$. In general, as time passes, $u(x,t)$ may undergo a complicated nonlinear evolution, but eventually it will emerge with its identity intact, up to translation. That is, for large values of t there will exist a function $\theta(t)$ such that $|u(x,t) - u_{c,\theta(t)}(x,t)|$ is small for all x in a large interval centered at $\theta(t)$.

The scenario just described gives only a crude idea of the stability properties of solitary-wave solutions of (1.1). More precise and refined investigations have occupied researchers for the past few decades. The first rigorous result in this direction was proved by Benjamin and Bona in the 1970's [Be, Bo]. They showed that if $u(x,0)$ is sufficiently close to a soliton $u_{c,\theta}(x,0)$ over the entire real line, or more precisely in the norm of the Sobolev space $H^1(\mathbf{R})$, then there exists a function $\theta : [0,\infty) \to \mathbf{R}$ such that u remains arbitrarily close to $u_{c,\theta(t)}$ in H^1 norm for all times $t > 0$. Later, Bona and Soyeur [BoSo] observed that a simple argument based on the Implicit Function Theorem is sufficient to obtain improved information on the behavior of the function $\theta(t)$: namely, that it can be taken to be a differentiable function of t, with the property that $\theta'(t)$ remains close to $-c$ for all $t > 0$. Other authors have used more sophisticated methods to obtain detailed information on the asymptotic behavior of $\theta(t)$ and $u(x,t)$ as $t \to \infty$. It is now known that when $u(x,0)$ is sufficiently close to $u_{c,\theta_0}(x,0)$, then for large values of t there exist functions $\bar{c}(t)$ and $\theta(t)$ such that

$$\lim_{t\to\infty} \bar{c}(t) = c_1,$$

$$\lim_{t\to\infty} \theta'(t) = -c_1,$$

and

$$\lim_{t\to\infty} \big(u(x,t) - u_{\bar{c}(t),\theta(t)}(x,t)\big) = 0 \quad \text{for } x \text{ near } -\theta(t).$$

Furthermore, in some sense these results are the best possible. For details the reader is referred to the papers [MM2, MM3, PW].

The results discussed in the preceding paragraph all deal with the case in which the initial data of a solution of (1.1) is a small perturbation of a single solitary wave. A more general case would be one in which multiple solitary waves of comparable size are present. Explicit examples of solutions of (1.1) which contain multiple solitary waves are the "multi-soliton" solutions, first identified by Gardner et al. in the 1960's

[GGKM]. To denote multi-soliton solutions we continue to use the notation $u_{c,\theta}$, but now the parameters $c = (c_1, \ldots, c_n)$ and $\theta = (\theta_1, \ldots, \theta_n)$ will be vectors in $\mathbf{R}^n$ whose components c_i and θ_i determine the speeds and initial locations of n individual solitons. A straightforward analysis of the explicit formula for multi-soliton solutions (cf. Lemma 3.6 below) shows that for each $i \in \{1, \ldots, n\}$,

$$\lim_{t\to-\infty} (u_{c,\theta}(x,t) - \phi_{c_i}(x - c_i t - \theta_i)) = 0$$

and

$$\lim_{t\to\infty} \left(u_{c,\theta}(x,t) - \phi_{c_i}(x - c_i t - \tilde{\theta}_i)\right) = 0,$$

uniformly in regions where x is comparable in size to $c_i t$. (Here $\tilde{\theta}_i$, $i = 1, \ldots, n$, are numbers which depend only on c and θ.) That is, $u_{c,\theta}(x,t)$ describes the interaction of n solitary waves, each with its own wavespeed c_i. At large negative values of t the solitary waves are well-separated, but as time evolves the faster ones overtake the slower ones and significant interactions occur. Eventually, for large positive values of t, n solitary waves emerge which have exactly the same speeds as the ones which entered the interaction: in fact, the only long-lasting effects of the interaction are the phase shifts represented by replacing θ_i with $\tilde{\theta}_i$.

The question naturally arises whether the behavior exhibited by multi-solitons is stable under small initial perturbations. This question has been addressed by Maddocks and Sachs [MS] and, more recently, by Martel, Merle and Tsai [MMT]. In particular, Martel et al. prove that the conclusion of Benjamin and Bona's stability result for single solitons holds as well for multi-solitons: if a solution u of (1.1) is sufficiently close in H^1 norm to a multi-soliton solution $u_{c,\theta}$ at time $t = 0$, then there exists $\theta : [0,\infty) \to \mathbf{R}^n$ such that u remains arbitrarily close to $u_{c,\theta(t)}$ in H^1 for all time (see Theorem 2.3 below). Moreover, Martel et al. are able to obtain detailed information on the asymptotic behavior of $\theta(t)$. For example, they prove that for large values of time t, $\theta(t)$ is a C^1 function of t, and that $\lim_{t\to\infty} \theta'(t) = -c_1$ for some $c_1 \in \mathbf{R}^n$ which is close to c. Moreover, as $t \to \infty$, $u - u_{c_1,\theta(t)}$ will tend to zero in L^2 norm on any interval which propagates to the right at a speed comparable to that of the slowest soliton component of $u_{c_1,\theta}$ (cf. Corollary 1 of [MMT]).

In proving their results, Martel et al. concentrate on the large-time behavior of $u(x,t)$; or more specifically, on the behavior of $u(x,t)$ for $t \geq T$, where T is taken so large that the numbers $\{c_1 T, c_2 T, \ldots, c_n T\}$ are widely separated. At such large values of time, the soliton components of $u_{c,\theta}$, having long ceased to interact with each other, are steadily propagating without change of form. Taking $t = T$ as their initial time, Martel et al. use results from [MM1, MM2] on the asymptotic behavior of individual solitary waves to analyze $u(x,t)$ for $t \geq T$ in the separate regions near each soliton component, under the assumption that $u(x,T)$ is sufficiently close to $u_{c,\theta}(x,T)$. Then this latter assumption is removed simply by observing that, according to standard results on the well-posedness of the initial-value problem for (1.1), $u(x,T)$ can be made arbitrarily close to $u_{c,\theta}(x,T)$ by taking $u(x,0)$ sufficiently close to $u_{c,\theta}(x,0)$.

A drawback of this approach, however, is that no information is obtained about the function $\theta(t)$ on the time interval $[0,T]$. In fact, it is not even clear whether $\theta(t)$ can be asserted to be a continuous function of t on $[0,T]$. This is an undesirable state of affairs, because it is on the time interval $[0,T]$, where all the soliton interactions take place, that the chief interest of the multi-soliton solution resides.

Our main result (Theorem 2.4 below) addresses this issue. We prove that if $u(x,0)$ is close, in an appropriate Sobolev space, to a multi-soliton profile $u_{c,\theta}(x,0)$ then there exists a C^1 function $\gamma : [0,\infty) \to \mathbf{R}^n$ such that the corresponding solution $u(x,t)$ of (1.1) remains close to $u_{c,\gamma(t)}(x,t)$ for all time, and such that $\gamma'(t)$ remains close to $-c$. Actually, this fact is derived as a consequence of the stability result of Martel et al., together with the Implicit Function Theorem. Thus the proof proceeds along the same lines as that used by Bona and Soyeur [BoSo] in the single-soliton case.

The stability results just discussed show that solutions of (1.1) with initial data that is a small H^1 perturbation of a multi-soliton will resolve asymptotically into solitary waves as $t \to \infty$, in domains that move to the right with at least the speed of the slowest solitary wave present. It remains an open question, however, whether this asymptotic behavior is still exhibited for solutions with general initial data in H^1. That this might be the case is suggested by the inverse scattering theory for solutions of (1.1); cf. the book of Schuur [S], where it is shown that smooth initial data with rapid decay at infinity give rise to solutions which behave asymptotically like multi-soliton solutions. To date, however, the methods of inverse scattering theory have not yielded results for more general classes of solutions.

The plan of this paper is as follows. In Section 2 we state and discuss our main result. In Section 3 some lemmas are established concerning manifolds of n-soliton solutions in H^1, and in Section 4, the main result is offered. An appendix contains a proof of the stability of n-solitons in higher-order Sobolev spaces.

Notation. The notation in force is standard. For $1 \le p < \infty$, L^p is the usual Banach space of measurable functions on $\mathbf{R}$ with norm given by $|f|_p = (\int_{-\infty}^{\infty} |f|^p \, dx)^{1/p}$. The space L^∞ consists of the measurable, essentially bounded functions f on $\mathbf{R}$ with norm $|f|_\infty = \text{ess sup}_{x\in\mathbf{R}}|f(x)|$. For $s \in \mathbf{R}$, the L^2-based Sobolev space $H^s = H^s(\mathbf{R})$ is the set of all tempered distributions f on $\mathbf{R}$ whose Fourier transforms $\hat{f}$ are measurable functions on $\mathbf{R}$ satisfying

$$\|f\|_s^2 = \int_{-\infty}^{\infty} (1+k^2)^s |\hat{f}(k)|^2 \, dk < \infty.$$

If X and Z are Banach spaces then $B(X,Z)$ denotes the space of all bounded linear maps l from X to Z, with norm

$$\|l\|_{B(X,Z)} = \sup_{\|x\|_X=1} \|l(x)\|_Z;$$

and $C([0,T],X)$ is the space of all continuous maps u from the interval $[0,T] \subset \mathbf{R}$ into X, with norm

$$\|u\|_{C([0,T],X)} = \sup_{t\in[0,T]} \|u(t)\|_X.$$

Finally, for matrices $Y \in B(\mathbf{R}^n, \mathbf{R}^n) = \mathbf{R}^{n^2}$, we sometimes use the norm

$$\|Y\|_\infty = \sup_{1\le l,m\le n} |Y_{lm}|.$$

2 Statement of the main result

To explain the results of the paper in detail, we begin by recalling the explicit formula for multi-solitons given by Hirota [Hi]. Let n be a given natural number, let $\theta = (\theta_1, \ldots, \theta_n)$ be a given vector in $\mathbf{R}^n$, and let c be a given element of the set

$$S_n = \{c = (c_1, \ldots, c_n) \in \mathbf{R}^n : \ c_i > 0 \text{ for } 1 \le i \le n \text{ and } c_i \neq c_j \text{ for } 1 \le i < j \le n\}.$$

Define a function of $x \in \mathbf{R}$ by

$$\phi^{(n)}(x;\theta,c) = 12\frac{d^2}{dx^2}\log\tau^{(n)}(x;\theta,c), \tag{2.1}$$

where

$$\tau^{(n)}(x;\theta,c) = \sum_{\epsilon\in\{0,1\}^n} \exp\left(\sum_{i=1}^n \epsilon_i\sqrt{c_i}(x+\theta_i) + \sum_{1\le i<j\le n} \epsilon_i\epsilon_j A_{ij}\right), \tag{2.2}$$

with

$$\exp(A_{ij}) = \left(\frac{\sqrt{c_i}-\sqrt{c_j}}{\sqrt{c_i}+\sqrt{c_j}}\right)^2.$$

The outermost sum in (2.2) is taken over all of the 2^n possible values of the n-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, where ϵ_i is equal to either 0 or 1 for $1 \le i \le n$.

The function $\phi^{(n)}(x;\theta,c)$ is called an n-soliton profile. Each n-soliton profile gives rise to a multi-soliton solution $u_{c,\theta}(x,t)$ of (1.1), defined by

$$u_{c,\theta}(x,t) = \phi^{(n)}(x;\theta - ct, c). \tag{2.3}$$

In other words, the n-soliton solution $u_{c,\theta}$ propagates in the set of n-soliton profiles $\{\phi^{(n)}(x;\bar{\theta},c) : \bar{\theta} \in R^n\}$, and the evolution of the phase parameter is linear: $\bar{\theta}(t) = \theta - ct$.

For ease of notation, when referring to $\phi^{(n)}(x;\theta,c)$ we will often drop one or more of the arguments x, θ, and c, as well as the superscript (n), when this will not cause confusion.

Since we want to discuss the stability of multi-solitons in the Sobolev spaces H^s, it is necessary to recall the well-posedness theory for (1.1) in these spaces. Observe first that the linear equation $v_t = v_{xxx}$ defines a unitary evolution operator $U(t)$ on H^s for every $s \in \mathbf{R}$; i.e., for each $t \geq 0$ one can define $U(t) : H^s \to H^s$ by setting $U(t)[f] = v(\cdot,t)$, where v is the solution of $v_t = v_{xxx}$ with $v(\cdot,0) = f$. In fact, $v(\cdot,t)$ is defined as a tempered distribution by $\hat{v}(k,t) = e^{ik^3t}\hat{f}(k)$, where the circumflex denotes the Fourier transform with respect to x, and k is the dual Fourier transform variable. It follows that $v \in C([0,T],H^s)$ and $v_t \in C([0,T],H^{s-3})$ for all $T > 0$.

Now suppose that $s \geq 1$, so that $u \in H^s$ implies that uu_x is well-defined as an element of H^{s-1}. In this case, define u to be a *strong* solution in $C([0,T],H^s)$ of (1.1) with initial data $u(0) = u_0$ if, for all $t \in [0,T]$,

$$u(t) = U(t)[u_0] - \int_0^t U(t-\tau)\,[u(\tau)u_x(\tau)]\;d\tau. \tag{2.4}$$

Note that if $u \in C([0,T],H^s)$ then the integrand on the right-hand side of (2.4) is a continuous function of τ with values in H^{s-1}, so the integral exists in H^{s-1} at least. Also, if $u \in C([0,T],H^s)$ satisfies (2.4), then as a distribution-valued function of t, u is differentiable, and its derivative u_t satisfies (1.1) in the sense of tempered distributions. It then follows from (1.1) that u_t is in fact in $C([0,T],H^{s-3})$.

The following well-posedness result is proved in [KPV1].

Theorem 2.1. *Suppose $s \geq 1$. For every $u_0 \in H^s$ and every $T > 0$ equation* (1.1) *has a unique strong solution $u \in C([0,T],H^s)$ with initial data $u(0) = u_0$. For this solution we have $u_t \in C([0,T],H^{s-3})$. Moreover, the map which takes the initial data u_0 to the solution u is continuous from H^s to $C([0,T],H^s)$.*

Remark 2.2. For $s < 1$, difficulties arise in making sense of the product uu_x appearing in (1.1). For $0 \leq s < 1$ one can interpret uu_x as the distributional derivative of the integrable function u^2, but when $s < 0$ not even this interpretation is available. Nevertheless, using ingenious arguments which take advantage of certain smoothing properties of (1.1), various authors have been able to formulate and prove well-posedness results in H^s for all $s > -3/4$. See, for example, [KPV2] and [CKSTT].

In what follows, we will typically use u_0 to denote initial data for (1.1) in H^s $(s \geq 1)$, and $u(t)$ to denote the corresponding solution of (1.1), guaranteed by Theorem 2.1 to exist in H^s for all $t > 0$.

We are now ready to state the following stability result for multi-solitons in H^1, which is taken from [MMT].

Theorem 2.3. *Let $c \in S_n$ and $\theta_0 \in \mathbf{R}^n$ be given. For every $\epsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1$ and $\|u_0 - \phi^{(n)}(\theta_0, c)\|_1 < \delta$, then for all $t > 0$ there exists $\theta(t) \in \mathbf{R}^n$ such that*

$$\|u(t) - \phi^{(n)}(\theta(t), c)\|_1 < \epsilon.$$

Our main result is as follows. It represents a generalization of the work of Bona and Soyeur, to whom the proof is due in case $n = 1$ [BoSo].

Theorem 2.4. *Let $c \in S_n$ and $\theta_0 \in \mathbf{R}^n$ be given. Then there exists a constant A with the following property. For every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1$ and $\|u_0 - \phi^{(n)}(\theta_0, c)\|_1 < \delta$, then there exists a C^1 function $\gamma : (0, \infty) \to \mathbf{R}^n$ such that for every $t > 0$,*

$$\|u(t) - \phi^{(n)}(\gamma(t), c)\|_1 < \epsilon \tag{2.5}$$

and

$$|\gamma'(t) + c| < A\epsilon. \tag{2.6}$$

Remark 2.5. Theorems 2.3 and 2.4 are still valid if the H^1 norm is replaced throughout by the H^k norm, for any integer $k \geq 1$. As has been observed in [BLN], this is a straightforward consequence of the infinite sequence of conservation laws for (1.1). For details the reader is referred to the Appendix. In this connection, it is interesting to note that Merle and Vega [MV] have proved stability of single-soliton solutions in $H^0 = L^2$.

Remark 2.6. It remains an open question whether, for fixed c and ϵ, the number δ in the statement of Theorem 2.3 can be chosen independently of θ_0. If this is indeed the case, then our proof below shows that δ can also be chosen independently of θ_0 in Theorem 2.4. See also the comments following the statement of Theorem 2.7 below.

The proof of Theorem 2.4 relies on an application of the Implicit Function Theorem, which is made possible by the fact that, according to Theorem 2.3, $u(t)$ is, for each $t > 0$, close enough to a multi-soliton profile to be within the domain of a function defined implicitly near that profile. By contrast, to prove a version of Theorem 2.4 on a finite time interval $[0, T]$, Theorem 2.3 would not be necessary, as the hypothesis required for our application of the Implicit Function Theorem would be provided by the well-posedness theory for (1.1).

Alternatively, one could envisage using the multi-soliton stability theory of Maddocks and Sachs [MS] in place of Theorem 2.3. However, their stability theory is not suitable for our purposes, for reasons which we now briefly digress to discuss.

The multi-soliton stability theory of [MS] is based on the infinite sequence of conserved functionals for (1.1), the first four of which are

$$\begin{aligned} I_1(u) &= \int_{-\infty}^{\infty} u\ dx, \\ I_2(u) &= \int_{-\infty}^{\infty} u^2\ dx, \\ I_3(u) &= \int_{-\infty}^{\infty} \left(u_x^2 - \frac{1}{3}u^3\right)\ dx, \\ I_4(u) &= \int_{-\infty}^{\infty} \left(u_{xx}^2 - \frac{5}{3}uu_x^2 + \frac{5}{36}u^4\right) dx. \end{aligned} \tag{2.7}$$

(Here, the functionals I_k $(k = 1, 2, \dots,)$ have been normalized so that, in each one, the term with the highest-order derivative appears with coefficient 1.) These functionals are conserved in the sense that $I_k(u(t))$ is independent of t whenever u is a strong solution of (1.1) in H^k (in the sense defined before Theorem 2.1). From this invariance property and the asymptotic analysis of multi-solitons, it follows easily (cf. [L, MS]) that $I_k(\phi^{(n)}(\theta, c))$ is independent of the phase parameter $\theta \in \mathbf{R}^n$. In fact, we have

$$I_k(\phi^{(n)}(\theta, c)) = (-1)^k \left(\frac{36}{2k-1}\right) \sum_{i=1}^{n} (c_i)^{(2k-1)/2}. \tag{2.8}$$

The stability properties of multi-solitons are closely related to the variational properties of the functionals I_k. Suppose $n \in \mathbf{N}$ and $c \in S_n$ are given. By (2.8), the set $G_c \subseteq H^n$ defined by

$$G_c = \{\psi \in H^n : I_k(\psi) = I_k(\phi^{(n)}(\theta, c)) \text{ for } 2 \le k \le n+2\}$$

is independent of $\theta \in \mathbf{R}^n$, and if we define

$$M_c = \{\psi \in H^n : \psi = \phi^{(n)}(\theta, c) \text{ for some } \theta \in \mathbf{R}^n\}, \tag{2.9}$$

then

$$M_c \subseteq G_c.$$

In the case $n = 1$ it is easy to see that $M_c = G_c$ for all $c > 0$. For $n > 1$, however, the question of whether $M_c = G_c$ appears to be open.

The stability result of Maddocks and Sachs [MS] is the following.

Theorem 2.7. *Let $n \geq 1$ and suppose $c \in S_n$. For every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^n$, $\theta_0 \in \mathbf{R}^n$, and $\|u_0 - \phi^{(n)}(\theta_0, c)\|_n < \delta$, then for all $t > 0$,*

$$\inf_{\psi \in G_c} \|u(t) - \psi\|_n < \epsilon.$$

Remark 2.8. Recently Neves and Lopes [NL] have given an alternate proof of Theorem 2.7, using a method which also leads to a similar result for the Benjamin-Ono equation in the double-soliton case.

Notice that the stability result of Theorem 2.7 is set in H^n, whereas the result of Theorem 2.3 is set in H^1, and is hence stronger (cf. Remark 2.5). Also, since it is not yet known whether $M_c = G_c$, Theorem 2.7 does not yet give a stability result for the set of multi-soliton profiles M_c, and hence cannot be used to prove a result like Theorem 2.4. If, on the other hand, it could be proved that $M_c = G_c$, then a proof of stability of multi-solitons (at least in the space H^n) could be based purely on consideration of the conserved functionals, without recourse to the detailed asymptotic analysis provided in [MMT]. Moreover, the stability result would have the advantage that, as in Theorem 2.7, the number δ corresponding to a given ϵ could be chosen independently of θ_0.

3 The embedding of M_c in H^1

In this section we prove several preliminary results which will be needed for the proof in Section 4 of the main result. Some of them can be given natural interpretations as statements about the geometric properties of the map $\beta : \mathbf{R}^n \to H^1$ defined by $\beta(\theta) = \phi^{(n)}(\theta, c)$. Thus Lemma 3.4 implies that β is an immersion, and Lemma 3.11 implies that β is one-to-one. Also, since Theorem 3.1 asserts the existence of a continuous map F which extends β^{-1} to a neighborhood U_δ of $M_c = \beta(\mathbf{R}^n)$, it follows that β is an embedding; i.e., an immersion which is a homeomorphism onto its image. An important technical point, which is crucial to the proof in Section 4, is that U_δ contains all the elements of H^1 within a distance δ of M_c; geometrically speaking, this means that U_δ contains a tubular neighborhood of M_c of uniform width in the direction "normal" to M_c.

For $n \in \mathbf{N}$ and $i \in \{1, \ldots, n\}$, define

$$\phi_i^{(n)}(x; \theta, c) = \frac{\partial \phi^{(n)}}{\partial \theta_i}(x; \theta, c)$$

and

$$\phi_{ij}^{(n)} = \frac{\partial^2 \phi^{(n)}}{\partial \theta_i \partial \theta_j}(x; \theta, c).$$

Similar notation will be used for the functions defined in (2.2). In addition, we occasionally use ∂_x to denote the operator of differentiation with respect to x, and ∂_{θ_i} to denote differentiation with respect to θ_i, and for a multi-index $N = (N_0, N_1, \ldots, N_n)$, where the N_j are non-negative integers, we define ∂^N to be the operator

$$\partial^N = \partial_x^{N_0} \partial_{\theta_1}^{N_1} \partial_{\theta_2}^{N_2} \ldots \partial_{\theta_n}^{N_n}.$$

Let $n \in \mathbf{N}$ and $c \in S_n$ be fixed. For each $\theta \in \mathbf{R}^n$, define $v_\theta \in H^n$ by

$$v_\theta(x) = \phi^{(n)}(x; \theta, c). \tag{3.1}$$

Then (2.9) becomes

$$M_c = \{v_\theta : \theta \in \mathbf{R}^n\}.$$

For $\delta > 0$, define

$$U_\delta = \left\{ u \in H^1 : \quad u \in B_\delta(v_\theta) \text{ for some } v_\theta \in M_c \right\},$$

where $B_\delta(v_\theta)$ denotes the open ball in H^1 with radius δ and center at v_θ.

Let $G : H^1 \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$G(u, \theta) = \left(\int u(x) \phi_1^{(n)}(x; \theta, c)\ dx, \ldots, \int u(x) \phi_n^{(n)}(x; \theta, c)\ dx \right).$$

Putting $k = 2$ in (2.8), we find that

$$I_2(\phi^{(n)}(x; \theta, c)) = \int_{-\infty}^{\infty} (\phi^{(n)}(x; \theta, c))^2\ dx$$

is independent of θ. Therefore, for each $i \in \{1, \ldots, n\}$, the derivative with respect to θ_i vanishes, *viz.*

$$\int_{-\infty}^{\infty} \phi^{(n)} \phi_i^{(n)}\ dx = 0. \tag{3.2}$$

Hence, for all $\theta \in \mathbf{R}^n$,

$$G(v_\theta, \theta) = 0, \tag{3.3}$$

where 0 denotes the zero vector in $\mathbf{R}^n$.

The main goal of this section is to prove the following Theorem.

Theorem 3.1. *There exist a number $\delta_0 > 0$ and a C^∞-map $F : U_{\delta_0} \mapsto \mathbf{R}^n$ such that for every $u \in U_{\delta_0}$,*

$$G(u, F(u)) = 0. \tag{3.4}$$

Remark 3.2. Actually, below we will only need that F is continuous on U_{δ_0}, but it is not more difficult to prove that F is C^∞.

To prove Theorem 3.1, we use the Implicit Function Theorem, which entails a study of G_θ, the partial derivative of G with respect to θ. Observe that G_θ is the map from $H^1 \times \mathbf{R}^n$ to $B(\mathbf{R}^n, \mathbf{R}^n)$ given by

$$G_\theta(u,\theta) = \begin{bmatrix} \int_{-\infty}^{\infty} u\phi_{11} & \int_{-\infty}^{\infty} u\phi_{12} & \dots & \int_{-\infty}^{\infty} u\phi_{1n} \\ \int_{-\infty}^{\infty} u\phi_{21} & \int_{-\infty}^{\infty} u\phi_{22} & \dots & \int_{-\infty}^{\infty} u\phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \int_{-\infty}^{\infty} u\phi_{n1} & \int_{-\infty}^{\infty} u\phi_{n2} & \dots & \int_{-\infty}^{\infty} u\phi_{nn} \end{bmatrix}, \tag{3.5}$$

where

$$\phi_{ij} = \frac{\partial^2 \phi^{(n)}(x;\theta,c)}{\partial\theta_i \partial\theta_j}.$$

The Implicit Function Theorem, together with (3.3), guarantees the existence of a solution $F(u)$ to (3.4) for u in some neighborhood $B_\delta(v_\theta)$ of v_θ, provided that the matrix $G_\theta(v_\theta, \theta)$ is nonsingular. To prove Theorem 3.1, however, we need to also verify that δ can be chosen independently of $\theta \in \mathbf{R}^n$. This requires keeping track of how δ depends on the size of G and its derivatives. For this purpose, the following version of the Implicit Function Theorem will be helpful.

Theorem 3.3. *Let X, Y, and Z be Banach spaces, and suppose $(x_0, y_0) \in X \times Y$. Suppose there exist a neighborhood U of x_0 in X, a neighborhood V of y_0 in Y, and a map $G : U \times V \to Z$ which is continuous on $U \times V$ and which has a continuous derivative with respect to y, G_y, on $U \times V$. Suppose also that $G(x_0, y_0) = 0$ and $G_y(x_0, y_0) : Y \to Z$ has a bounded inverse. Then*

(i) There exists $\eta_0 > 0$ with the following property. For every $\eta \in (0, \eta_0]$, there exists $\delta = \delta(\eta) > 0$ such that $B_\delta(x_0) \subseteq U$, $B_\eta(y_0) \subseteq V$, and for each $x \in B_\delta(x_0)$ there is exactly one point $F(x)$ in $B_\eta(y_0)$ such that $G(x, Fx) = 0$. The map F is continuous from $B_\delta(x_0)$ to $B_\eta(y_0)$.

(ii) *If*

$$K_1 = \|G_y(x_0, y_0)^{-1}\|_{B(Z,Y)};$$

and K_2 *and* K_3 *are constants such that*

$$\|G_y(x, y) - G_y(x_0, y_0)\|_{B(Y,Z)} \leq K_2 \left(\|x - x_0\|_X + \|y - y_0\|_Y\right)$$

and

$$\|G(x, y_0) - G(x_0, y_0)\|_Z = \|G(x, y_0)\|_Z \leq K_3\|x - x_0\|_X$$

for all $x \in U$ *and all* $y \in V$*, then the number* η_0 *and the function* $\delta(\eta)$ *in part (i) can be chosen to depend only on* K_1*,* K_2*, and* K_3*.*

(iii) *If, in addition,* G *is* C^1 *on* $U \times V$*, then there exists* $\eta_1 > 0$*, possibly smaller than* η_0*, such that for all* $\eta \in (0, \eta_1]$*, the function* F *is* C^1 *on* $B_{\delta(\eta)}(x_0)$*. Furthermore, if* G *is* C^k *on* $U \times V$ *for any* $k \geq 1$*, then for all* $\eta \in [0, \eta_1]$*,* F *is* C^k *on* $B_{\delta(\eta)}(x_0)$*. (The number* η_1 *and the function* $\delta(\eta)$ *do not depend on* k*.)*

(iv) *If*

$$K_4 = \|G_x(x_0, y_0)\|_{B(X,Z)}$$

and K_5 *is a constant such that*

$$\|G_x(x, y) - G_x(x_0, y_0)\|_{B(X,Z)} \leq K_5 \left(\|x - x_0\|_X + \|y - y_0\|_Y\right)$$

for all $x \in U$ *and* $y \in V$*, then the number* η_1 *in part (iii) can be chosen to depend only on the constants* K_i*,* $1 \leq i \leq 5$*.*

Parts (i) and (iii) of this theorem are proved in Theorem 15.1 and Corollary 15.1 of [D]. Parts (ii) and (iv) are implicit in the proofs of Theorem 15.1 and Corollary 15.1 of [D], and can be established by keeping track of the constants involved in these proofs. We omit the details. (See also [H].)

From Theorem 3.3 we see that, in order to prove Theorem 3.1, it will be necessary to obtain θ-independent bounds on the size of $G_\theta(v_\theta, \theta)^{-1}$, as well as on the size of G itself and its derivatives. These bounds will be obtained below in Lemmas 3.4 through 3.10.

Lemma 3.4. *For each fixed* $\theta \in \mathbf{R}^n$ *and* $c \in S_n$*, the collection* $\{\phi_i^{(n)}(x; \theta, c) : 1 \leq i \leq n\}$ *forms a linearly independent set of functions of* x*.*

Proof. Suppose there exist constants $\alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^n \alpha_i \phi_i(x) = 0$ for all $x \in \mathbf{R}$; we wish to show that $\alpha_i = 0$ for $i = 1, \ldots, n$. Integrating twice with respect to x and using (2.1), it is discovered that

$$\sum_{i=1}^{n} \alpha_i \tau_i(x) = C_1 x\tau + C_2 \tau, \tag{3.6}$$

where τ_i denotes $\frac{\partial \tau}{\partial \theta_i}$, and C_1 and C_2 are constants. It is straightforward to see that the functions $x\tau$ and τ are linearly independent from each other and from the τ_i, so it follows from (3.6) that $C_1 = C_2 = 0$. Hence

$$\sum_{i=1}^{n} \alpha_i \tau_i(x) = 0 \tag{3.7}$$

for all $x \in \mathbf{R}$.

Now observe that each τ_i can be written in the form

$$\tau_i(x) = \sum_{j=1}^{m_i} a_{ij} \exp(b_{ij} x), \tag{3.8}$$

where a_{ij} and b_{ij} are constants, with $b_{i1} = \sqrt{c_i}$, $b_{ij} > \sqrt{c_i}$ for $2 \le j < m_i$, and $a_{i1} = \sqrt{c_i} > 0$. If at least one of the α_i is nonzero, let i_0 be such that $c_{i_0} = \min\{c_i : \alpha_i \ne 0\}$. Then it follows from (3.7) and (3.8) that $\exp\left(\sqrt{c_{i_0}}\, x\right)$ can be expressed as a linear combination of functions of the form $\exp(bx)$ with $b > \sqrt{c_{i_0}}$. This contradiction shows that each of the α_i must be equal to zero. □

Lemma 3.5. *Suppose $n \in \mathbf{N}$ and $c \in S_n$ are given. Then for every multi-index N there exist constants $A = A(c, n, N)$ and $B = B(c, n, N)$ such that*

$$|\partial^N \tau^{(n)}(x; \theta, c)| \le A \tau^{(n)}(x; \theta, c) \tag{3.9}$$

and

$$|\partial^N \phi^{(n)}(x; \theta, c)| \le B \tag{3.10}$$

for all $x \in \mathbf{R}$ and $\theta \in \mathbf{R}^n$.

Proof. The estimate (3.9) follows immediately from the definition of τ in (2.2). Also, from (2.1), notice that

$$\phi = 12 \left(\frac{\tau\tau'' - (\tau')^2}{\tau^2} \right)$$

(where primes denote derivatives with respect to x), so (3.10) follows immediately in the case when $N = (0, 0, \ldots, 0)$. A similar argument establishes (3.10) for any derivative of ϕ. □

Here is some convenient notation for dealing with the decomposition of an n-soliton profile into a k-soliton and an $(n-k)$-soliton profile. Suppose k is fixed in $\{1,\ldots,n-1\}$. For each $\theta=(\theta_1,\ldots,\theta_n)\in\mathbf{R}^n$ and each $c=(c_1,\ldots,c_n)\in S_n$, define

$$\phi^*(x;\theta,c)=\phi^{(k)}(x;(\theta_1,\ldots,\theta_k),(c_1,\ldots,c_k)) \tag{3.11}$$

and

$$\phi^{**}(x;\theta,c)=\phi^{(n-k)}(x;(\theta_{k+1},\ldots,\theta_n),(c_{k+1},\ldots,c_n)). \tag{3.12}$$

In particular, whenever N is such that $N_j\neq 0$ for some $j\in\{k+1,\ldots,n\}$, it must be the case that

$$\partial^N\phi^*\equiv 0;$$

and whenever N is such that $N_j\neq 0$ for some $j\in\{1,\ldots,k\}$, it is correspondingly true that

$$\partial^N\phi^{**}\equiv 0.$$

The next lemma gives expression to the well-known fact that when the sets $\{\theta_1,\theta_2,\ldots,\theta_k\}$ and $\{\theta_{k+1},\ldots,\theta_n\}$ are widely separated, $\phi^{(n)}$ is well approximated by $\phi^*+\phi^{**}$.

Lemma 3.6. *Suppose $n\in\mathbf{N}$ and $c\in S_n$ are given, and let*

$$D=\min\{\sqrt{c_1},\ldots,\sqrt{c_n}\}.$$

Then for every multi-index N there exists a constant $C=C(c,n,N)$ such that the following is true. Let $k\in\{1,\ldots,n-1\}$ be given, and define ϕ^* and ϕ^{**} as in (3.11) *and* (3.12)*. Suppose $\theta=(\theta_1,\ldots,\theta_n)\in\mathbf{R}^n$ is such that*

$$\theta_1\leq\theta_2\leq\cdots\leq\theta_n,$$

and define

$$\tilde\theta_i=\begin{cases}\theta_i+\frac{1}{\sqrt{c_i}}\sum_{j=k+1}^n A_{ij} & (i=1,\ldots,k)\\ \theta_i & (i=k+1,\ldots,n).\end{cases} \tag{3.13}$$

Then, it follows that

(i) $|\partial^N\phi^{(n)}(x;\theta,c)-\partial^N\phi^*(x;\tilde\theta,c)|\leq C\exp(-D(x+\theta_{k+1}))$ *for all* $x\geq-\theta_{k+1}$.

(ii) $|\partial^N\phi^{**}(x;\tilde\theta,c)|\leq C\exp(-D(x+\theta_{k+1}))$ *for all* $x\geq-\theta_{k+1}$.

(iii) $|\partial^N\phi^{(n)}(x;\theta,c)-\partial^N\phi^{**}(x;\tilde\theta,c)|\leq C\exp(D(x+\theta_k))$ *for all* $x\leq-\theta_k$.

(iv) $|\partial^N \phi^*(x;\tilde{\theta},c)| \le C\exp(D(x+\theta_k))$ *for all* $x \le -\theta_k$.

Hence, in particular,

$$\left|\partial^N \phi^{(n)}(x;\theta,c) - \left(\partial^N \phi^*(x;\tilde{\theta},c) + \partial^N \phi^{**}(x;\tilde{\theta},c)\right)\right| \le 2C\exp(-Dp_\theta(x)) \tag{3.14}$$

for all $x \in \mathbf{R}$, *where*

$$p_\theta(x) = \max\{|x+\theta_k|, |x+\theta_{k+1}|\} = \left|x + \left(\frac{\theta_{k+1}+\theta_k}{2}\right)\right| + \left(\frac{\theta_{k+1}-\theta_k}{2}\right). \tag{3.15}$$

Proof. Define

$$\tilde{\tau}(x) = \tau^{(n)}(x)\cdot\exp(-L(x)), \tag{3.16}$$

where

$$L(x) = \sum_{i=k+1}^{n} \sqrt{c_i}(x+\theta_i) \ + \sum_{k+1\le i<j\le n} A_{ij}. \tag{3.17}$$

Since $L(x)$ is a linear function of x, the equations (2.1) and (3.16) imply that

$$\phi^{(n)}(x) = 12\frac{d^2}{dx^2}\log\tilde{\tau}(x). \tag{3.18}$$

Using (2.2), (3.13), and (3.17), we can expand the right-hand side of (3.16) as follows:

$$\begin{aligned}\tilde{\tau}(x) = \sum_{\epsilon^*\in\{0,1\}^k} \Bigg\{ &\exp\left(\sum_{i=1}^{k}\epsilon_i\sqrt{c_i}(x+\tilde{\theta}_i) \ + \sum_{1\le i<j\le k}\epsilon_i\epsilon_j A_{ij}\right)\times \\ &\sum_{\epsilon^{**}\in\{0,1\}^{n-k}} \exp\left(\ \sum_{i=k+1}^{n}(\epsilon_i-1)\sqrt{c_i}(x+\theta_i) + B(\epsilon)\right)\Bigg\};\end{aligned} \tag{3.19}$$

where

$$B(\epsilon) = \sum_{i=1}^{k}\ \sum_{j=k+1}^{n}\epsilon_i(\epsilon_j-1)A_{ij} \ + \sum_{k+1\le i<j\le n}(\epsilon_i\epsilon_j-1)A_{ij},$$

and $\epsilon = (\epsilon^*,\epsilon^{**})$ with $\epsilon^* = (\epsilon_1,\ldots,\epsilon_k)$ and $\epsilon^{**} = (\epsilon_{k+1},\ldots,\epsilon_n)$.

Notice first that if τ^* is defined by

$$\begin{aligned}\tau^*(x;\theta,c) &= \tau^{(k)}(x;(\theta_1,\ldots,\theta_k),(c_1,\ldots,c_k)) \\ &= \sum_{\epsilon\in\{0,1\}^k}\exp\left(\sum_{i=1}^{k}\epsilon_i\sqrt{c_i}(x+\theta_i) + \sum_{1\le i<j\le k}\epsilon_i\epsilon_j A_{ij}\right),\end{aligned} \tag{3.20}$$

then from (3.19), it follows that for all $x \in \mathbf{R}$,

$$\tau^*(x;\tilde{\theta},c) \leq \tilde{\tau}(x;\theta,c). \tag{3.21}$$

Assume that $x+\theta_{k+1} \geq 0$, and consider the inner sum, indexed by ϵ^{**}, in (3.19). For each term in this sum, there are two possibilities. Either $\epsilon_i = 1$ for all $i \in \{k+1,\ldots,n\}$, in which case the value of the corresponding term is just $\exp(0)=1$; or, $\epsilon_i = 0$ for some $i \in \{k+1,\ldots,n\}$, in which case the corresponding term can be bounded above by a constant times $\exp(-D(x+\theta_{k+1}))$. It therefore follows that for all $x \geq -\theta_{k+1}$, we have

$$|\tilde{\tau}(x;\theta,c) - \tau^*(x;\tilde{\theta},c)| \leq C\tau^*(x;\tilde{\theta},c)\exp(-D(x+\theta_{k+1})).$$

(Here, and in what follows, we use C to denote various constants which are independent of x and θ; the value of C may differ from line to line.)

Consider next the equation obtained from (3.19) by differentiating any number of times with respect to x or the variables θ_1, θ_2, ... , θ_n. In the resulting equation, the only terms on the right-hand side which are not exponentially small are those in which $\epsilon^{**} = (1,1,\ldots,1)$ and none of the derivatives are applied within the inner sum. Thus, for any multi-index M, the inequality

$$|\partial^M \tilde{\tau}(x;\theta,c) - \partial^M \tau^*(x;\tilde{\theta},c))| \leq C\tau^*(x;\tilde{\theta},c)\exp(-D(x+\theta_{k+1})) \tag{3.22}$$

holds for all $x \geq -\theta_{k+1}$, with a constant C that depends only on c, n, and M.

From (3.18), it transpires that

$$\partial^N \phi^{(n)} = \frac{P(\tilde{\tau})}{\tilde{\tau}^{|N|+2}} \tag{3.23}$$

where $|N| = N_0 + N_1 + \cdots + N_n$ and $P(\tilde{\tau})$ is a homogeneous polynomial of order $|N|+2$ in $\tilde{\tau}$ and its derivatives. Similarly from (2.1), (3.11) and (3.20), there follows the relation

$$\phi^*(x;\theta,c) = 12\frac{d^2}{dx^2}\log\tau^*(x;\theta,c),$$

so that

$$\partial^N \phi^* = \frac{P(\tau^*)}{(\tau^*)^{|N|+2}}. \tag{3.24}$$

Write

$$\partial^N \phi^{(n)} - \partial^N \phi^* = \frac{P(\tilde{\tau}) - P(\tau^*)}{\tilde{\tau}^{|N|+2}} + P(\tau^*)\left(\frac{1}{\tilde{\tau}^{|N|+2}} - \frac{1}{(\tau^*)^{|N|+2}}\right), \tag{3.25}$$

and consider the two terms on the right-hand side separately.

To estimate the first term, express the numerator in the form

$$P(\tilde{\tau}) - P(\tau^*) = \sum_M (\partial^M \tilde{\tau} - \partial^M \tau^*) Q_M(\tilde{\tau}, \tau^*) \tag{3.26}$$

where each $Q_M(\tilde{\tau}, \tau^*)$ is a homogeneous polynomial of degree $|N| + 1$ in $\tilde{\tau}$, τ^*, and their derivatives. From (3.9), (3.21), (3.22), and (3.26), it follows that

$$|P(\tilde{\tau}) - P(\tau^*)| \leq C\tilde{\tau}^{|N|+2} \exp(-D(x + \theta_{k+1})). \tag{3.27}$$

To estimate the second term, write

$$\begin{aligned} \left| P(\tau^*) \left(\frac{1}{\tilde{\tau}^{|N|+2}} - \frac{1}{(\tau^*)^{|N|+2}} \right) \right| &= \left| \frac{P(\tau^*)(\tau^* - \tilde{\tau})((\tau^*)^{|N|+1} + \cdots + \tilde{\tau}^{|N|+1})}{(\tau^* \tilde{\tau})^{|N|+2}} \right| \\ &\leq \frac{C(\tau^*)^{|N|+2}(\tau^* \exp(-D(x + \theta_{k+1})))(\tilde{\tau}^{|N|+1})}{(\tau^* \tilde{\tau})^{|N|+2}} \\ &\leq C \exp(-D(x + \theta_{k+1})) \end{aligned} \tag{3.28}$$

where again we have used (3.9), (3.21), and (3.22).

Statement (i) of the lemma then follows from (3.25), (3.27), and (3.28).

Attention is now turned to part (iii) of the lemma. Begin by rewriting (2.2) in the form

$$\begin{aligned} \tau(x; \theta, c) = \sum_{\epsilon^{**} \in \{0,1\}^{(n-k)}} &\left\{ \exp\left(\sum_{i=k+1}^{n} \epsilon_i \sqrt{c_i}(x + \theta_i) + \sum_{k+1 \leq i < j \leq n} \epsilon_i \epsilon_j A_{ij} \right) \times \right. \\ &\left. \sum_{\epsilon^* \in \{0,1\}^k} \exp\left(\sum_{i=1}^{k} \epsilon_i \sqrt{c_i}(x + \theta_i) + \tilde{B}(\epsilon) \right) \right\} \end{aligned} \tag{3.29}$$

where ϵ^* and ϵ^{**} are defined as before and

$$\tilde{B}(\epsilon) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \epsilon_i \epsilon_j A_{ij} + \sum_{1 \leq i < j \leq k} \epsilon_i \epsilon_j A_{ij}.$$

For each term in the inner sum (indexed by ϵ^*) of (3.29), there are two possibilities: either $\epsilon_i = 0$ for all $i \in \{1, \ldots, k\}$, in which case the corresponding term is $\exp(0) = 1$; or, $\epsilon_i = 1$ for some $i \in \{1, \ldots, k\}$, in which case the corresponding term is bounded above by a constant times $\exp(D(x + \theta_k))$, provided that $x + \theta_k \leq 0$. Therefore if τ^{**} is defined by

$$\begin{aligned} \tau^{**}(x; \theta, c) &= \tau^{(n-k)}(x; (\theta_{k+1}, \ldots, \theta_n), (c_{k+1}, \ldots, c_n)) \\ &= \sum_{\epsilon \in \{0,1\}^{n-k}} \exp\left(\sum_{i=k+1}^{n} \epsilon_i \sqrt{c_i}(x + \theta_i) + \sum_{k+1 \leq i < j \leq n} \epsilon_i \epsilon_j A_{ij} \right), \end{aligned}$$

then for all $x \leq -\theta_k$, it follows, as in (3.21) and (3.22) that

$$\tau^{**}(x;\theta,c) \leq \tau(x;\theta,c) \tag{3.30}$$

and, for any multi-index M,

$$|\partial^M \tau(x;\theta,c) - \partial^M \tau^{**}(x;\tilde{\theta},c))| \leq C\tau^{**}(x)\exp(D(x+\theta_k)). \tag{3.31}$$

The arguments used above to deduce part (i) from (3.21) and (3.22) now allow us to deduce part (iii) from (3.30) and (3.31).

Next, observe that from (3.20) it follows easily that for each multi-index M such that $|M| \geq 1$, there exists a constant C such that, for all $x \leq -\theta_k$,

$$|\partial^M \tau^*(x;\theta,c)| \leq C\exp(D(x+\theta_k)). \tag{3.32}$$

Since $\tau^*(x;\theta,c) \geq 1$ for all x, (3.24) and (3.32) together imply statement (iv) of the lemma. The proof of statement (ii) of the lemma is similar: one starts from

$$\partial^N \phi^{**} = \frac{P(\tilde{\tau}^{**})}{(\tilde{\tau}^{**})^{|N|+2}},$$

where

$$\begin{aligned}\tilde{\tau}^{**} &= \tau^{**} \cdot \exp\left(-\sum_{i=k+1}^{n} \sqrt{c_i}(x+\theta_i))\right)\\ &= \sum_{\epsilon^{**}\in\{0,1\}^{n-k}} \exp\left(\sum_{i=k+1}^{n}(\epsilon_i - 1)\sqrt{c_i}(x+\theta_i) + \sum_{k+1\leq i<j\leq n}\epsilon_i\epsilon_j A_{ij}\right),\end{aligned}$$

(compare with (3.19)), and uses the estimate

$$|\partial^M \tilde{\tau}^{**}(x;\theta,c)| \leq C\exp(-D(x+\theta_{k+1})),$$

which is valid for all $|M| \geq 1$ and all $x \geq -\theta_{k+1}$.

Finally, (3.14) follows immediately from (i)-(iv). □

A related result which will find use below is the following.

Lemma 3.7. *Let n, c, and D be as defined in Lemma 3.6. There exists a constant $C = C(c,n,N)$ such that if $\theta = (\theta_1,\ldots,\theta_n) \in \mathbf{R}^n$ with $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$, then for all $x \geq -\theta_1$,*

$$|\partial^N \phi^{(n)}(x;\theta,c)| \leq C\exp(-D(x+\theta_1)) \tag{3.33}$$

and for all $x \leq -\theta_n$,

$$|\partial^N \phi^{(n)}(x;\theta,c)| \leq C\exp(D(x+\theta_n)). \tag{3.34}$$

Proof. Reference to (2.2) reveals that for any multi-index M, there exists C such that for all $x \le -\theta_n$,

$$|\partial^M \tau(x;\theta,c)| \le C(\exp(D(x+\theta_k)).$$

Since $\tau \ge 1$, (3.34) then follows from the formula

$$\partial^N \phi^{(n)} = \frac{P(\tau)}{\tau^{|N|+2}},$$

in which P is the same as in (3.23).

To prove (3.33), start from

$$\partial^N \phi^{(n)} = \frac{P(\tilde\tau)}{\tilde\tau^{|N|+2}},$$

where

$$\tilde\tau = \tau \cdot \exp\left(-\sum_{i=1}^{n} \sqrt{c_i}(x+\theta_i)\right).$$

As in the proof of part (ii) of Lemma 3.6, we obtain that

$$|\partial^M \tilde\tau(x;\theta,c)| \le C(\exp(-D(x+\theta_1))$$

for all multi-indices M and all $x \ge -\theta_1$; inequality (3.33) follows since $\tilde\tau \ge 1$. □

Lemma 3.8. *Suppose $n \in \mathbf{N}$ and $c \in S_n$ are given. For every multi-index N there exists a constant $\tilde C = \tilde C(c,n,N)$ such that for every $\theta \in \mathbf{R}^n$,*

$$\int_{-\infty}^{\infty} \left(\sup_{|\zeta-\theta|\le 1} \left|\partial^N \phi^{(n)}(x;\zeta,c)\right| \right) \, dx \le \tilde C. \tag{3.35}$$

In particular,

$$\int_{-\infty}^{\infty} \left|\partial^N \phi^{(n)}(x;\theta,c)\right| \, dx \le \tilde C.$$

Proof. Fix N and use induction on n. For $n=1$, $\phi^{(1)}(x;\theta,c) = \phi(x+\theta;0,c)$ for all $x,\theta \in \mathbf{R}$, so the integral on the left-hand side of (3.35) is independent of θ. We may assume therefore that $\theta = 0$, and hence the supremum in the integrand is taken over $|\zeta| \le 1$. But if $|\zeta| \le 1$, then from Lemma 3.7, it follows that

$$|\partial^N \phi^{(1)}(x;\zeta,c)| \le \begin{cases} C\exp(-D(x-1)) & \text{for all } x \ge 1 \\ C\exp(D(x+1)) & \text{for all } x \le -1. \end{cases}$$

For the remaining values, $-1 \le x \le 1$, (3.10) implies that

$$|\partial^N \phi^{(1)}(x;\zeta,c)| \le B,$$

and hence (3.35) follows.

Next, assume the desired constants $\tilde{C}(c,i,N)$ have been proved to exist for all $i \in \{1,2,\ldots,n-1\}$ and $c \in S_i$. On this basis, we now establish the existence of $\tilde{C}(c,n,N)$ for all $c \in S_n$.

Let $c \in S_n$ and $\theta \in \mathbf{R}^n$ be given; by relabelling the indices, assume that $\theta_i \le \theta_{i+1}$ for $1 \le i \le n-1$. Moreover, since

$$\phi^{(n)}(x;\theta,c) = \phi^{(n)}(x+\theta_1;(0,\theta_2-\theta_1,\ldots,\theta_n-\theta_1),c),$$

we may also assume that $\theta_1 = 0$.

Suppose first that $\theta_{k+1}-\theta_k \le 2$ for all $k = 1,\ldots,n-1$; then $\theta_n < 2n$, and so $|\zeta-\theta| \le 1$ implies $\zeta_1 \ge -1$ and $\zeta_n \le 2n+1$. Hence from Lemma 3.7, it is deduced that

$$|\partial^N \phi^{(1)}(x;\zeta,c)| \le \begin{cases} C\exp(-D(x-(2n+1))) & \text{for all } x \ge 2n+1 \\ C\exp(D(x+1)) & \text{for all } x \le -1, \end{cases}$$

and, as in the case when $n = 1$, this estimate together with (3.10) yields the desired result.

Now suppose on the contrary that there exists some $k \in \{1,\ldots,n-1\}$ such that $\theta_{k+1}-\theta_k > 2$. If $\zeta \in \mathbf{R}^n$ satisfies $|\zeta-\theta| \le 1$ it follows that $\zeta_i < \zeta_{k+1}$ for all $i \in \{1,\ldots,k\}$ and $\zeta_i > \zeta_{k+2}$ for all $i \in \{k+1,\ldots,n\}$. Hence if we define $\phi^*(x;\zeta,c)$, $\phi^{**}(x;\zeta,c)$ and $\tilde{\zeta}$ by replacing θ_i by ζ_i on the right-hand sides of (3.11), (3.12), and (3.13), respectively, then the conclusions of Lemma 3.6 hold with θ and $\tilde{\theta}$ replaced by ζ and $\tilde{\zeta}$.

Applying the induction hypothesis, there obtains the estimates

$$\int_{-\infty}^{\infty} \left(\sup_{|\tilde{\zeta}-\tilde{\theta}| \le 1} \left| \partial^N \phi^*(x;\tilde{\zeta},c) \right| \right) \, dx \le \tilde{C}_1$$

and

$$\int_{-\infty}^{\infty} \left(\sup_{|\tilde{\zeta}-\tilde{\theta}| \le 1} \left| \partial^N \phi^{**}(x;\tilde{\zeta},c) \right| \right) \, dx \le \tilde{C}_2.$$

where $\tilde{C}_1 = \tilde{C}((c_1,\ldots,c_k),k,N)$ and $\tilde{C}_2 = \tilde{C}((c_{k+1},\ldots,c_n),n-k,N)$. Moreover, it is easy to see that since $|\zeta-\theta| \le 1$, the function p_ζ defined by (3.15) satisfies $p_\zeta(x) \ge q(x)$ for all $x \in \mathbf{R}$, where

$$q(x) = \left| x + \left(\frac{\theta_{k+1}+\theta_k}{2} \right) \right|,$$

and hence $\int_{-\infty}^{\infty} \exp(-Dq(x))\ dx = \int_{-\infty}^{\infty} \exp(-D|u|)\ du = 2/D$. It therefore follows from (3.14) that

$$\int_{-\infty}^{\infty} \left(\sup_{|\zeta-\theta|\leq 1} \left|\partial^N \phi^{(n)}(x;\zeta,c)\right| \right)\ dx \leq \tilde{C}_1 + \tilde{C}_2 + C(c,n,N)D$$

where $C(c,n,N)$ is as defined in Lemma 3.6. The induction is completed by defining $\tilde{C}(c,n,N)$ to equal $\tilde{C}_1 + \tilde{C}_2 + C(c,n,N)D$. □

Lemma 3.9. *Suppose $n \in \mathbf{N}$ and $c \in S_n$ are given. For every multi-index N there exists $C > 0$ such that*

$$\int_{-\infty}^{\infty} \left(\sup_{|\zeta-\theta|\leq 1} \left|\partial^N \phi^{(n)}(x;\theta,c)\right| \right)^2\ dx \leq C,$$

and in particular

$$\int_{-\infty}^{\infty} \left|\partial^N \phi^{(n)}(x;\theta,c)\right|^2\ dx \leq C,$$

for all $\theta \in \mathbf{R}^n$.

Proof. This follows immediately from Lemmas 3.5 and 3.8. □

Lemma 3.10. *Suppose $n \in \mathbf{N}$ and $c \in S_n$ are given. For $\theta \in \mathbf{R}^n$, define $d^{(n)}(\theta,c)$ to be the determinant of $G_\theta(v_\theta,\theta)$. There exists $\alpha = \alpha(c,n) > 0$ such that for all $\theta \in \mathbf{R}^n$,*

$$\left|d^{(n)}(\theta,c)\right| > \alpha.$$

Proof. First, rewrite the matrix G_θ in a more convenient form. Notice that taking the derivative of (3.2) with respect to θ_j yields, for each i and j in $\{1,\dots,n\}$, the equation

$$\int_{-\infty}^{\infty} \phi\phi_{ij}\ dx = -\int_{-\infty}^{\infty} \phi_i\phi_j\ dx. \tag{3.36}$$

Therefore, if $P^{(n)}(\theta,c)$ is defined to be the matrix whose (i,j) entry is

$$P_{ij}^{(n)}(\theta,c) = \int_{-\infty}^{\infty} \phi_i\phi_j\ dx, \tag{3.37}$$

then

$$G_\theta(v_\theta,\theta) = -P^{(n)}(\theta,c). \tag{3.38}$$

Hence, in particular, $|d^{(n)}(\theta,c)| = |\det P^{(n)}(\theta,c)|$.

As in the proof of Lemma 3.8, use induction on n, although here the argument is a bit more elaborate. First, since $d^{(1)}(\theta,c)$ is independent of $\theta \in \mathbf{R}$, the desired conclusion obviously holds for $n=1$. Assume that the desired numbers $\alpha(c,i)$ have been proved to exist for $i=1,2,\ldots,n-1$ (for all $c \in S_i$); we intend to prove that for all $c \in S_n$, an appropriate constant $\alpha(c,n)$ exists. Let $c \in S_n$ and $\theta \in \mathbf{R}^n$ be given; by relabelling the indices, assume that $\theta_i \le \theta_{i+1}$ for $1 \le i \le n-1$. Also, as in the proof of Lemma 3.8, assume that θ_1=0.

Let $M = M(\theta)$ be defined by

$$M = \max_{1\le i\le n-1} (\theta_{i+1} - \theta_i),$$

and choose $k \in \{1,\ldots,n-1\}$ so that $\theta_{k+1} - \theta_k = M$. For this k, let ϕ^* and ϕ^{**} be as in Lemma 3.6. For $1 \le l,m \le n$, define

$$\beta_{lm} = \begin{cases} P_{lm}^{(n)} - \int_{-\infty}^{\infty} \phi_l^* \phi_m^* \, dx & \text{if } l,m \in \{1,\ldots,k\} \\ P_{lm}^{(n)} - \int_{-\infty}^{\infty} \phi_l^{**} \phi_m^{**} \, dx & \text{if } l,m \in \{k+1,\ldots,n\} \\ P_{lm}^{(n)} & \text{if } l \in \{1,\ldots,k\} \text{ and } m \in \{k+1,\ldots,n\} \\ P_{lm}^{(n)} & \text{if } l \in \{k+1,\ldots,n\} \text{ and } m \in \{1,\ldots,k\}, \end{cases}$$

where $P_{lm}^{(n)}$ is as defined in (3.37) and, as usual, the subscripts on ϕ^* and ϕ^{**} denote partial derivatives.

We now claim that the estimate

$$|\beta_{lm}| \le C e^{-DM/2} \tag{3.39}$$

holds for all $\theta \in \mathbf{R}^n$ and for all $l,m \in \{1,\ldots,n\}$, with a constant C which is independent of θ. To prove this, consider first the case when $l,m \in \{1,\ldots,k\}$. From Lemma 3.5, we have the estimate

$$\begin{aligned} |\beta_{lm}| &= \left| \int_{-\infty}^{\infty} \left(\phi_l^{(n)} - \phi_l^*\right) \phi_m^{(n)} \, dx + \int_{-\infty}^{\infty} \phi_l^* \left(\phi_m^{(n)} - \phi_m^*\right) \, dx \right| \\ &\le C \int_{-\infty}^{\infty} |\phi_l^{(n)} - \phi_l^*| \, dx + C \int_{-\infty}^{\infty} |\phi_m^{(n)} - \phi_m^*| \, dx. \end{aligned} \tag{3.40}$$

Using parts (i), (iii), and (iv) of Lemma 3.6, and observing that $\phi_l^{**} \equiv 0$ leads to the conclusion

$$\begin{aligned}\int_{-\infty}^{\infty} |\phi_l^{(n)} - \phi_l^*|\, dx &\le \int_{-\infty}^{-\theta_k - M/2} \left(|\phi_l^{(n)} - \phi_l^{**}| + |\phi_l^*|\right)\, dx + \int_{-\theta_k - M/2}^{\infty} |\phi_l^{(n)} - \phi_l^*|\, dx \\ &\le C \int_{-\infty}^{-M/2} e^{Du}\, du + C \int_{M/2}^{\infty} e^{-Du}\, du \\ &\le (C/D) e^{-DM/2}.\end{aligned}$$

The same estimate applies of course to the second integral on the right-hand side of (3.40). It follows that (3.39) holds in this case.

In the case when $l, m \in \{k+1, \dots, n\}$, (3.39) follows from a similar argument, this time using the fact that $\phi_l^* \equiv 0$ to write

$$\int_{-\infty}^{\infty} |\phi_l^{(n)} - \phi_l^{**}|\, dx \le \int_{-\infty}^{-\theta_k - M/2} |\phi_l^{(n)} - \phi_l^{**}|\, dx + \int_{-\theta_k - M/2}^{\infty} \left(|\phi_l^{(n)} - \phi_l^*| + |\phi_l^{**}|\right)\, dx,$$

and then using parts (i), (ii), (iii) of Lemma 3.6. Finally, if $l \in \{1, \dots, k\}$ and $m \in \{k+1, \dots, n\}$, then by Lemma (3.5), it is the case that

$$\begin{aligned}|\beta_{lm}| &\le C \int_{-\infty}^{-\theta_k - M/2} |\phi_l^{(n)}| + C \int_{-\theta_k - M/2}^{\infty} |\phi_m^{(n)}| \\ &= C \int_{-\infty}^{-\theta_k - M/2} |\phi_l^{(n)} - \phi_l^{**}| + C \int_{-\theta_k - M/2}^{\infty} |\phi_m^{(n)} - \phi_m^*|.\end{aligned}$$

It is therefore concluded from parts (i) and (iii) of Lemma 3.6 that (3.39) holds. The same argument obviously applies when the roles of l and m are reversed. Thus (3.39) is proved in all cases.

Now the $n \times n$ matrix $S^{(n)}$ with entries defined by $S_{lm}^{(n)} = P_{lm}^{(n)} - \beta_{lm}$ can be written in block form as

$$S^{(n)} = \begin{bmatrix} P^{(k)}(\theta^*, c^*) & 0 \\ 0 & P^{(n-k)}(\theta^{**}, c^{**}) \end{bmatrix}.$$

In consequence,

$$\det S^{(n)} = \left(\det P^{(k)}(\theta^*, c^*)\right) \left(\det P^{(n-k)}(\theta^{**}, c^{**})\right),$$

and so by the induction hypothesis

$$|\det S^{(n)}| > \alpha(c^*, k) \cdot \alpha(c^{**}, n-k). \tag{3.41}$$

From Lemma 3.9 and (3.39), it is seen that the matrix norms $\|P^{(n)}\|_\infty$ and $\|S^{(n)}\|_\infty$ are bounded independently of θ. Since the determinant of a matrix is a polynomial function of the entries of the matrix, it follows easily from the Mean Value Theorem that

$$|\det P^{(n)} - \det S^{(n)}| \le C\|P^{(n)} - S^{(n)}\|_\infty \tag{3.42}$$

where C depends only on $\|P^{(n)}\|_\infty$ and $\|S^{(n)}\|_\infty$, and therefore is independent of θ. Combining (3.39), (3.41), and (3.42) yields the estimate

$$|d^{(n)}(\theta, c)| = |\det P^{(n)}| > \alpha_1 - Ce^{-DM/2} \tag{3.43}$$

where

$$\alpha_1 = \inf_{1 \le k \le n-1} \{\alpha(c^*, k) \cdot \alpha(c^{**}, n-k)\} > 0.$$

Choose M_0 so large that the right-hand side of (3.43) is greater than $\alpha_1/2$ for $M \ge M_0$. For any given $\theta \in \mathbf{R}^n$, there are two possibilities: either $\theta_n > M_0 n$ or $\theta_n \le M_0 n$. Since $\theta_0 = 0$, then in the first case, it must be the case that

$$M = \max_{1 \le i \le n-1} (\theta_{i+1} - \theta_i) \ge M_0,$$

so (3.43) yields

$$|d^{(n)}(\theta, c)| > \alpha_1/2.$$

In the second case, the vector θ is an element of the subset

$$K = \{\theta \in \mathbf{R}^n : \max_{1 \le i \le n} |\theta_i| \le M_0 n\}.$$

Observe, however, that $d^{(n)}(\theta, c) \ne 0$ for all $\theta \in \mathbf{R}^n$. This follows from Lemma 3.4 and the elementary fact that whenever $v_1, \dots v_n$ are linearly independent vectors in an inner product space, then

$$\det\langle v_i, v_j\rangle_{i,j=1,n} \ne 0$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. Therefore, since K is compact and $|d^{(n)}(\theta, c)|$ is continuous and positive everywhere on K, there exists $\alpha_2 > 0$ such that $|d^{(n)}(\theta, c)| > \alpha_2$ for all $\theta \in K$. This completes the proof of the Lemma with

$$\alpha(c, n) = \min(\alpha_1/2, \alpha_2).$$

□

Next are established a couple of lemmas which will be needed to piece together the local functions obtained from Theorem 3.3 to obtain a global function defined on a neighborhood of M_c. The following notation will be convenient when we have to deal simultaneously with k-soliton solutions corresponding to different values of k in $\{1, \ldots, n\}$. Let $c = (c_1, \ldots, c_n) \in S_n$ be fixed. For each $\theta = (\theta_1, \ldots, \theta_n) \in (\mathbf{R} \cup \{\infty\})$, let $I_\theta = \{i \in \{1, \ldots, n\} : \theta_i < \infty\}$ and let $k = |I_\theta|$, the number of elements in I_θ. If $k = 0$, define $v_\theta = 0$, otherwise define

$$v_\theta(x) = \phi^{(k)}(x; \theta^\#, c^\#)$$

where $\theta^\#$ and $c^\#$ are the ordered k-tuples obtained by removing the infinite components from θ and c. Thus, for example, when $n = 7$ and $I_\theta = \{1, 4, 6, 7\}$, then $v_\theta = \phi^{(4)}(x; (\theta_1, \theta_4, \theta_6, \theta_7), (c_1, c_4, c_6, c_7))$. In the case when $\theta \in \mathbf{R}^n$, this definition of v_θ coincides with that given above in (3.1).

Lemma 3.11. *Suppose $\theta, \tilde{\theta} \in (\mathbf{R} \cup \{\infty\})^n$. Then $v_\theta = v_{\tilde{\theta}}$ only if $\theta = \tilde{\theta}$.*

Proof. For $\theta \in (\mathbf{R} \cup \{\infty\})^n$ and $\epsilon \in \{0, 1\}^n$, define $a(\epsilon, \theta)$ by setting $a(\epsilon, \theta) = 0$ if $\epsilon_i = 1$ for some $i \notin I_\theta$, and $a(\epsilon, \theta) = 1$ otherwise. Write v_θ in the form

$$v_\theta = 12 \frac{d^2}{dx^2} \log \tau_\theta(x)$$

where

$$\tau_\theta(x) = \sum_{\epsilon \in \{0,1\}^n} a(\epsilon, \theta) \exp\left(\sum_{i=1}^n \epsilon_i \sqrt{c_i}(x + \theta_i) + \sum_{1 \le i < j \le n} \epsilon_i \epsilon_j A_{ij} \right),$$

and use the convention that terms of the form $0 \cdot \infty$ are equal to zero. If $b(\epsilon, \theta)$ and d_ϵ are given by

$$b(\epsilon, \theta) = \exp\left(\sum_{i=1}^n \epsilon_i \sqrt{c_i} \theta_i + \sum_{1 \le i < j \le n} \epsilon_i \epsilon_j A_{ij} \right)$$

and

$$d_\epsilon = \sum_{i=1}^n \epsilon_i \sqrt{c_i},$$

then

$$\tau_\theta(x) = \sum_{\epsilon \in \{0,1\}^n} a(\epsilon, \theta) b(\epsilon, \theta) e^{d_\epsilon x}.$$

If $v_\theta = v_{\tilde{\theta}}$, it follows that for all $x \in \mathbf{R}$,

$$\tau_\theta(x) = \tau_{\tilde{\theta}}(x) e^{px+q},$$

where p and q are constants. Hence

$$\sum_{\epsilon\in\{0,1\}^n} a(\epsilon,\theta)b(\epsilon,\theta)e^{d_\epsilon x} = \sum_{\epsilon\in\{0,1\}^n} a(\epsilon,\tilde{\theta})b(\epsilon,\tilde{\theta})e^{q}e^{(p+d_\epsilon)x}. \tag{3.44}$$

Since the functions e^{rx} and e^{sx} are linearly independent on $\mathbf{R}$ whenever $r \neq s$, every exponential term which appears on the left of (3.44) with a non-zero coefficient must also appear on the right. In particular, the left side of (3.44) contains the term $e^{0\cdot x}$ corresponding to $\epsilon = \epsilon_0 = (0, 0, \ldots, 0)$ (notice that $a(\epsilon_0, \theta) = b(\epsilon_0, \theta) = 1$, and that $d_\epsilon > 0$ if $\epsilon \neq \epsilon_0$). Hence, there must exist at least one $\epsilon_1 \in \{0,1\}^n$ for which $p + d_{\epsilon_1} = 0$. But ϵ_1 must equal ϵ_0, for otherwise $d_{\epsilon_1} > 0$ would imply $p < 0$, and then the term $e^{(p+d_{\epsilon_0})x} = e^{px}$ appearing on the right-hand side of (3.44) would not correspond to any term on the left-hand side of (3.44). Therefore, $p = -d_{\epsilon_0} = 0$, and comparing coefficients of $e^{0\cdot x}$ on both sides of (3.44) then gives $q = 0$. It is thus demonstrated that

$$\sum_{\epsilon\in\{0,1\}^n} a(\epsilon,\theta)b(\epsilon,\theta)e^{d_\epsilon x} = \sum_{\epsilon\in\{0,1\}^n} a(\epsilon,\tilde{\theta})b(\epsilon,\tilde{\theta})e^{d_\epsilon x} \tag{3.45}$$

holds for all $x \in \mathbf{R}$.

Now consider the terms in (3.45) corresponding to $\epsilon = e_i$, where e_i is the standard basic n-tuple defined by $(e_i)_j = \delta_{ij}$. By permuting the indices, it may be assumed that $\sqrt{c_1} < \cdots < \sqrt{c_n}$. Then, $d_{e_1} < d_\epsilon$ for all $\epsilon \in \{0,1\}^n$ such that $\epsilon \neq \epsilon_0$ and $\epsilon \neq e_1$. The identity (3.45) implies $a(e_1,\theta)b(e_1,\theta) = a(e_1,\tilde{\theta})b(e_1,\tilde{\theta})$. From the definitions of $a(\epsilon,\theta)$ and $b(\epsilon,\theta)$, we see that this in turn implies that $\theta_1 = \tilde{\theta}_1$.

To finish, use induction to prove $\theta_k = \tilde{\theta}_k$ for all $k \in \{1,\ldots,n\}$. Assume $\theta_i = \tilde{\theta}_i$ for all $1 \leq i \leq k-1$, and let $r = d_{e_k}$. If $d_\epsilon = r$ for any $\epsilon \in \{0,1\}^n$ with $\epsilon \neq e_k$, then we must have $\epsilon_i = 0$ for $i \geq k$, in which case it follows from the induction hypothesis that $a(\epsilon,\theta)b(\epsilon,\theta) = a(\epsilon,\tilde{\theta})b(\epsilon,\tilde{\theta})$. Therefore all the terms on the left of (3.45) which contain e^{rx} and correspond to $\epsilon \neq e_k$ will balance with equal terms on the right of (3.45). But then the identity (3.45) implies that the terms corresponding to $\epsilon = e_k$ must be equal as well, which implies that $\theta_k = \tilde{\theta}_k$. □

Lemma 3.12. *Suppose $n \in \mathbf{N}$, let $c \in S_n$ be fixed, and for $\theta \in (\mathbf{R}\cup\{\infty\})^n$ define I_θ and $\theta^\#$ as before Lemma 3.11. For every $\eta > 0$ there exists $\delta > 0$ such that if $\theta, \tilde{\theta} \in (\mathbf{R}\cup\{\infty\})^n$ with $|I_\theta| + |I_{\tilde{\theta}}| \geq 1$ and $\|v_\theta - v_{\tilde{\theta}}\|_1 < \delta$, then $I_\theta = I_{\tilde{\theta}}$ and $|\theta^\# - \tilde{\theta}^\#| < \eta$.*

Proof. It is required to show that for every $\eta > 0$ there exists $\delta > 0$ such that if $I_\theta \neq I_{\tilde{\theta}}$, or $I_\theta = I_{\tilde{\theta}}$ and $|\theta^\# - \tilde{\theta}^\#| \geq \eta$, then $\|v_\theta - v_{\tilde{\theta}}\|_1 \geq \delta$. To prove this, we use induction on $|I_\theta| + |I_{\tilde{\theta}}|$.

Suppose first that $|I_\theta| + |I_{\tilde{\theta}}| = 1$; then necessarily $I_\theta \neq I_{\tilde{\theta}}$. Without loss of generality, assume that $|I_{\tilde{\theta}}| = 0$, $v_{\tilde{\theta}} = 0$, and $|I_\theta| = 1$. Since the set $S =$

$\{\|v_\theta\|_1 : \theta \in (\mathbf{R} \cup \{\infty\})^n$ and $|I_\theta| = 1\}$ contains only a finite number of positive elements, clearly $\delta = \min S > 0$, and since $\|v_\theta - v_{\tilde\theta}\|_1 = \|v_\theta\|_1 \geq \delta$, the result follows in this case.

Make the induction hypothesis that for every $i \in \{1, \ldots, l-1\}$ and every $\eta > 0$ there exists $\delta_i(\eta) > 0$ such that if $|I_\theta| + |I_{\tilde\theta}| = i$, and either $I_\theta \neq I_{\tilde\theta}$ or $I_\theta = I_{\tilde\theta}$ and $|\theta^\# - \tilde\theta^\#| \geq \eta$, we have $\|v_\theta - v_{\tilde\theta}\|_1 \geq \delta_i(\eta)$. In particular, taking $\tilde\theta = (\infty, \ldots, \infty)$ so that $v_{\tilde\theta} = 0$ and $|I_{\tilde\theta}| = 0$, it follows from the induction hypothesis that for every $i \in \{1, \ldots . l-1\}$ and every $\eta > 0$,

$$\|v_\theta\|_1 = \|v_\theta - v_{\tilde\theta}\|_1 \geq \delta_i(\eta) \tag{3.46}$$

whenever $|I_\theta| = i$.

Let $\eta > 0$ be given and assume $|I_\theta| + |I_{\tilde\theta}| = l \geq 1$. We aim to show the existence of the desired $\delta > 0$ both in case $I_\theta \neq I_{\tilde\theta}$ and in case $I_\theta = I_{\tilde\theta}$ and $|\theta^\# - \tilde\theta^\#| \geq \eta$.

Assume first that $I_\theta = I_{\tilde\theta}$ and $|\theta^\# - \tilde\theta^\#| \geq \eta$. For ease of notation, write θ in place of $\theta^\#$ and $\tilde\theta$ in place of $\tilde\theta^\#$, so that $\theta, \tilde\theta \in \mathbf{R}^m$ where $2m = l$. By subtracting a common constant from all components of θ and $\tilde\theta$, it can be presumed that

$$\min\{\tilde\theta_1, \ldots, \tilde\theta_m\} = 0.$$

Let $M > 0$ be fixed but arbitrary for the moment (a value of M will be chosen later). To obtain estimates on $\|v_\theta - v_{\tilde\theta}\|_1$, several cases are considered, according to the location of the components of θ and $\tilde\theta$ with respect to the interval $[0, M]$.

Case I. Suppose $\{\tilde\theta_1, \ldots, \tilde\theta_m\} \subseteq [0, M]$ and $\{\theta_1, \ldots, \theta_m\} \subseteq [-M, 2M]$. Define the function $f(\theta, \tilde\theta) : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ by $f(\theta, \tilde\theta) = \|v_\theta - v_{\tilde\theta}\|_1$; then f is continuous and is positive on $\{(\theta, \tilde\theta) : |\theta - \tilde\theta| \geq \eta\}$ by Lemma 3.11. Since the set

$$T = \{(\theta, \tilde\theta) : \ \{\tilde\theta_1, \ldots, \tilde\theta_m\} \subseteq [0, M],\ \{\theta_1, \ldots, \theta_m\} \subseteq [-M, 2M], \text{ and } |\theta - \tilde\theta| \geq \eta\}$$

is compact in $\mathbf{R}^m \times \mathbf{R}^m$, there exists $\delta_0 = \delta_0(M) > 0$ such that $f(\theta, \tilde\theta) > \delta_0$ for all $(\theta, \tilde\theta) \in T$.

Case II. Suppose $\{\tilde\theta_1, \ldots, \tilde\theta_m\} \subseteq [0, M]$ and $\theta_i < -M$ for some i. By permuting the indices we may assume that $\theta_1 \leq \cdots \leq \theta_m$ and $\theta_1 < -M$. Split the interval $[-M, 0]$ into m subintervals of equal length M/m. At least one of these subintervals must have interior disjoint from the set $\{\theta_1, \ldots, \theta_m\}$. Therefore either $\theta_m \leq -M/m$, or there exists $k \in \{1, \ldots, m-1\}$ such that $\theta_{k+1} - \theta_k \geq M/m$. If $\theta_m \leq -M/m$, we obtain from Lemmas 3.7 and 3.9 that there exists a constant C, which is independent of θ and $\tilde\theta$, such that

$$\|v_\theta - v_{\tilde\theta}\|_1^2 \geq \|v_\theta\|_1^2 + \|v_{\tilde\theta}\|_1^2 - C\exp(-DM/(2m)).$$

It follows that for M sufficiently large,

$$\|v_\theta - v_{\tilde\theta}\|_1 \geq \frac{1}{4}\left(\|v_\theta\|_1 + \|v_{\tilde\theta}\|_1\right) \geq \frac{1}{2}\inf\{\|v_\theta\|_1 : \theta \in \mathbf{R}^m\} \geq \frac{1}{2}\delta_m(\eta)$$

where (3.46) has been used.

If, on the other hand, $\theta_{k+1} - \theta_k \geq M/m$ for some $k \in \{1, \ldots, m-1\}$, then it follows from Lemma 3.6 that

$$\|v_\theta - v_{\tilde{\theta}}\|_1^2 \geq \|\phi^*\|_1^2 + \|\phi^{**} - v_{\tilde{\theta}}\|_1^2 - C\exp(-DM/(2m))$$

where ϕ^* and ϕ^{**} are as defined in (3.11) and (3.12), and C is independent of θ and $\tilde{\theta}$. Hence for M sufficiently large,

$$\|v_\theta - v_{\tilde{\theta}}\|_1 \geq \frac{1}{2}\|\phi^*\|_1.$$

Since $\phi^* = v_{\theta^*}$, where

$$\theta_i^* = \begin{cases} \theta_i & \text{if } 1 \leq i \leq k \\ \infty & \text{if } k+1 \leq i \leq m, \end{cases}$$

(3.46) then implies that

$$\|v_\theta - v_{\tilde{\theta}}\|_1 \geq \frac{1}{2}\delta_k(\eta). \tag{3.47}$$

Case III. Suppose $\{\tilde{\theta}_1, \ldots, \tilde{\theta}_m\} \subseteq [0, M]$ and $\theta_i > 2M$ for some i. Again by permuting the indices, it may be taken that $\theta_1 \leq \cdots \leq \theta_m$ and $\theta_m > 2M$. Split the interval $[M, 2M]$ into m subintervals of equal length, at least one of which must have interior disjoint from the set $\{\theta_1, \ldots, \theta_m\}$. Hence, either $\theta_1 \geq M + M/m$, or there exists $k \in \{1, \ldots, m-1\}$ such that $\theta_{k+1} - \theta_k \geq M/m$. The same argument as used in Case II then shows that for M sufficiently large,

$$\|v_\theta - v_{\tilde{\theta}}\|_1 \geq \frac{1}{2}\min(\delta_m(\eta), \delta_{m-k}(\eta)). \tag{3.48}$$

Case IV. Suppose $\tilde{\theta}_i > M$ for some $i \in \{1, \ldots, m\}$. As usual, assume without loss of generality that $0 = \tilde{\theta}_1 \leq \cdots \leq \tilde{\theta}_m$ and $\tilde{\theta}_m > M$. Then there exists $k \in \{1, \ldots, m-1\}$ such that $\tilde{\theta}_{k+1} - \tilde{\theta}_k > M/m$. Split $[\tilde{\theta}_k, \tilde{\theta}_{k+1}]$ into $m+1$ subintervals of equal length. At least one of these subintervals has interior disjoint from $\{\theta_1, \ldots, \theta_m\}$. Choose such a subinterval and denote it by $[a, b]$; of course, $b - a > M/(m(m+1))$. Define $\theta_i^* = \theta_i$ if $\theta_i \leq a$ and $\theta_i^* = \infty$ otherwise, and define $\theta_i^{**} = \theta_i$ if $\theta_i \geq b$ and $\theta_i^{**} = \infty$ otherwise. Similarly, define $\tilde{\theta}_i^* = \tilde{\theta}_i$ for $1 \leq i \leq k$ and $\tilde{\theta}_i^* = \infty$ otherwise; and define $\tilde{\theta}_i^{**} = \tilde{\theta}_i$ for $k+1 \leq i \leq m$ and $\tilde{\theta}_i^{**} = \infty$ otherwise. Then from Lemma 3.6, it may be concluded that

$$\|v_{\tilde{\theta}} - v_\theta\|_1^2 = \|v_{\tilde{\theta}^*} - v_{\theta^*}\|_1^2 + \|v_{\tilde{\theta}^{**}} - v_{\theta^{**}}\|_1^2 - C\exp(-DM/m(m+1))$$

where C is independent of θ and $\tilde{\theta}$. Hence, for M sufficiently large, we have

$$\|v_{\tilde{\theta}} - v_\theta\|_1 \geq \frac{1}{4}\left(\|v_{\tilde{\theta}^*} - v_{\theta^*}\|_1 + \|v_{\tilde{\theta}^{**}} - v_{\theta^{**}}\|_1\right). \tag{3.49}$$

In the current situation,

$$1 \leq |I_{\tilde{\theta}^*}| + |I_{\theta^*}| \leq k + m \leq 2m - 1 < l \quad \text{and}$$
$$1 \leq |I_{\tilde{\theta}^{**}}| + |I_{\theta^{**}}| \leq (m-k) + m \leq 2m - 1 < l,$$

so the induction hypothesis can be applied to both the terms on the right-hand side of (3.49). If $I_{\tilde{\theta}^*} \neq I_{\theta^*}$ or $I_{\tilde{\theta}^{**}} \neq I_{\theta^{**}}$, then (3.49) yields

$$\|v_{\tilde{\theta}} - v_\theta\|_1 \geq \frac{1}{2}\left(\min_{1\leq i\leq l-1} \delta_i(\eta)\right). \tag{3.50}$$

The remaining possibility is that $I_{\tilde{\theta}^*} = I_{\theta^*}$ and $I_{\tilde{\theta}^{**}} = I_{\theta^{**}}$. But, in that case, since $|\theta - \tilde{\theta}| \geq \eta$, we must have either $|\theta^* - \tilde{\theta}^*| \geq \eta/2$ or $|\theta^{**} - \tilde{\theta}^{**}| \geq \eta/2$, and so (3.49) yields

$$\|v_{\tilde{\theta}} - v_\theta\|_1 \geq \frac{1}{4}\left(\min_{1\leq i\leq l-1} \delta_i(\eta/2)\right). \tag{3.51}$$

Now choose M so large that all the estimates in Cases II through IV are valid (notice this can be done with an M whose value is independent of θ and $\tilde{\theta}$), and for such an M define δ to be the smallest of $\delta_0(M)$ and the numbers on the right-hand sides of (3.47), (3.48), (3.50), and (3.51). We then have $\|v_{\tilde{\theta}} - v_\theta\|_1 \geq \delta$ whenever $I_\theta = I_{\tilde{\theta}}$ and $|\theta^\# - \tilde{\theta}^\#| \geq \eta$, completing the inductive step in this case.

It remains to consider the possibility that $I_\theta \neq I_{\tilde{\theta}}$. Let $m = |I_\theta|$ and $\tilde{m} = |I_{\tilde{\theta}}|$, so that $m + \tilde{m} = l$. By switching m and $\tilde{m}$ if necessary, assume $m > 0$. Now the arguments used above in Cases I through IV can be repeated unchanged, with the understanding that $v_{\tilde{\theta}} = 0$ when $\tilde{m} = 0$, and the replacement of the set T in Case I by

$$T = \{(\theta, \tilde{\theta}) : \ \{\tilde{\theta}_1, \ldots, \tilde{\theta}_m\} \subseteq [0, M], \ \{\theta_1, \ldots, \theta_m\} \subseteq [-M, 2M]\}.$$

(Note in particular that Case IV can only arise when $\tilde{m} \geq 2$, and that in Case IV, the situation wherein $I_{\tilde{\theta}^*} = I_{\theta^*}$ and $I_{\tilde{\theta}^{**}} = I_{\theta^{**}}$ cannot now arise, since it would contradict $I_\theta \neq I_{\tilde{\theta}}$.) The induction is complete and the lemma proved. □

Proof of Theorem 3.1. From Lemma 3.9 and (3.38), it is known that the entries of $G_\theta(v_\theta, \theta)$ are bounded independently of $\theta \in \mathbf{R}^n$. Therefore, from Cramer's rule and Lemma 3.10, the entries of the inverse matrix $G_\theta(v_\theta, \theta)^{-1}$ are also bounded independently of $\theta \in \mathbf{R}^n$. Hence the quantity

$$K_1 = \sup_{\theta\in\mathbf{R}^n} \|G_\theta(v_\theta, \theta)^{-1}\|_{B(\mathbf{R}^n, \mathbf{R}^n)} \tag{3.52}$$

is finite.

Now let $\theta_0 \in \mathbf{R}^n$ be fixed but arbitrary, and let $u_0 = v_{\theta_0}$. For all $(u,\theta) \in H^1 \times \mathbf{R}^n$, (3.5) implies that

$$\begin{aligned} \|G_\theta(u,\theta) - G_\theta(u_0,\theta_0)\|_{B(\mathbf{R}^n,\mathbf{R}^n)} &\leq \sup_{1\leq i,j\leq n} \left| \int_{-\infty}^{\infty} u\phi_{ij}(\theta) - \int_{-\infty}^{\infty} u_0\phi_{ij}(\theta_0) \right| \\ &\leq \sup_{1\leq i,j\leq n} \left(|u|_2 |\phi_{ij}(\theta) - \phi_{ij}(\theta_0)|_2 + |u-u_0|_2 |\phi_{ij}(\theta_0)|_2 \right) \end{aligned} \tag{3.53}$$

where the L^2-norms are taken in the x variable.

We claim that

$$|\phi_{ij}(\theta) - \phi_{ij}(\theta_0)|_2 \leq M|\theta - \theta_0|, \tag{3.54}$$

where M is independent of θ. To prove (3.54), we might as well assume that $|\theta - \theta_0| \leq 1$, since $|\phi_{ij}(\theta)|_2$ is bounded independently of θ by Lemma 3.9. For each $x \in \mathbf{R}$, the Mean Value Theorem provides a $\zeta_x \in \mathbf{R}^n$ on the line segment between θ and θ_0, such that

$$\phi_{ij}(x;\theta,c) - \phi_{ij}(x;\theta_0,c) = \sum_{k=1}^{n} \phi_{ijk}(x;\zeta_x,c)\,(\theta-\theta_0)_k\,.$$

It follows that

$$|\phi_{ij}(x;\theta,c) - \phi_{ij}(x;\theta_0,c)| \leq \sup_k \sup_{|\zeta-\theta_0|\leq 1} |\phi_{ijk}(x;\zeta,c)| \cdot |\theta - \theta_0|,$$

and (3.54) then follows from Lemma 3.9.

Now let U be the ball of radius 1 centered at u_0 in H^1, so that for all $u \in U$, $|u|_2 \leq 1 + |u_0|_2 = 1 + |\phi_{\theta_0}|_2$. The inequalities (3.53), (3.54) and Lemma 3.9 yield that

$$\|G_\theta(u,\theta) - G_\theta(u_0,\theta_0)\|_{B(\mathbf{R}^n,\mathbf{R}^n)} \leq K_2\left(\|u-u_0\|_1 + |\theta-\theta_0|\right)$$

for all $(u,\theta) \in U \times \mathbf{R}^n$, where the constant K_2 can be taken to be independent of $\theta_0 \in \mathbf{R}^n$. Similarly, we have that

$$|G(u,\theta_0)| = |G(u,\theta_0) - G(u_0,\theta_0)| \leq |u-u_0|_2 \left(\sup_{1\leq i\leq n} |\phi_i|_2 \right) \leq K_3|u-u_0|_2,$$

for all $(u,\theta) \in U \times \mathbf{R}^n$, where K_3 can be chosen independently of θ_0.

Next, observe that

$$K_4 = \|G_u(u_0,\theta_0)\|_{B(H^1,\mathbf{R}^n)} \leq \sup_{1\leq i\leq n} |\phi_i(\theta_0)|_2,$$

which is uniformly bounded in θ_0 by Lemma 3.9; and for all $\psi \in H^1$ we have

$$
\begin{aligned}
|G_u(u,\theta)[\psi] - G_u(u_0,\theta_0)[\psi]| &\leq \sup_i \int_{-\infty}^{\infty} \psi\,[\phi_i(\theta) - \phi_i(\theta_0)] \\
&\leq |\psi|_2 \left(\sup_i |\phi_i(\theta) - \phi_i(\theta_0)|_2 \right).
\end{aligned}
\tag{3.55}
$$

The same argument used to prove (3.54) shows that the right-hand side of (3.55) is bounded by $K_5\|\psi\|_1|\theta - \theta_0|$, where K_5 can be chosen independently of θ_0. Therefore

$$
\|G_u(u,\theta) - G_u(u_0,\theta_0)\|_{B(H^1,\mathbf{R}^n)} \leq K_5\left(\|u - u_0\|_1 + |\theta - \theta_0|\right)
$$

for all $u \in H^1$ and $\theta \in \mathbf{R}^n$.

As a consequence of Theorem 3.3, there exist a number $\eta_1 > 0$ and a function $\delta(\eta)$ defined for $\eta \in (0,\eta_1]$ such that, for every $\theta \in \mathbf{R}^n$ and every $u \in B_{\delta(\eta)}(v_\theta)$, there is a unique point $F_\theta(u) \in B_\eta(\theta)$ such that $G(u, F_\theta(u)) = 0$. Moreover, since $G(u,\theta)$ is clearly C^∞ on $H^1 \times \mathbf{R}^n$, the map $F_\theta : B_{\delta(\eta)}(v_\theta) \to B_\eta(\theta)$ is C^∞.

Next, we claim that there exists a number $\delta_0 > 0$ with the property that whenever $\theta_1, \theta_2 \in \mathbf{R}^n$ and $u \in B_{\delta_0}(v_{\theta_1}) \cap B_{\delta_0}(v_{\theta_2})$, then $F_{\theta_1}(u) = F_{\theta_2}(u)$. To see this, let $\tilde{\eta} = \frac{1}{2}\eta_1$. By Lemma 3.12, there is a $\tilde{\delta} > 0$ such that if $|v_{\theta_1} - v_{\theta_2}| < \tilde{\delta}$, then $|\theta_1 - \theta_2| < \tilde{\eta}$. Define the quantity δ_0 by

$$
\delta_0 = \min(\delta(\tilde{\eta}), \delta(\eta_1), \frac{1}{2}\tilde{\delta}),
$$

and suppose $u \in B_{\delta_0}(v_{\theta_1}) \cap B_{\delta_0}(v_{\theta_2})$. Then $u \in B_{\delta(\eta_1)}(\theta_2)$, so $F_{\theta_2}(u)$ is the unique point in $B_{\eta_1}(\theta_2)$ such that $G(u, F_{\theta_2}(u)) = 0$. On the other hand, we have

$$
|v_{\theta_1} - v_{\theta_2}| \leq |v_{\theta_1} - u| + |v_{\theta_2} - u| < 2\delta_0 \leq \tilde{\delta},
$$

so $|\theta_1 - \theta_2| < \tilde{\eta}$. Moreover, $F_{\theta_1}(u) \in B_{\tilde{\eta}}(\theta_1)$ and hence

$$
|F_{\theta_1}(u) - \theta_2| \leq |F_{\theta_1}(u) - \theta_1| + |\theta_1 - \theta_2| < 2\tilde{\eta} = \eta_1.
$$

Therefore $F_{\theta_1}(u) \in B_{\eta_1}(\theta_2)$. But since $G(u, F_{\theta_1}(u)) = 0$, and $y = F_{\theta_2}(u)$ is the unique solution of $G(u,y) = 0$ in $B_{\eta_1}(\theta_2)$, it must be the case that $F_{\theta_1}(u) = F_{\theta_2}(u)$, as desired.

It follows from what has just been proved that the maps $F_\theta : B_{\delta_0}(v_\theta) \to \mathbf{R}^n$ piece together to form a globally defined map on the neighborhood U_{δ_0} of M_c. In other words, there is a well-defined map $F : U_{\delta_0} \to \mathbf{R}^n$ obtained by setting

$$
F(u) = F_\theta(u) \quad \text{if } u \in B_{\delta_0}(\theta).
$$

Since each F_θ is C^∞ and satisfies (3.4), the same is true of F. □

4 Proof of Theorem 2.4

Without loss of generality, presume that

$$\epsilon < \frac{\alpha(c,n)}{2C_1(c,n)} \ , \tag{4.1}$$

where $\alpha(c,n)$ is defined in Lemma 3.10, and $C_1(c,n)$ is the number defined below in (4.8). From Lemma 3.9 and the Mean Value Theorem (cf. the proof of (3.54)), there is an $\eta > 0$ such that whenever $\theta, \gamma \in \mathbf{R}^n$ satisfy $|\theta - \gamma| < \eta$, then $\|v_\theta - v_\gamma\|_1 < \epsilon/2$. Let $\delta_1 = \delta(\eta)$, where $\delta(\eta)$ is the function defined above in the proof of Theorem 3.1, so that whenever $\theta \in \mathbf{R}^n$ and $u \in B_{\delta_1}(v_\theta)$, then $F(u) \in B_\eta(\theta)$. Finally, let δ_0 be the number defined in the statement of Theorem 3.1. By Theorem 2.3, there is a $\delta > 0$ such that if $u_0 \in H^1$ and $\|u_0 - v_{\theta_0}\|_1 < \delta$ for some $\theta_0 \in \mathbf{R}^n$, then for all $t > 0$ there exists $\theta(t) \in \mathbf{R}^n$ such that

$$\|u(t) - v_{\theta(t)}\|_1 < \min(\delta_0, \delta_1, \epsilon/2). \tag{4.2}$$

In particular, $u(t) \in U_{\delta_0}$ for all $t > 0$, so by Theorem 3.1 we can define a function $\gamma : (0,\infty) \to \mathbf{R}^n$ by setting $\gamma(t) = F(u(t))$. Also, since the map $t \mapsto u(t)$ is continuous from $(0,\infty)$ to H^1 by Theorem 2.1, and $F : U_{\delta_0} \to \mathbf{R}^n$ is continuous by Theorem 3.1, then $\gamma(t)$ is a continuous function of t on $(0,\infty)$.

From (4.2) and the definition of the function F on U_{δ_0}, it follows that $\gamma(t) = F_{\theta(t)}(u(t))$ for all $t > 0$. Moreover, (4.2) implies that $u(t) \in B_{\delta_1}(\theta(t))$, or, in other words, $|\gamma(t) - \theta(t)| < \delta_1$, and hence that $\|v_{\theta(t)} - v_{\gamma(t)}\|_1 < \epsilon/2$. Therefore

$$\|u(t) - v_{\gamma(t)}\|_1 \le \|u(t) - v_{\theta(t)}\|_1 + \|v_{\theta(t)} - v_{\gamma(t)}\|_1 < \epsilon/2 + \epsilon/2 = \epsilon,$$

so proving (2.5).

It remains to show that $\gamma(t)$ is a C^1 function and satisfies (2.6). For this purpose, consider the function $H : (0,\infty) \times \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$H(t,\theta) = \left(\int_{-\infty}^{\infty} u(x,t)\phi_1^{(n)}(x;\theta,c)\ dx, \ldots, \int_{-\infty}^{\infty} u(x,t)\phi_n^{(n)}(x;\theta,c)\ dx \right).$$

The idea is to apply the Implicit Function Theorem to H at the points $(t, \gamma(t))$ where it is known that

$$H(t,\gamma(t)) = G(u(t), F(u(t))) = 0. \tag{4.3}$$

Fix $t_0 > 0$. Observe first that if U is any neighborhood of t_0 in $(0,\infty)$, then H is C^1 on $U \times \mathbf{R}^n$. In fact, the derivatives of H with respect to components of θ clearly exist up to any order. As for derivatives with respect to t, it is known from Theorem 2.1 that u is differentiable as a distribution-valued function of t with $u_t \in C([0,T], H^{-2})$. Hence for any function ψ in the Schwarz class $\mathcal{S}(\mathbf{R})$, the action

$$\langle u, \psi \rangle = \int_{-\infty}^{\infty} u(x,t)\psi(x)\, dx$$

of $u(t)$ on ψ will be a differentiable function of t, with derivative

$$\langle u_t, \psi \rangle = \langle -uu_x - u_{xxx}, \psi \rangle,$$

which is a continuous function of t. But since $\phi_i = \phi_i^{(n)}(x;\theta,c) \in \mathcal{S}(\mathbf{R})$ for each $i \in \{1,\dots,n\}$, it follows that H is continuously differentiable with respect to t, with the i^{th} component of H_t given by

$$H_t(t,\theta)_i = \langle u_t, \phi_i \rangle = \langle -u(t)u_x(t) - u_{xxx}(t), \phi_i \rangle. \tag{4.4}$$

Next, it is shown that the partial derivative $H_\theta(t_0, \gamma(t_0))$ is an invertible map from $\mathbf{R}^n$ to $\mathbf{R}^n$. To see this, observe that

$$H_\theta(t,\theta) = G_\theta(u(t),\theta)$$

for all $(t,\theta) \in (0,\infty) \times \mathbf{R}^n$, and hence

$$H_\theta(t_0,\gamma(t_0)) = G_\theta(u(t_0),\gamma(t_0)) = G_\theta(v_{\gamma(t_0)},\gamma(t_0)) + G(h, v_{\gamma(t_0)}), \tag{4.5}$$

where $h = u(t_0) - v_{\gamma(t_0)}$. But, we also know that

$$\left|\det G_\theta(v_{\gamma(t_0)},\gamma(t_0))\right| = \left|d^{(n)}(\gamma(t_0),c)\right| > \alpha(c,n) \tag{4.6}$$

by Lemma 3.10, and for all $i,j \in \{1,\dots,n\}$,

$$\left|G(h,v_{\gamma(t_0)})_{ij}\right| = \left|\int_{-\infty}^{\infty} h\phi_{ij}^{(n)}(x;\gamma(t_0),c)\, dx\right| \le |h|_2 |\phi_{ij}|_2 \le C\epsilon, \tag{4.7}$$

where C depends only on n and c, by Lemma 3.9 and (2.5). Combining (3.42), (4.5), (4.6), and (4.7) (and recalling that the matrix norm $\|G_\theta(v_\theta,\theta)\|_\infty$ is bounded independently of θ), it is deduced that there exists a number $C_1(c,n)$ such that

$$|\det H_\theta(t_0,\gamma(t_0))| \ge \alpha(c,n) - C_1(c,n)\epsilon. \tag{4.8}$$

From (4.1) it now follows that $H_\theta(t_0,\gamma(t_0))$ is invertible.

It follows from what has just been proved and Theorem 3.3 that there exist numbers $\eta > 0$ and $\delta > 0$ such that for every $t \in (t_0 - \delta, t_0 + \delta)$ there is exactly one vector $\zeta(t) \in B_\eta(\gamma(t_0))$ such that $H(t, \zeta(t)) = 0$, and the map $t \mapsto \zeta(t)$ is C^1. On the other hand, since $\gamma(t)$ is continuous, then there exists $\delta_1 \in (0, \delta)$ such that $\gamma(t) \in B_\eta(\gamma(t_0)$ for all $t \in (t_0 - \delta_1, t_0 + \delta_1)$. It then follows from (4.3) and the uniqueness of $\zeta(t)$ that $\gamma(t) = \zeta(t)$ for $t \in (t_0 - \delta_1, t_0 + \delta_1)$. Since $\zeta(t)$ is C^1 near t_0, this implies that $\gamma(t)$ is C^1 near t_0 as well. Since t_0 was arbitrary, we have proved that $\gamma(t)$ is C^1 on $(0, \infty)$.

It remains to prove (2.6). Differentiating (4.3) with respect to t yields

$$H_t(t, \gamma(t)) + H_\theta(t, \gamma(t)) \cdot \gamma'(t) = 0. \tag{4.9}$$

For each $t > 0$, define $h(x, t)$ as an element of H^1 by

$$h(x, t) = u(x, t) - v_{\gamma(t)}(x) = u(x, t) - \phi^{(n)}(x; \gamma(t), c).$$

Then we have

$$-(uu_x + u_{xxx}) = -(\phi\phi_x + \phi_{xxx} + \phi h_x + \phi_x h + hh_x + h_{xxx}), \tag{4.10}$$

where both sides represent distributions in H^{-2}. But substituting (2.3) into (1.1) gives the equation

$$-\sum_{j=1}^{n} \phi_j c_j + \phi\phi_x + \phi_{xxx} = 0,$$

and therefore, from (4.10),

$$-(uu_x + u_{xxx}) = -\sum_{j=1}^{n} \phi_j c_j + \phi h_x + \phi_x h + hh_x + h_{xxx}. \tag{4.11}$$

Substituting (4.11) into (4.4) and using the fact that

$$\langle h_{xxx}, \phi_i \rangle = -\int_{-\infty}^{\infty} h(\phi_i)_{xxx}\, dx,$$

yields

$$H_t(t, \gamma(t))_i = -\sum_{j=1}^{n} c_j \left(\int_{-\infty}^{\infty} \phi_i \phi_j \, dx \right) + R_i(t), \tag{4.12}$$

where

$$R_i(t) = \int_{-\infty}^{\infty} (h\phi_x\phi_i + h_x\phi\phi_i + hh_x\phi_i - h(\phi_i)_{xxx})\, dx.$$

Define $M(t) = G_\theta(v_{\gamma(t)}, \gamma(t))$, so that

$$M_{ij} = -\int_{-\infty}^{\infty} \phi_i \phi_j \; dx,$$

by (3.37) and (3.38). Then (4.12) can be written in vector form as

$$H_t(t, \gamma(t)) = Mc + R. \tag{4.13}$$

Observe that for each $i, j \in \{1, \ldots, n\}$,

$$H_\theta(t, \gamma(t))_{ij} = \int_{-\infty}^{\infty} u(t)\phi_{ij} \; dx = \int_{-\infty}^{\infty} \phi\phi_{ij} \; dx + \tilde{R}_{ij}(t),$$

where

$$\tilde{R}_{ij}(t) = \int_{-\infty}^{\infty} h\phi_{ij} \; dx.$$

From (3.36) we have then that, as matrices,

$$H_\theta(t, \gamma(t)) = M + \tilde{R}. \tag{4.14}$$

Equations (4.9), (4.13), and (4.14) together imply that

$$\gamma' = -(M + \tilde{R})^{-1}(Mc + R) = -(I + M^{-1}\tilde{R})^{-1}(c + M^{-1}R). \tag{4.15}$$

But since $\|h\|_1 < \epsilon$ by (2.5), then Lemmas 3.5, 3.8, and 3.9 imply that $|R| \leq C\epsilon$ and $\|\tilde{R}\|_\infty < C\epsilon$, where C is a constant that depends only on c and n. Moreover, from (3.52) we have that $\|M^{-1}\|_{B(\mathbf{R}^n, \mathbf{R}^n)} \leq K_1$, where K_1 depends only on c and n. The estimate (2.6) therefore follows from (4.15) and elementary considerations.

A Appendix

In this appendix we prove the statement made in Remark 2.5. The following lemma regarding the invariant functionals I_k mentioned in (2.7) is needed. The proof of the lemma is essentially contained in Section 3 of [BLN].

Lemma A.1. *Suppose $k \geq 1$ is an integer. For all $h \in H^k$ and $\phi \in H^{k+1}$, we have*

$$|I_{k+2}(\phi + h) - I_{k+2}(\phi)| \leq C_1 \left(\|h\|_k + \|h\|_k^{k+2} \right) \tag{A.1}$$

and

$$|I_{k+2}(\phi + h) - I_{k+2}(\phi)| \geq \|h\|_k^2 - C_2 \left(\|h\|_{k-1} + \|h\|_{k-1}^{k+2} \right), \tag{A.2}$$

where C_1 and C_2 depend only on $\|\phi\|_{k+1}$.

Following the argument of [BLN], we will use induction to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^k$, $\theta_0 \in \mathbf{R}^n$, and $\|u_0 - \phi^{(n)}(\theta_0, c)\|_k < \delta$, then for all $t > 0$,

$$\|u(t) - \phi^{(n)}(\gamma(t), c)\|_k < \epsilon,$$

where γ is the same function defined in Theorem 2.4. Theorem 2.4 already takes care of the case $k = 1$, so it suffices to prove that the statement holds for k, under the assumption that it holds for $k - 1$.

Fix $c \in S_n$ and let $\epsilon > 0$ be given. By Lemma 3.9, there is a uniform upper bound for $\|\phi^{(n)}(\theta, c)\|_{k+1}$ as θ ranges over all of $\mathbf{R}^n$. Therefore, we can choose constants C_1 and C_2 such that if we set $\phi = \phi^{(n)}(\theta, c)$, the estimates (A.1) and (A.2) in Lemma A.1 hold for all $\theta \in \mathbf{R}^n$. Choose $\alpha > 0$ such that

$$C_2(\alpha + \alpha^{k+2}) < \epsilon^2/2. \tag{A.3}$$

By the induction assumption, there is a $\delta_1 > 0$ such that if $\|u_0 - \phi^{(n)}(\theta_0, c)\|_{k-1} < \delta_1$, then for all $t > 0$,

$$\|u(t) - \phi^{(n)}(\gamma(t), c)\|_{k-1} < \alpha. \tag{A.4}$$

Choose $\delta_2 > 0$ such that

$$C_1(\delta_2 + \delta_2^{k+2}) < \epsilon^2/2.$$

Finally, define $\delta = \min(\delta_1, \delta_2)$.

If $\|u_0 - \phi^{(n)}(\theta_0, c)\|_k < \delta$, then applying (A.1) to $\phi = \phi^{(n)}(\theta_0, c)$ and $h = u_0 - \phi$ gives

$$|I_{k+2}(u_0) - I_{k+2}(\phi^{(n)}(\theta_0, c))| \leq \epsilon^2/2.$$

Since I_{k+2} is a conserved functional for (1.1), and $I_{k+2}(\phi^{(n)}(\theta, c))$ is independent of $\theta \in \mathbf{R}^n$, it follows that for all $t > 0$,

$$|I_{k+2}(u(t)) - I_{k+2}(\phi^{(n)}(\gamma(t), c))| \le \epsilon^2/2. \tag{A.5}$$

Now let $\phi = \phi^{(n)}(\gamma(t), c)$ and $h = u(t) - \phi$. Since

$$\|u_0 - \phi^{(n)}(\theta_0, c)\|_{k-1} \le \|u_0 - \phi^{(n)}(\theta_0, c)\|_k < \delta \le \delta_1,$$

(A.4) holds. It then follows from (A.2), (A.3), and (A.5) that

$$\|h\|_k^2 < \epsilon^2/2 + \epsilon^2/2 = \epsilon^2,$$

as desired.

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