Sharp Well-posedness Results for the BBM Equation

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Abstract

The regularized long-wave or BBM equation

$$u_t + u_x + uu_x - u_{xxt} = 0$$

was derived as a model for the unidirectional propagation of long-crested, surface water waves. It arises in other contexts as well, and is generally understood as an alternative to the Korteweg-de Vries equation. Considered here is the initial-value problem wherein u is specified everywhere at a given time $t = 0$, say, and inquiry is then made into its further development for $t > 0$. It is proven that this initial-value problem is globally well posed in the L^2-based Sobolev class H^s if $s \geq 0$. Moreover, the map that associates the relevant solution to given initial data is shown to be smooth. On the other hand, if $s < 0$, it is demonstrated that the correspondence between initial data and putative solutions cannot be even of class C^2. Hence, it is concluded that the BBM equation cannot be solved by iteration of a bounded mapping leading to a fixed point in H^s-based spaces for $s < 0$. One is thus led to surmise that the initial-value problem for the BBM equation is not even locally well posed in H^s for negative values of s.

1 Introduction

The propagation of unidirectional, one-dimensional, small-amplitude long waves in nonlinear dispersive media is sometimes well approximated by the Korteweg-de Vries equation or its regularized counterpart the BBM-equation (see e.g. [8], [16], [26]). Starting in the latter half of the 1960's and in the 1970's, the mathematical theory for such nonlinear, dispersive wave equations came to the fore as a major topic within nonlinear analysis. Much effort has been expended on various aspects of the pure initial value problems

$$u_t + u_x + uu_x + u_{xxx} \,=\, 0 \tag{1}$$

or

$$u_t + u_x + uu_x - u_{xxt} \,=\, 0 \tag{2}$$

with

$$u(x, 0) \,=\, u_0(x), \tag{3}$$

though these are not always the most physically relevant formulations (see the discussions in [3], [4], [5] [8], [10]). For the Korteweg-de Vries equation (1), recent theory has shown the pure initial-value problem to be globally well posed in the L^2-based Sobolev classes H^s if $s > -\frac{3}{4}$ (see [15]). This result turns out to be sharp in the sense that below this value of s, the initial-value problem cannot be solved by a Picard iteration, as is shown in the recent work of Christ, Colliander and Tao [14] (see also [19, 12, 22] and further comments in the concluding section). The prospect in view here is a similar theory for the initial-value problem for (2)-(3).

First, it will be established in Section 2 that (2) is globally well posed in H^s provided only that $s \geq 0$. Let $\mathcal{U}$ be the mapping that associates to a given function $u_0 = u_0(x)$ the unique solution $u = u(t,x)$ of (2) with that initial data. The proof of local well-posedness leads to the conclusion that $\mathcal{U}$ is smooth as a mapping of the relevant Banach spaces. In particular, $\mathcal{U}$ is well defined on L^2 and hence on a dense subset of any Sobolev space H^s for $s < 0$. In Section 3, it is demonstrated that when considered as a putative mapping of H^s for $s < 0$, $\mathcal{U}$ cannot be even C^2. It is thus projected that $s = 0$ is the sharp value for which well-posedness holds for (2). Section 4 contains a short conclusion and additional commentary.

Previously, the initial-value problem for (2) was known to be globally well-posed in H^k for integers $k \geq 1$ (see Benjamin *el al.* [2]). The argument for local well-posedness in L^2 follows the analysis of Boussinesq systems by Bona, Chen and Saut [6, 7]. Our result follows from an elaboration of their argument using a short- and long-wavelength decomposition as in Bourgain [13]. Analyticity of the solution map is an elementary consequence of the local existence theory, but appears not to have been noted in the literature. For (1), analyticity of the solution map was first established by Zhang [23, 24, 25].

Notation. We denote by $\widehat{\cdot}$ or $\mathcal{F}$ the Fourier transform and by $\mathcal{F}^{-1}$ the inverse transform. In case the variable(s) on either side of this transformation need emphasis, the notation $\mathcal{F}_{x\mapsto\xi}$ or $\mathcal{F}_{(x,t)\mapsto(\xi,\tau)}$ will be employed. In general, the norm in a Banach space X is denoted $\|\cdot\|_X$. Thus the symbol $\|\cdot\|_{L^p}$ denotes the norm in the Lebesgue space L^p while $\|\cdot\|_{H^s}$ is the norm in the L^2 – based Sobolev space $H^s = H^s(\mathbb{R})$. The notation $A \approx B$ means that there exists a constant $c \geq 1$ such that $\frac{1}{c}|A| \leq |B| \leq c|A|$. For any positive A and B, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there exists a positive constant c such that $A \leq cB$ (resp. $A \geq cB$). The symbol $\langle\xi\rangle$ connotes $(1+\xi^2)^{\frac{1}{2}}$. The characteristic function of an interval $I \subset \mathbb{R}$ is written $\mathbb{1}_I$. The convolution over $\mathbb{R}$ of two functions f and g is written $f \star g$. The only inner product intervening in our analysis is that of L^2. If $f, g \in L^2$, then $\langle f, g\rangle$ is the L^2 - inner product of f and g.

2 Well-posedness for data in H^s, $s \geq 0$

As already mentioned, the initial-value problem (2)-(3) is known to be globally well posed in H^k for any integer $k = 1, 2, \cdots$. A straightforward application of nonlinear interpolation theory (see [9] and the references therein) extends this result to H^s for any $s \geq 1$. Thus our interest devolves to the range $0 \leq s \leq 1$.

2.1 Bilinear Estimates

The goal of this subsection is to prove two helpful inequalities (see also [7]).

Lemma 1 *Let $u, v \in H^s(\mathbb{R})$, $s \geq 0$. Then*

$$\|\varphi(D_x)(uv)\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{H^s}, \tag{4}$$

where $\varphi(\xi) = \frac{\xi}{1+\xi^2}$ and $\varphi(D_x)$ is the Fourier multiplier operator defined by $\widehat{\varphi(D_x)u}(\xi) = \varphi(\xi)\widehat{u}(\xi)$.

Proof. Expressing $\varphi(D_x)uv$ in terms of Fourier transformed variables and using duality and a polarization argument, one may write (4) in the equivalent form

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi\langle\xi\rangle^s}{(1+\xi^2)\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}\hat{u}(\xi_1)\hat{v}(\xi-\xi_1)\overline{\hat{w}(\xi)}d\xi d\xi_1\right| \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{5}$$

For $s \geq 0$, $\langle\xi\rangle^s \lesssim \langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s$. In consequence, the combination $\langle\xi\rangle^s/\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s$ is bounded and may be ignored. Set $w_1(\xi) = \frac{\xi}{1+\xi^2}\overline{\hat{w}(\xi)}$. With this notation, the left-hand side of (5) is simply $\langle\hat{u}\star\hat{v},\bar{w}_1\rangle$. Define u_1 to be the reverse of $\hat{u}$, so $u_1(x) = \hat{u}(-x)$ for all x. Clearly, u and u_1 have the same L^2-norm. Then $\langle\hat{u}\star\hat{v},\bar{w}_1\rangle = \langle u_1\star w_1,\bar{\hat{v}}\rangle$ and hence one may bound (5) via the Cauchy-Schwarz inequality by $\|u_1\star w_1\|_{L^2}\|v\|_{L^2}$. A use of Young's inequality then gives $\|u_1\star w_1\|_{L^2} \leq \|u\|_{L^2}\|w_1\|_{L^1}$. The proof is completed by estimating the L^1-norm of w_1 by $\left\|\frac{\xi}{1+\xi^2}\right\|_{L^2_\xi}\|w\|_{L^2}$. □

Remark. The estimate (5) is not valid for $s < 0$. This becomes clear upon letting $\hat{u}$ be the characteristic function of the interval $[N-1, N+1]$, $\hat{v}$ the characteristic function of the interval $[-N-1, -N+1]$ and $\hat{w}(\xi)$ the characteristic function of the interval $[-1, 1]$ in (5). Then the left-hand side of (5) behaves as N^{-2s}, while the right-hand side is a constant independent of N. Hence, if $s < 0$, no matter how large is the implied constant, (5) fails for $N \gg 1$.

Lemma 2 *Let $f \in H^r(\mathbb{R})$ and $g \in H^s(\mathbb{R})$ for some r and s with $0 \leq s \leq r$ and $\frac{1}{2} < r$. Then, $\varphi(D_x)fg \in H^{s+1}(\mathbb{R})$ and there is a constant $C = C(r, s)$ such that*

$$\|\varphi(D_x)(fg)\|_{H^{s+1}(\mathbb{R})} \leq C\|f\|_{H^r(\mathbb{R})}\|g\|_{H^s(\mathbb{R})}, \tag{6}$$

where φ is as in Lemma 1.

Proof. Since $r > \frac{1}{2}$ and $r \geq s \geq 0$, the elements of $H^r(\mathbb{R})$ are multipliers in $H^s(\mathbb{R})$, which is to say,

$$\|fg\|_{H^s(\mathbb{R})} \leq C\|f\|_{H^r(\mathbb{R})}\|g\|_{H^s(\mathbb{R})}$$

where $C = C(r, s)$. Since $\varphi(D_x)$ smooths by exactly one derivative in the L^2-based Sobolev classes $H^s(\mathbb{R})$, the result follows. □

2.2 Local Well-posedness

Write (2)-(3) in the form

$$\begin{cases} iu_t = \varphi(D_x)u + \frac{1}{2}\varphi(D_x)u^2, \\ u(0,x) = u_0(x). \end{cases} \tag{7}$$

Let $S(t) = \exp(-it\varphi(D_x))$ be the unitary group defining the associated free evolution; that is, $S(t)u_0$ solves the linear initial-value problem

$$\begin{cases} iu_t = \varphi(D_x)u, \\ u(0,x) = u_0(x). \end{cases} \tag{8}$$

Then, (7) may be rewritten as the integral equation

$$u(t,x) = S(t)u_0(x) - \frac{i}{2}\int_0^t S(t-t')\varphi(D_x)u^2(t',x)dt' = \mathcal{A}(u,u_0)(x,t). \tag{9}$$

This latter integral equation may be solved locally in time by performing a Picard iteration (*e.g.* by a fixed-point argument) in the space X_T^s of continuous functions defined on $[-T,T]$ with values in H^s, equipped its usual norm

$$\|u\|_{X_T^s} = \sup_{t\in[-T,T]} \|u(t,\cdot)\|_{H^s}.$$

More precisely, argue as follows. The H^s norm is clearly preserved by the free evolution since its symbol has absolute value equal to 1. Thus, for any $t \geq 0$ and $s \in \mathbb{R}$,

$$\|S(t)u_0\|_{H^s} = \|u_0\|_{H^s},$$

and consequently, for any $T > 0$,

$$\|S(t)u_0\|_{X_T^s} = \|u_0\|_{H^s}.$$

The second term on the right-hand side of (9) may be bounded by using again the fact that $S(t)$ is a unitary operator in $H^s(\mathbb{R})$ for each value of its argument and Lemma 1, *viz.*

$$\frac{1}{2}\left\|\int_0^t S(t-t')\varphi(D_x)u^2(t',x)dt'\right\|_{X_T^s} \leq \frac{1}{2}C_sT\|u\|_{X_T^s}^2 \tag{10}$$

where C_s is a constant depending only on s. Similarly there obtains

$$\frac{1}{2}\left\|\int_0^t S(t-t')\varphi(D_x)\Big(u^2(t',x) - v^2(t',x)\Big)dt'\right\|_{X_T^s} \leq \frac{1}{2}C_sT\|u-v\|_{X_T^s}\|u+v\|_{X_T^s}. \tag{11}$$

These inequalities imply the following local well-posedness result.

Theorem 3 *Fix $s \geq 0$. For any $u_0 \in H^s(\mathbb{R})$, there exists a $T = T(u_0) > 0$ and a unique solution $u \in X_T^s$ of the initial-value problem (2)-(3). The maximal existence time $T = T_s$ for the solution has the property that*

$$T_s \geq \frac{1}{4C_s \|u_0\|_{H^s(\mathbb{R})}} \tag{12}$$

where the positive constant C_s depends only on s.

For $R > 0$, let $\mathcal{B}_R$ denote the ball of radius R centered at the origin in $H^s(\mathbb{R})$ and let $T = T(R) > 0$ denote a uniform existence time for (2)- (3) with $u_0 \in \mathcal{B}_{\mathcal{R}}$. Then the correspondence $u_0 \mapsto u$ that associates to u_0 the solution u of (2)-(3) with initial value u_0 is a real analytic mapping of $\mathcal{B}_{\mathcal{R}}$ to X_T^s.

Proof. Existence, uniqueness and the lower bound on the existence time follow from the contraction mapping principle applied to the closed ball $\mathcal{B}_M$ centered at the origin in X_T^s, where we may choose

$$M = 2\|u_0\|_{H^s(\mathbb{R})} \quad \text{and} \quad T = \frac{1}{2C_s M} = \frac{1}{4C_s\|u_0\|_{H^s(\mathbb{R})}}, \tag{13}$$

for example. It follows readily from (10) and (11) that the mapping $\mathcal{A}$ in (9) is a contraction mapping of $\mathcal{B}_M$ and thus $\mathcal{A}$ has a unique fixed point which is easily seen to comprise a solution of (2)-(3) on the time interval $[0, T]$. Clearly $T_s \geq 1/4C_s\|u_0\|_{H^s(\mathbb{R})}$. (By *solution*, we mean in the first instance a solution in the sense of tempered distributions, say, but it is straightforward to ascertain that in fact, all the terms in the differential equation lie in X_T^{s-2} and that the equation is satisfied identically at least in this space.) Note that in this case, the equivalence of the integral equation and the initial value problem in the space X_T^s is clear, at least for $s \geq 0$.

Attention is now turned to the smoothness issue for the solution map $\mathcal{U}$. This result is local in the sense that if it can be established for T sufficiently small, it is generally true because of uniqueness of solutions to the initial-value problem and the semigroup property. Let $\Lambda : H^s \times X_T^s \longrightarrow X_T^s$ be defined as

$$\Lambda(u_0, v(t)) = v(t) - S(t)u_0 - \frac{1}{2}\int_0^t S(t - t')\varphi(D_x)v^2(t')dt',$$

where the spatial variable x has been suppressed throughout. Due to Lemma 1, Λ is a smooth map from $H^s \times X_T^s$ to X_T^s, provided $s \geq 0$. Let $\Lambda(u_0, u(t)) = 0$, which is to say, suppose $u(t)$ is a solution of (2) with initial data u_0. Then the Fréchet derivative of Λ with respect to the second variable is calculated to be the linear map

$$\Lambda'_u(u_0, u(t))[h] = h - \int_0^t S(t - t')\varphi(D_x)u(t')h(t')dt'.$$

Since, as in the proof of Theorem 3,

$$\left\| \int_0^t S(t - t')\varphi(D_x)h(t')u(t')dt' \right\|_{X_T^s} \lesssim T\|u\|_{X_T^s}\|h\|_{X_T^s},$$

it is deduced that for T small enough $\Lambda'_u(u_0, u(t))$ is invertible since it is of the form $I + K$ and

$$\|K\|_{\mathcal{B}(X^s_T, X^s_T)} < 1,$$

where $\mathcal{B}(X^s_T, X^s_T)$ is the Banach space of bounded linear operators on X^s_T . Thus the second assertion of Theorem 3 follows from the Implicit Function Theorem. □

Remark The argument for smoothness of the flow map $\mathcal{U}$ is quite general and can be found in many particular contexts (see, *e.g.* Bekiranov [1], Kenig, Ponce and Vega [18, 19], Bona, Sun and Zhang [10] and Zhang [23, 24, 25] to name but a few). Indeed, whenever one solves a partial differential equation in a suitable functional framework by a Picard iteration scheme applied to an integral equation formulation, [1] there automatically obtains strong regularity information on the corresponding flow map. This is in contrast with the methods and results for quasi-linear partial differential equations which provide local existence, uniqueness and continuity of the flow map, but the flow map may lack smoothness; in fact, it may not even be uniformly continuous on bounded sets.

2.3 Global Well-Posedness

In this subsection, the local theory is extended using a low-frequency–high-frequency decomposition. The outcome is a satisfactory global well-posedness theorem.

Theorem 4 *In Theorem 3, the value of T may be taken arbitrarily large and hence the Cauchy problem (2)–(3) is globally well-posed in H^s for any $s \geq 0$.*

Proof of Theorem 4. Fix $T > 0$. The aim is to show that corresponding to any initial data $u_0 \in H^s$, there is a unique solution u of (2) that lies in X^s_T, and that u depends continuously upon u_0. Because of Theorem 3, this result is clear for data that is small enough in H^s. Moreover, as continuous dependence, uniqueness and the analytic dependence on the data of the flow map are all properties that are local in time, the issue is to prove existence of a solution corresponding to initial data of arbitrary size. Fix $u_0 \in H^s$ and let $N \gg 1$ be such that

$$\int_{|\xi| \geq N} \langle \xi \rangle^{2s} |\widehat{u_0}(\xi)|^2 d\xi \lesssim T^{-2}. \tag{14}$$

Such values of N exist since $\langle \xi \rangle^{2s} |\widehat{u_0}(\xi)|^2$ is an L^1-function. With N fixed as above, define

$$v_0(x) := \int_{|\xi| \geq N} e^{ix\xi}\, \widehat{u_0}(\xi) d\xi\,.$$

Using Theorem 3, a solution $v \in X^s_T$ of (2) is adduced having as initial data v_0. Split the initial datum u_0 into two pieces, namely

$$u_0 = v_0 + w_0,$$

[1] Exactly as we do in the fundamental Cauchy-Lipschitz theorem for ODE's

and consider the initial-value problem

$$\begin{cases} w_t - w_{xxt} + w_x + ww_x + (vw)_x = 0, \\ w(0,x) = w_0(x). \end{cases} \tag{15}$$

If there is a solution of (15) in X_T^s, then $v + w$ will be a solution of (2) in X_T^s and the result will be established. Observe that $w_0 \in H^1(\mathbb{R})$. (Indeed, w_0 lies in $H^r(\mathbb{R})$ for any r.) The analogue of the integral equation (9) in this context can be solved locally in time in X_S^1 for small values of S, by the same sort of contraction mapping argument used to prove Theorem 3. The function v is fixed in this discussion of courses, and use has been made of Lemma 2 with $r = 1$. If one had in hand an *a priori* bound on the H^1-norm of w showing it was bounded on the interval $[-T, T]$, then it would follow that the contraction argument could be iterated and a solution on $[-T, T]$ thereby obtained.

An *a priori* bound for w is now provided which implies that the local existence result continues to hold on any time interval over which $\|v(t,\cdot)\|_{L^2}$ remains finite, and in particular on $[-T, T]$. The formal steps to this inequality are as follows. Multiply the first equation in (15) by w and integrate over the entire real line $\mathbb{R}$. After integrations by parts, there appears the identity

$$\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}(w(t,x)^2 + w_x^2(t,x))dx\right) - \int_{-\infty}^{\infty} v(t,x)w(t,x)w_x(t,x)dx = 0. \tag{16}$$

Due to the Sobolev and Hölder inequalities, the estimate

$$\left|\int_{-\infty}^{\infty} v(t,x)w(t,x)w_x(t,x)dx\right| \lesssim \|w(t,\cdot)\|_{H^1}^2 \|v(t,\cdot)\|_{L^2} \tag{17}$$

is valid. Then (16), (17) and Gronwall's inequality yield

$$\|w(t,\cdot)\|_{H^1} \lesssim \|w_0\|_{H^1} \exp\left(\int_0^t \|v(t',\cdot)\|_{L^2} dt'\right). \tag{18}$$

The key *a priori* bound (18) now enables one to deduce that $w(t)$ exists at least on the time interval $[-T, T]$. The justification of these formal steps is made by regularizing; for smooth solutions the calculations are secure. One then finishes by using the continuous dependence result to infer that (18) remains valid in the limit wherein the smoothing disappears.
This completes the proof of Theorem 4. □

3 Local "ill-posedness" below L^2

Here, an ill-posedness result is established which indicates the sharpness of Theorems 3 and 4. This result suggests that one can not expect to obtain via iteration arguments even local solutions of the BBM equation for data in H^s if $s < 0$.

Theorem 5 *For any $s < 0$, $T > 0$ the flow map Φ established in Theorems 3 and 4 is not of class C^2 from H^s to X_T^s.*

The proof is made in several steps outlined below. The analysis turns upon the explicit arithmetic relation

$$\frac{\xi_1}{1+\xi_1^2} + \frac{\xi-\xi_1}{1+(\xi-\xi_1)^2} - \frac{\xi}{1+\xi^2} = \frac{\xi\xi_1(\xi-\xi_1)(\xi^2-\xi\xi_1+\xi_1^2+3)}{(1+\xi_1^2)(1+(\xi-\xi_1)^2)(1+\xi^2)} := \theta(\xi,\xi_1) \tag{19}$$

for the symbol φ.

3.1 Reduction of Theorem 5 to disproving a bilinear estimate

Consider the Cauchy problem

$$\begin{cases} iu_t = \varphi(D_x)u + \frac{1}{2}\varphi(D_x)(u^2), \\ u(0,x) = \eta u_0(x), \end{cases} \tag{20}$$

where $\eta > 0$ is a parameter. As in (9), write (20) as an integral equation, *viz.*

$$u(t,x) = \eta S(t)u_0(x) - i\int_0^t S(t-t')\varphi(D_x)(\frac{u^2(t',x)}{2})dt' ,$$

where, as before, $S(t) = \exp(-it\varphi(D_x))$ is the unitary group defining the solution of the linear BBM-equation. The solution of (20) is a function of three variables, $u = u(\eta,t,x)$. Clearly $u(0,t,x) = 0$. Furthermore, the formal first two derivatives of $u(\eta,t,x)$ with respect to η at $\eta = 0$ are

$$\frac{\partial u}{\partial \eta}(0,t,x) = S(t)u_0(x) := u_1(t,x)$$

and

$$\frac{\partial^2 u}{\partial \eta^2}(0,t,x) = -2i\int_0^t S(t-t')\varphi(D_x)(\frac{u_1^2(t',x)}{2})dt' := u_2(t,x),$$

respectively. If it is presumed that the map $\mathcal{U}$ exists and is of class C^2 from H^s to X_T^s, then the estimate

$$\|u_2\|_{X_T^s} \lesssim \|u_0\|_{H^s}^2 \tag{21}$$

necessarily holds. The strategy used to prove Theorem 5 is to find a u_0 such that (21) fails if $s < 0$, no matter how small might be $T > 0$.

3.2 The choice of u_0 and a representation for u_2

Let u_0 be a defined via its Fourier transform as follows;

$$\widehat{u_0}(\xi) = \gamma^{-\frac{1}{2}}N^{-s}\left(\mathbb{1}_{I_1}(\xi) + \mathbb{1}_{I_2}(\xi)\right)$$

where $I_1 = [-N-\gamma, -N+\gamma]$ and $I_2 = [N-\gamma, N+\gamma]$. Here, the positive constant N is large and the positive constant γ is small. The relation between N and γ will be fixed presently. It is clear that $\|u_0\|_{H^s} \approx 1$. Moreover, notice that u_0 is real-valued since $\widehat{u_0}$ is an even, real-valued function.

Consider the first iteration u_1 as a function of N, γ and s. Because

$$\mathcal{F}_{x\mapsto\xi}(u_1)(t,\xi) = \exp(-it\varphi(\xi))\widehat{u_0}(\xi),$$

it is immediate that up to a factor of 2π,

$$u_1(t,x) = \gamma^{-\frac{1}{2}} N^{-s} \int_{\xi \in I_1 \cup I_2} \exp(ix\xi - it\varphi(\xi)) d\xi.$$

To compute u_2, the following technical lemma is helpful.

Lemma 5 *Let $F(t,x)$ be given and define $v(t,x)$ by*

$$v(t,x) = \int_0^t S(t-t')F(t',x)dt'$$

where $S(t) = \exp(-it\varphi(D_x))$ as before. Then, formally, v may be expressed in the form

$$v(t,x) = \frac{1}{4i\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(ix\xi - it\varphi(\xi)) \frac{e^{it(\tau+\varphi(\xi))} - 1}{\tau + \varphi(\xi)} \widehat{F}(\tau,\xi) d\tau d\xi.$$

Proof of Lemma 5. We follow closely the proof of Lemma 4 in [20]. By the group law for $S(t)$, $v(t,x)$ may be written as

$$v(t,x) = S(t) \int_0^t S(-t')F(t',x)dt'. \tag{22}$$

Set $H(t',x) = S(-t')F(t',x)$. A calculation reveals that

$$\int_0^t S(-t')F(t',x)dt' = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\tau} - 1}{i\tau} \mathcal{F}_{t' \mapsto \tau}(H)(\tau,x) d\tau. \tag{23}$$

Indeed, both sides of (23) vanish at $t = 0$, and the derivative of the right-hand side with respect to t is $H(t,x)$ by the formula of the inverse Fourier transform. Define G by

$$G(t,t',x) = S(t-t')F(t',x).$$

Using (22), (23) and the fact that $S(t)$ is, for each t, a Fourier multiplier operator, one arrives at the representation

$$v(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\tau} - 1}{i\tau} \mathcal{F}_{t' \mapsto \tau}(G)(t,\tau,x) d\tau$$

for v. The inverse Fourier transform formula with respect to x then yields

$$v(t,x) = \frac{1}{4i\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{it\tau} - 1}{\tau} \mathcal{F}_{(t',x) \mapsto (\tau,\xi)}(G)(t,\tau,\xi) e^{ix\xi} d\tau d\xi. \tag{24}$$

Now, it is clear that

$$\mathcal{F}_{x \mapsto \xi}(G)(t,t',\xi) = \exp(-i(t-t')\varphi(\xi)) \mathcal{F}_{x \mapsto \xi}(F)(t',\xi),$$

and therefore

$$\begin{aligned}\mathcal{F}_{(t',x)\mapsto(\tau,\xi)}(G)(t,\tau,\xi) &= e^{-it\varphi(\xi)}\int_{-\infty}^{\infty} e^{-it'(\tau-\varphi(\xi))}\mathcal{F}_{x\mapsto\xi}(F)(t',\xi)dt' \\ &= e^{-it\varphi(\xi)}\widehat{F}(\tau-\varphi(\xi),\xi). \qquad (25)\end{aligned}$$

Substituting (25) into (24) gives

$$v(t,x) = \frac{1}{4i\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(ix\xi - it\varphi(\xi))\frac{e^{it\tau}-1}{\tau}\widehat{F}(\tau-\varphi(\xi),\xi)d\tau d\xi.$$

It remains to perform the change of variable

$$\tau - \varphi(\xi) \mapsto \tau_1$$

to complete the proof of Lemma 5. □

Using Lemma 5, the following representation for $u_2(t,x)$ emerges; up to a constant,

$$u_2(t,x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(ix\xi - it\varphi(\xi))\frac{e^{it(\tau+\varphi(\xi))}-1}{\tau+\varphi(\xi)}\varphi(\xi)\left(\widehat{u_1}\star\widehat{u_1}\right)(\tau,\xi)d\tau d\xi,$$

in which justifying the formal steps leading to the representation presents no difficulty because of the particular form of what was called F in the Lemma. Because of the explicit formula $\widehat{u_1}(\tau,\xi) = \delta(\tau+\varphi(\xi))\widehat{u_0}(\xi)$ (δ connoting the Dirac delta function) the convolution $\widehat{u_1}\star\widehat{u_1}$ is seen to have the form

$$\left(\widehat{u_1}\star\widehat{u_1}\right)(\tau,\xi) = \int_{-\infty}^{\infty}\delta(\tau+\varphi(\xi_1)+\varphi(\xi-\xi_1))\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)d\xi_1.$$

Using this relationship and the definition of θ in the arithmetic formula (19) pertaining to the symbol φ, there obtains

$$u_2(t,x) = \gamma^{-1}N^{-2s}\int_{\xi_1\in I_1\cup I_2,\xi-\xi_1\in I_1\cup I_2}\varphi(\xi)\exp(ix\xi - it\varphi(\xi))\frac{e^{-it\theta(\xi,\xi_1)}-1}{\theta(\xi,\xi_1)}d\xi d\xi_1\,,$$

up to a constant. In Fourier transformed variables, and still ignoring constants of no consequence, this amounts to

$$\begin{aligned}\mathcal{F}_{x\mapsto\xi}(u_2)(t,\xi) &= \gamma^{-1}N^{-2s}\exp(-it\varphi(\xi))\varphi(\xi)\int_{\xi_1\in I_1\cup I_2,\xi-\xi_1\in I_1\cup I_2}\frac{e^{-it\theta(\xi,\xi_1)}-1}{\theta(\xi,\xi_1)}d\xi_1 \\ &= \gamma^{-1}N^{-2s}\exp(-it\varphi(\xi))\varphi(\xi)\left\{\int_{A_1(\xi)}\dots+\int_{A_2(\xi)}\dots\right\} \\ &:= g_1(t,\xi)+g_2(t,\xi)\end{aligned}$$

where

$$A_1(\xi) = \left\{\xi_1 \ : \ \xi_1\in I_1,\ \xi-\xi_1\in I_1 \text{ or } \xi_1\in I_2,\ \xi-\xi_1\in I_2\right\}$$

and

$$A_2(\xi) = \Big\{\xi_1 \ : \ \xi_1 \in I_1, \ \xi - \xi_1 \in I_2 \text{ or } \xi_1 \in I_2, \ \xi - \xi_1 \in I_1\Big\}.$$

Let $f_j = \mathcal{F}^{-1}_{\xi \mapsto x}(g_j)$, $j = 1, 2$. Then, we see that

$$u_2 = f_1 + f_2$$

and, due to the support properties of the $g_j(t, \xi)$, $j = 1, 2$, it transpires that

$$\|u_2(t, \cdot)\|_{H^s(\mathbb{R})} \approx \|f_1(t, \cdot)\|_{H^s} + \|f_2(t, \cdot)\|_{H^s}. \tag{26}$$

3.3 Proof of Theorem 5

As we will see momentarily, with a proper choice of γ and N, the main contribution to the $H^s(\mathbb{R})$ norm of $u_2(t, \cdot)$ arises from f_2. If $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$ and $\xi - \xi_1 \in I_1$, then $|\xi_1| \approx |\xi - \xi_1| \approx N$ and $|\xi| \le 2\gamma$. An examination of the formula for θ in (19) together with the preceding orders of magnitude reveals that $|\theta(\xi, \xi_1)| \lesssim \gamma$. Let $0 < \varepsilon \ll 1$ and let $\gamma = N^{-\varepsilon}$. With this choice, if $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$ and $\xi - \xi_1 \in I_1$, one has

$$\left| \frac{e^{-it\theta(\xi,\xi_1)} - 1}{\theta(\xi, \xi_1)} + it \right| \lesssim \gamma t^2,$$

provided $N \gg 1$. It follows that

$$\begin{aligned} \|f_2(t, .)\|_{H^s} &\gtrsim |t|\, \gamma^{-1} N^{-2s} \Big\{ \int_{|\xi| \approx \gamma} |\varphi(\xi)|^2 \langle \xi \rangle^{2s} \gamma^2 d\xi \Big\}^{1/2} \\ &\gtrsim |t|\, \gamma^{-1} N^{-2s} \gamma \Big\{ \int_{|\xi| \approx \gamma} |\xi|^2 d\xi \Big\}^{1/2} \\ &\gtrsim |t|\, \gamma^{-1} N^{-2s} \gamma\, \gamma^{\frac{3}{2}} \\ &= |t|\, N^{-2s} N^{-\frac{3}{2}\varepsilon}. \end{aligned}$$

The way is now prepared to contradict the inequality in (21). Suppose indeed that the flow map $\mathcal{U}$ is C^2 as a mapping of $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$ for some $s < 0$ and some $T > 0$. Then (21) holds for such values of s and T. Fix a non-zero value of t in $[-T, T]$. Then using (26) and the just derived lower bound for $\|f_2(t, .)\|_{H^s}$, it is seen that

$$1 \gtrsim \|u_0\|^2_{H^s} \gtrsim \|u_2\|_{X^s_T} \ge \|u_2(t, .)\|_{H^s} \gtrsim \|f_2(t, .)\|_{H^s} \gtrsim N^{-2s} N^{-\frac{3}{2}\varepsilon}.$$

Since $s < 0$, one can make $\|u_2(t, \cdot)\|_{H^s}$ larger than any fixed positive real number by first taking $\varepsilon > 0$ small enough and then choosing N large enough. This contradiction disproves the validity of (21) and the proof of Theorem 5 is complete. □

4 Conclusion

In the body of the paper, it has been shown that the BBM-equation is globally well posed in H^s for any $s \ge 0$. Moreover, the solutions are remarkably smooth in their temporal variable, but there is not even local smoothing in the spatial variable (see [2]). It was also shown that the method for establishing this result, namely converting to an integral equation, solving the integral equation locally in time by a Picard iteration, and then showing the local solution can be continued by a long-wave–short-wave splitting argument, will not be successful if $s < 0$. Indeed, what was shown was that the prospect of using a Picard iteration must necessarily fail if $s < 0$.

One would be tempted to then assert that the BBM-equation is ill-posed in H^s for negative values of s. However, recent experience with the Korteweg-de Vries equation suggests one should be cautious about asserting ill-posedness in the face of a result like Theorem 5. As mentioned in the introduction, a similar result obtains for the Korteweg-de Vries equation posed on the real line $\mathbb{R}$, in which the cut-off index is $s = -\frac{3}{4}$. However, the recent work of Kappeler and Topalov [17] has shown the initial-value problem to be well posed below $H^{-\frac{3}{4}+\epsilon}$, at least for the periodic initial-value problem. (The flow map is continuous, but not even uniformly continuous on bounded sets in this case.)

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