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# Pole dynamics of interacting solitons and blowup of complex-valued solutions of KdV

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**Abstract**
The pole dynamics in the complex plane associated with the two-soliton solution of the Korteweg–de Vries equation are studied in detail. The poles trace smooth curves as time evolves and fall into one of three categories: those which are asymptotic for large negative time to the faster soliton, but for large positive time are asymptotic to the slower soliton, those which follow the opposite pattern of the previous class, and those which are asymptotic for large positive and negative time to the faster soliton. Furthermore, the precise position and time of the interaction is identified. Finally, new examples of finite time blowup of complex-valued solutions of the Korteweg–de Vries equation are found and their asymptotic behaviours at blowup are determined.

It is suggested that these findings lend support to the assertions that the leading, slower moving soliton transforms during the interaction into the faster moving soliton, and that a mass–energy transfer takes place between the two solitons.

Mathematics Subject Classification: 35Q53, 35Q51, 37K40, 35A21

## 1. Introduction

This paper has its origins in some remarks of Kruskal (1974) about the soliton solutions of the Korteweg–de Vries equation and the present authors' ruminations concerning the formation of singularities for nonlinear dispersive wave equations. Kruskal's general idea was that interesting properties of solutions of the equation might be deduced from the behaviour of the poles of their analytic extensions in the spatial variable. Specifically, he had in mind the interaction of two solitons.

  

On the other hand, a major open problem is whether or not the generalized Korteweg–de Vries equation admits blowing up, real-valued solutions for supercritical powers. In Bona and Weissler (2001), it was shown that a large class of nonlinear dispersive wave equations, including the generalized Korteweg–de Vries equation (subcritical, critical and supercritial powers alike) have solutions that blow up in finite time in response to complex-valued initial data. Our idea is that, in the case of real-valued initial data, finite time blowup in the supercritical case could arise as a result of singularities of its analytic extension touching the real axis.

To carry out such a program, one would need to study meromorphic solutions of the generalized Korteweg–de Vries equation in the complex plane, including the dynamics of their singularities. The equation is normally posed with real-valued initial data, but some general theory has been developed concerning complex-valued solutions. One of the earliest papers to consider complex-valued solutions of the Korteweg–de Vries equation was Kato and Masuda (1986), which showed short time existence of solutions which are analytic in a complex strip around the real axis. More recently, Grujic and Kalisch (2002), Bona and Grujic (2003) and Bona *et al* (2005) have studied the behaviour of global solutions which, at each time $t$, are analytic in a strip. In particular, the dependence on time of the width of the strip of analyticity has been estimated. Unfortunately, the detailed dynamics of complex singularities of these solutions has not yet been successfully analysed. In this paper, we carry out such an analysis for a particular class of solutions of the Korteweg–de Vries equation. The recent works of Yuan and Wu (2005) and Wu and Yuan (2007) are also concerned with complex-valued solutions of the Korteweg–de Vries equation, but analyticity plays no role in their analysis, which is set in Sobolev classes.

The Korteweg–de Vries equation (henceforth also referred to as the KdV equation),

$$u_t + u_{xxx} + uu_x = 0, \tag{1.1}$$

where $u = u(x, t)$, is well known to possess single and multiple soliton solutions, all of which are given by explicit formulae. The two-soliton solution behaves asymptotically for large (positive and negative) time like two independent solitons of different speeds. The faster one trails the slower one for large negative time, and at some point overtakes it creating an interaction. Two solitons emerge from the interaction, having the same speeds as the original two solitons, but now the faster one is of course leading. One curious fact is that the faster soliton has been positively shifted in space due to the interaction, while the slower soliton has been negatively shifted in space. The nature of the interaction is not completely understood. One early interpretation of the interaction is that the two solitons retain their individual integrity during the interaction, even as the faster one overtakes the slower one. On the other hand, the original numerical work by Zabusky and Kruskal (1965), as well as theoretical work of Lax (1968), show that if the speeds of the two solitons are close to each other, then the two-soliton solution has two local spatial maxima at all times, corresponding to the maxima of the two independent solitons at $-\infty$. Thus, the two solitons remain apart, and seem to exchange roles as a result of the interaction. If the speeds are very different, however, then the faster soliton appears to swallow the slower soliton during the interaction, thereby passing through the slower soliton and leaving it intact except for the spatial shift. In this case, the two maxima come together creating a solution with a single spatial maximum during the interaction.

Later, Bowtell and Stuart (1983) suggested that for all possible relative speeds, the two solitons exchange roles during the interaction via some sort of energy transfer. On the other hand, Hodnett and Maloney (1989) (see also Leveque 1987) argue an opposite perspective, suggesting that the two solitons 'always pass through one another, irrespective of the amplitude ratio'. We will have more to say about these two papers shortly.

The papers of Bryan and Stuart (1992) and Benes *et al* (2006) propose decompositions of the two-soliton solution into three components, with one of the components representing a transfer of energy between the two solitons. Moreover, the paper of Benes *et al* (2006) includes a detailed comparison of all the various decompositions of the two-soliton solution proposed by a number of authors.

One of the goals of this paper is to better understand the nature of this interaction. Our approach, following Kruskal's lead, is to consider these solutions $u(x, t)$ to be defined for $x \in \mathbb{C}$ and taking on complex values. (The variable $t$ is still restricted to the reals.) In particular, the two-soliton solutions are viewed as time dependent meromorphic functions.

The main technical accomplishment here is obtaining a rather complete picture of the behaviour of the poles of the two-soliton solutions as a function of time. For large positive and negative time, the poles separate into two distinct groups, travelling at speeds which asymptotically approach the speeds of the two independent solitons. With the exception of one special case, the poles all trace out smooth curves in the complex plane for all time, allowing each pole to be followed individually. The poles whose asymptotic speed for large negative time is the speed of the faster soliton remain for all time in a trailing position behind the poles whose asymptotic speed for large negative time is the speed of the slower soliton. Moreover, not only do the poles which are slower moving for large negative time remain in front but for large positive time their speeds approach the speed of the faster soliton. Furthermore, the poles which move faster for large negative time are divided into two classes, namely, those which switch roles during the interaction and become slower moving for large positive time, and those whose speeds, also for large positive time, approach the speed of the faster soliton.

In fact, we are able to identify which poles are in each class in terms of their vertical spacing. Each pole which is slower moving for large negative time is paired with the faster moving pole whose imaginary part is closest to it for large negative time. These two poles move vertically closer as time evolves, exchanging their relative vertical position during the interaction, but retaining their original horizontal ordering. The pole which was slower moving for large negative time becomes a faster moving one for large positive time and vice versa. The remaining poles, which are unpaired and which move at the faster speed for large negative and postive time, also move vertically in time within a specified horizontal strip. There is a unique time which we call the *interaction time* which has the following property. For all other times, no two poles are vertically aligned, except for poles related by symmetries of the equation (complex conjugation and periodicity in the imaginary direction). At the interaction time, all the poles which are associated with the fast soliton for both large positive and large negative time are vertically aligned, while the other poles are symmetrically distributed on either side of the aligned poles.

In the authors' view, these results support the interpretation of a change in roles of the two solitons and a transfer of energy between the two solitons during the interaction for any combination of speeds. We refer the reader to the last section of the paper for a discussion of this point.

Another consequence of our analysis is the discovery of new examples of finite time blowup of complex-valued solutions of the KdV equation and the precise description of their asymptotic behaviour at blowup. Previously, finite time blowup of complex-valued solutions of KdV has been shown in Birnir (1987), Bona and Weissler (2001), Yuan and Wu (2005) and Li (2007). The behaviour of the solutions we exhibit is self-similar, but at a faster blowup rate than would be predicted from the scaling properties of the equation. Moreover, we give examples of both single-point blowup and two-point blowup. In all our examples, the solutions continue in a natural way beyond the blow-up time. On the other hand, global complex-(non-real) valued solutions are also found to exist.

Finally, since the interaction time is well defined, one can analytically describe the fusion of the two solitons, which in turn can be studied as a function of the different soliton speeds.

Another paper which is a direct predecessor to ours is that of Thickstun (1976). This work, which undertakes the analysis of the dynamics of the poles of the two-soliton solution, is also motivated by the remarks in Kruskal (1974). The results of Thickstun (1976) have some overlap with the present analysis. However, a considerably more complete picture is obtained here. In particular, Thickstun's paper does not contain the detailed information about how the poles with given asymptotic speeds for large negative time evolve into the poles with the two asymptotic speeds for large positive time. Nor does it contain the specific interaction time. Finally, it treats essentially only what we refer to below as the commensurable case.

Another line of inquiry related to this paper is the study of the pole dynamics for rational solutions and elliptic solutions of the Korteweg–de Vries equation. We refer the reader to Airault *et al* (1977) and Deconinck and Segur (2000), as well as the references cited therein.

In the next section, notation is established and the principal results of the paper are stated precisely. Following that the outline of the rest of the paper is given.

## 2. Statement of main results

We begin by recalling some well-known facts about the KdV equation, and in particular about the two-soliton solution. In the KdV equation (1.1), if $u = v_x$, then $u$ satisfies (1.1) if and only if

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(v_t + v_{xxx} + \frac{1}{2}v_x^2\right) = 0. \tag{2.1}$$

If $v$ is written in the form $v = 12(\log F)_x$, it follows that $v$ satisfies

$$v_t + v_{xxx} + \tfrac{1}{2}v_x^2 = 0 \tag{2.2}$$

if and only if $F$ satisfies

$$F(F_t + F_{xxx})_x - F_x(F_t + F_{xxx}) + 3(F_{xx}^2 - F_x F_{xxx}) = 0. \tag{2.3}$$

Note that a zero of $F$ of order $m$ corresponds to a simple pole of $v$ with residue $12m$.

It is well known that if

$$f(x,t) = \exp(-k(x - x_0) + k^3 t),$$

where $k > 0$, then

$$F(x,t) = 1 + f(x,t)$$

is a solution of (2.3). In this case, the solution $u$ of (1.1) is given by

$$\begin{aligned} u(x,t) = \frac{12k^2 f(x,t)}{(1 + f(x,t))^2} &= \frac{12k^2}{(\mathrm{e}^{k(x-x_0-k^2t)/2} + \mathrm{e}^{-k(x-x_0-k^2t)/2})^2} \\ &= 3k^2 \mathrm{sech}^2\left(\frac{k}{2}(x - x_0 - k^2 t)\right). \end{aligned}$$

This is the soliton solution. Note that for a given $t$, its amplitude is maximal when $f(x,t) = 1$, i.e. at

$$x = x_0 + k^2 t.$$

Furthermore, the zeros of $F$ in the complex plane occur when $f(x,t) = -1$, that is to say, when

$$x = x_0 + k^2 t + \frac{m\pi\mathrm{i}}{k}, \qquad m \text{ odd}.$$

In other words, the poles of the soliton solution $u$ of (1.1) are all vertically aligned with the point of maximum amplitude of the soliton, and they all move rigidly to the right at the same speed, maintaining the same vertical spacing. Thus, the movement of the soliton, as a real-valued solution of KdV on $\mathbb{R}$ is mirrored by the movement of the poles of that same solution, considered as a complex-valued solution of KdV on $\mathbb{C}$.

Next, it is known (and easy to verify) that if

$$f_1(x,t) = \exp(-k_1(x-x_1) + k_1^3 t) = \mathrm{e}^{k_1 x_1}\exp(-k_1 x + k_1^3 t), \tag{2.4}$$

$$f_2(x,t) = \exp(-k_2(x-x_2) + k_2^3 t) = \mathrm{e}^{k_2 x_2}\exp(-k_2 x + k_2^3 t), \tag{2.5}$$

where $0 < k_1 < k_2$ (for example), then

$$F(x,t) = 1 + f_1(x,t) + f_2(x,t) + \frac{(k_2-k_1)^2}{(k_2+k_1)^2} f_1(x,t) f_2(x,t) \tag{2.6}$$

is also a solution of (2.3). This gives the two-soliton solution of the KdV equation. The zeros of $F(\cdot,t)$ provide the poles of the solution in the complex plane. The goal of this paper is to see whether or not for the two-soliton solution, the movement of the two solitons, as a real-valued solution of KdV on $\mathbb{R}$, is somehow reflected in the dynamics of the poles of that same solution, considered as complex-valued solution of KdV on $\mathbb{C}$. Evidently, the two-soliton solution is considerably more complex than the single soliton. The expectation is that the analysis of the poles of the solution will shed some light on the properties of the interacting solitons.

In the analysis of the moving poles, it will sometimes be necessary to distinguish two cases corresponding to whether or not the real numbers $k_1$ and $k_2$ with $0 < k_1 < k_2$ are commensurable. If they are, then $F$ given by (2.6) has a periodic structure. More precisely, if $k_1$ and $k_2$ are commensurable, there exist positive integers $p_1$ and $p_2$ such that

$$\frac{k_2}{k_1} = \frac{p_2}{p_1}, \qquad \gcd(p_1,p_2) = 1. \tag{2.7}$$

It follows that $F(x,t)$ is periodic in $x$ with minimal (imaginary) period $2\pi\lambda\mathrm{i}$ where

$$\lambda = \frac{p_1}{k_1} = \frac{p_2}{k_2}. \tag{2.8}$$

In this case, which we shall refer to as the *commensurable* case, since $F(\cdot,t)$ clearly has no real zeros, it suffices to study the zeros of $F(\cdot,t)$ in the fundamental strip

$$S = \{x \in \mathbb{C} : 0 < \operatorname{Im} x < 2\pi\lambda\}. \tag{2.9}$$

Furthermore, $F$ is a polynomial of degree $p_1 + p_2$ in $\mathrm{e}^{-x/\lambda}$, with coefficients depending on $t$, and therefore at any given time has precisely $p_1 + p_2$ zeros, counted by multiplicity, in the fundamental strip. If $k_1$ and $k_2$ with $0 < k_1 < k_2$ are not commensurable, then $F$ has no periodic structure, and so the zeros will be studied in all of $\mathbb{C}$, or equivalently in either the upper- or lower-half plane.

Our first result concerns the interaction of the two solitons. It turns out that there is a unique value of $t$ where a large proportion of the poles are vertically aligned in the complex plane. We refer to this time as the *interaction time* and the location of this vertical alignment as the *interaction centre*.

**Theorem 1.** *Let $F$ be given by (2.6). There is a unique $t_0 \in \mathbb{R}$, the interaction time, and a unique $x_0 \in \mathbb{R}$, the interaction centre, given explicitly by*

$$t_0 = -\frac{x_2 - x_1}{k_2^2 - k_1^2} - \frac{1}{(k_2+k_1)k_1k_2} \log \frac{k_2+k_1}{k_2-k_1}, \tag{2.10}$$

$$x_0 = \frac{k_2^2 x_1 - k_1^2 x_2}{k_2^2 - k_1^2} - \frac{k_1^2 + k_1k_2 + k_2^2}{(k_2+k_1)k_1k_2} \log \frac{k_2+k_1}{k_2-k_1}, \tag{2.11}$$

*and characterized as follows.*

- *(i) If $t \neq t_0$, then all zeros of $F(\cdot, t)$ are simple, and have different real parts, unless related by periodicity or by complex conjugacy.*
- *(ii) If $t = t_0$ and if $k_1$ and $k_2$ with $0 < k_1 < k_2$ are commensurable, with $p_1$ and $p_2$ the positive integers satisfying (2.7), then if $\lambda$ is as in (2.8), precisely $p_2 - p_1$ zeros in the strip $\{0 < \operatorname{Im} x < 2\pi\lambda\}$, counted without multiplicity, are vertically aligned, all with real part equal to $x_0$. If $p_1$ is odd and $p_2$ is even, then one of these zeros is order 3. Otherwise they are all simple zeros. The other zeros (all simple) are symmetrically located with respect to (but not on) the vertical axis $\operatorname{Re} x = x_0$.*
- *(iii) If $t = t_0$ and if $k_1$ and $k_2$ with $0 < k_1 < k_2$ are not commensurable, then there are infinitely many zeros of $F(\cdot, t_0)$ with $\operatorname{Re} x = x_0$. These zeros have an asymptotic density of $(k_2 - k_1)/2\pi$. The other zeros are symmetrically located with respect to (but not on) the vertical axis $\operatorname{Re} x = x_0$ and have a vertical asymptotic density of $2k_1/2\pi$. All the zeros of $F(\cdot, t_0)$ are simple.*
- *(iv) The function $u(\cdot, t_0)$ is symmetric about the point $x_0$ on both $\mathbb{R}$ and $\mathbb{C}$. If $t \neq t_0$, then $u(\cdot, t)$ is not symmetric on $\mathbb{R}$ about any point.*

In Whitham (1974, chapter 17, p 585), the two-soliton interaction is said to occur in a neighbourhood of a certain time and $x$-value, these values being just the first of the two terms in (2.10) and (2.11). As theorem 1 gives a precise interaction time, it is certainly interesting to investigate the shape of the solution at this time. Calculations with `MAPLE` done by L Gouarin show that if $k_2/k_1 \geqslant \sqrt{3}$, then $u(\cdot, t_0)$ has one hump, but if $k_2/k_1 < \sqrt{3}$, then $u(\cdot, t_0)$ has two humps. The value $\sqrt{3}$ was found by Lax (1968).

In the function $F(\cdot, t)$ given by (2.6), $x_1$ and $x_2$ simply shift the interaction in space and time, and can be chosen in any convenient way with no loss of mathematical generality. In light of theorem 1, it is natural to require that the interaction take place at the origin, and at time 0, i.e. $x_0 = 0$ and $t_0 = 0$. It is straightforward to ascertain that these conditions are equivalent to requiring

$$e^{k_1 x_1} = e^{k_2 x_2} = \frac{k_2 + k_1}{k_2 - k_1}. \tag{2.12}$$

With these choices of $x_1$ and $x_2$, the function $F$ given in (2.6) becomes

$$F(x,t) = 1 + \gamma e^{-k_1 x + k_1^3 t} + \gamma e^{-k_2 x + k_2^3 t} + e^{-(k_1+k_2)x + (k_1^3 + k_2^3)t}, \tag{2.13}$$

where

$$\gamma = \frac{k_2 + k_1}{k_2 - k_1} > 1. \tag{2.14}$$

In what follows, we shall suppose that the choice (2.12) has been made, so that $F$ is defined as in (2.13). (In fact, the `MAPLE` calculations just mentioned were done for $F$ given by (2.13).)

**Theorem 2.** *Consider the function $F$ given by (2.13). With the exception of the commensurable case with $p_1$ odd and $p_2$ even, all zeros of $F(\cdot, t)$ are given by analytic curves defined for all $t \in \mathbb{R}$. In the exceptional case ($k_1$ and $k_2$ commensurable, $p_1$ odd and $p_2$ even) the same is true except for three of the zeros in the fundamental strip $S$ (or in any periodically equivalent strip $S + 2\pi\lambda m\mathrm{i}$ for some integer $m$). These zeros are described by three non-intersecting analytic curves defined separately for $t < 0$ and $t > 0$, all of which converge to the third order zero $x = \pi\lambda\mathrm{i}$ (or a periodically equivalent zero) of $F(\cdot, 0)$ as $t \to 0$. Furthermore, the asymptotic behaviours of all these curves as $t \to -\infty$ and as $t \to \infty$ are described as follows.*

*(i) For every odd integer $m \in \mathbb{Z}$ there exists a unique curve $x_{m,s^-}(t)$ of zeros of $F(\cdot, t)$ such that*

$$x_{m,s^-}(t) = k_1^2 t + \frac{1}{k_1}\log\gamma + \frac{m\pi\mathrm{i}}{k_1} + o(1), \tag{2.15}$$

*as $t \to -\infty$.*

*(ii) For every odd integer $m \in \mathbb{Z}$ there exists a unique curve $x_{m,s^+}(t)$ of zeros of $F(\cdot, t)$ such that*

$$x_{m,s^+}(t) = k_1^2 t - \frac{1}{k_1}\log\gamma + \frac{m\pi\mathrm{i}}{k_1} + o(1), \tag{2.16}$$

*as $t \to \infty$.*

*(iii) For every odd integer $n \in \mathbb{Z}$ there exists a unique curve $x_{n,f^-}(t)$ of zeros of $F(\cdot, t)$ such that*

$$x_{n,f^-}(t) = k_2^2 t - \frac{1}{k_2}\log\gamma + \frac{n\pi\mathrm{i}}{k_2} + o(1), \tag{2.17}$$

*as $t \to -\infty$.*

*(iv) For every odd integer $n \in \mathbb{Z}$ there exists a unique curve $x_{n,f^+}(t)$ of zeros of $F(\cdot, t)$ such that*

$$x_{n,f^+}(t) = k_2^2 t + \frac{1}{k_2}\log\gamma + \frac{n\pi\mathrm{i}}{k_2} + o(1), \tag{2.18}$$

*as $t \to \infty$.*

*For all $t < 0$ we have*

$$\operatorname{Re} x_{n,f^-}(t) < k_2^2 t < k_1^2 t < \operatorname{Re} x_{m,s^-}(t), \tag{2.19}$$

*and for all $t > 0$, we have*

$$\operatorname{Re} x_{m,s^+}(t) < k_1^2 t < k_2^2 t < \operatorname{Re} x_{n,f^+}(t). \tag{2.20}$$

*Finally, the above possibilities describe all curves of zeros of $F(\cdot, t)$.*

The poles described in (i) and (ii) above are associated with the slower soliton as $t \to \mp\infty$, respectively. However, it turns out that the zeros of $F(\cdot, t)$ which behave as (2.15) as $t \to -\infty$ do not move continuously into the zeros of $F(\cdot, t)$ which behave as (2.16) as $t \to \infty$. Note, however, that (2.15) and (2.16) show the well-known backward shift of the slower wave (see Whitham 1974, chapter 17, p 585). Likewise, the poles described in (iii) and (iv) above are associated with the faster soliton as $t \to \mp\infty$, respectively. However, as we shall see, while some of the zeros of $F(\cdot, t)$ which behave as (2.17) as $t \to -\infty$ move continuously into the zeros of $F(\cdot, t)$ which behave as (2.18) as $t \to \infty$, others 'change affiliation' and become associated with the slower soliton as $t \to \infty$. But again, (2.17) and (2.18) show the forward shift of the faster wave (see, again, Whitham 1974, chapter 17, p 585).

The next main result describes in much more detail the movement of the poles as $t$ moves from $-\infty$, past 0 and towards $\infty$. In order to state this result, we need to define a pairing which associates with each 'slow' pole (as $t \to -\infty$) a certain 'fast' pole, the one which is vertically closest as $t \to -\infty$. For every odd integer $m \in \mathbb{Z}$, let $d_m$ denote the asymptotic vertical distance (as $t \to -\infty$) of the 'slow' pole which satisfies (2.15) to the set of poles associated with the faster soliton, namely,

$$d_m = \inf_{n\in Z, n \text{ odd}} \left| \frac{m}{k_1} - \frac{n}{k_2} \right|.$$

With one exception, which is described in proposition 4.5, the minimum is realized by a unique odd integer, which we call $n_m$. The fast pole which behaves like (2.17) with $n = n_m$ as $t \to -\infty$ is said to be *paired* with this slow pole. In the exceptional case, which can occur only if $k_1$ and $k_2$ are commensurable and $p_1$ is odd and $p_2$ is even, the minimum is realized by two odd integers $n$. In this exceptional case, we let $n_m$ denote either of the two minimizers, and the slow pole is 'associated' with the two corresponding fast poles. This exceptional situation occurs once in each periodic strip. Thus, as $t \to -\infty$, the poles fall into one of three categories: poles associated with the slow soliton, poles associated with the fast soliton which are paired with slow poles and the poles associated with the fast soliton, but which are not paired with a slow pole. Finally, given $n \in \mathbb{Z}$ the asymptotic vertical distance (as $t \to -\infty$) of the 'fast' pole which satisfies (2.17) to the set of poles associated with the slow soliton is

$$D_n \equiv \inf_{m\in Z, m \text{ odd}} \left| \frac{m}{k_1} - \frac{n}{k_2} \right|.$$

**Theorem 3.**

(i) *If $t \neq 0$, no two poles have the same real part, except those related by complex conjugacy or by periodicity (in the commensurable case). If $t < 0$ the real parts of the poles are ordered as follows. The poles associated with the slow soliton are to the right of all the poles associated with the fast soliton. Among themselves, the real parts of the slow poles are in opposite order of the $d_m$. In other words, the closer the imaginary part of a given slow pole is to the imaginary parts of the set of fast poles, asymptotically as $t \to -\infty$, the larger is its real part for all $t < 0$. On the other hand, the real parts of the poles associated with the fast soliton are in the same order as the $D_n$. In other words, the closer the imaginary part of a fast pole is to the imaginary parts of the set of slow poles, asymptotically as $t \to -\infty$, the more to the left (negative) is its real part, for all $t < 0$. In particular, all the unpaired fast poles remain to the right of all the paired fast poles, for all $t < 0$.*

(ii) *Let $x(t)$ represent a curve of zeros of $F(\cdot, t)$. In the non-commensurable case,* $\operatorname{Im} x'(t) \neq 0$ *for all $t < 0$. Furthermore, paired slow and fast poles move vertically towards each other for all $t < 0$, in such a way that at $t = 0$, they have the same imaginary parts and non-zero real parts which differ only by a sign. It is still true that* $\operatorname{Im} x'(0) \neq 0$ *for the paired poles. The paired pole which has been associated with the slow soliton for $t < 0$ has a positive real part at $t = 0$, and the paired pole which has been associated with the fast soliton for $t < 0$ has a negative real part at $t = 0$. These paired poles are symmetrically located about, but not on, the imaginary axis. At $t = 0$, the unpaired fast poles all have real part 0 and* $\operatorname{Im} x'(0) = 0$. *In the commensurable case, the same is true with the following exceptions. If* $\operatorname{Im} x(t) \to \pi\lambda$ *as $t \to -\infty$ (where $\lambda$ is given by (2.8)), then in fact,* $\operatorname{Im} x(t) = \pi\lambda$ *for all $t \in \mathbb{R}$, and so $x'(t) \in \mathbb{R}$ when it is defined. If $p_1$ and $p_2$ are both odd, then there are two such poles, which form a paired couple of slow and fast poles, which move in perfectly horizontal fashion with $x'(t) > 0$ for all $t \in \mathbb{R}$, for both poles. At $t = 0$ they are symmetrically located about, but not on, the imaginary axis. If $p_1$ is even and $p_2$ is odd, there is one such pole, which is an unpaired fast pole with $x'(t) > 0$ for all $t \in \mathbb{R}$. At $t = 0$ it is purely imaginary. Finally, if $p_1$ is* *odd and $p_2$ is even, there is one such pole, which is the slow pole 'paired' with two fast poles. At $t = 0$ this slow pole and its two associated fast poles all converge to the point $\pi\lambda\mathrm{i}$. The slow pole approaches $\pi\lambda\mathrm{i}$ from the right, and $x'(t) \to -\infty$ as $t \to 0^-$. The two curves of associated fast poles approach $\pi\lambda\mathrm{i}$ from the angles $2\pi/3$ and $4\pi/3$, their derivatives becoming infinite as $t \to 0^-$. The asymptotic behaviour of the three poles as they converge to $\pi\lambda\mathrm{i}$ is independent of the values of $p_1$ and $p_2$. This exceptional situation is repeated in a $2\pi\lambda\mathrm{i}$ periodic fashion.*

*(iii) Let $x(t)$ represent a curve of zeros of $F(\cdot, t)$ which corresponds to an unpaired fast pole for $t < 0$. Then $x(t)$ remains an unpaired fast pole for all $t > 0$ and $x(t) = -\overline{x(-t)}$ for all $t \in \mathbb{R}$. Let $x(t)$ and $y(t)$ represent two curves of zeros of $F(\cdot, t)$ which correspond when $t < 0$ to a paired set of poles, one slow and one fast. Then $x(t) = -\overline{y(-t)}$ for all $t \in \mathbb{R}$. In particular, the curve corresponding to a slow pole when $t < 0$ then corresponds to a fast pole when $t > 0$, and vice versa. Finally, in the exceptional case where $k_1$ and $k_2$ are commensurable and $p_1$ is odd and $p_2$ is even, let $x(t)$ denote the slow pole with* $\operatorname{Im} x(t) = \pi\lambda$ *for all $t < 0$, and let $y_1(t)$ and $y_2(t)$ represent the two associated fast poles for $t < 0$. It follows that* $\operatorname{Im}(y_1(t) + y_2(t)) = 2\pi\lambda$. *Furthermore, all three curves can be extended continuously by requiring $x(0) = y_1(0) = y_2(0) = \pi\lambda\mathrm{i}$ and $x(t) = -\overline{x(-t)}$ and $y_1(t) = -\overline{y_2(-t)}$ for all $t \in \mathbb{R}$. With this convention, $x(t)$ remains a slow pole for $t > 0$, and the other two curves are its associated fast poles for $t > 0$. Of course this last configuration of poles repeats itself with period $2\pi\lambda\mathrm{i}$.*

**Remark.**The situation where $k_1$ and $k_2$ are commensurable and $p_1$ is odd while $p_2$ is even will often be referred to as the 'exceptional case' or 'exceptional situation'.

It is interesting to observe that Kruskal (1974) anticipated parts of theorem 3 in two examples. The first example he considered is $k_1 = 2$ and $k_2 = 4$ in our notation. In this case, the fundamental strip $S$ is $0 < \operatorname{Im} x < \pi$; it contains precisely three poles, counted with multiplicity, at any given time. This is an example of the exceptional case described at the end of part (ii) of theorem 3. Kruskal correctly observed that these 'three poles coalesce equi-angularly'. The second example he considered is $k_1 = 6$ and $k_2 = 7$. In this case, he correctly noted that 'some of the poles of the larger soliton (those whose imaginary parts are farthest from those of the smaller)' transfer from the trailing to the leading soliton.

In the exceptional situation, one of the poles actually turns around and moves in the direction opposite to the motion of the solitons. It is worth noting that this phenomenon is in fact more general. Indeed, the location of zeros of $F(\cdot, t)$ depends continuously on the parameters $k_1$ and $k_2$ (by Rouché's theorem), and so if $k_1$ and $k_2$ are close to the exceptional case where this turnaround occurs, then necessarily some curve of zeros of $F(\cdot, t)$ will be moving backwards. (This phenomenon was previously observed by Thickstun (1976).) Unfortunately, we have not found satisfactory criteria to describe this phenomenon in general (see proposition 3.9 for some partial results). Also, it would be interesting to find a manifestation of the backward movements of a pole in the trace of the movement of the two solitons, when restricted to $x \in \mathbb{R}$.

In addition, the fact that the asymptotic nature of the triple pole in the exceptional case does not depend on the particular values of $k_1$ and $k_2$ leads to the idea that this singularity is intrinsic to equation (1.1). Indeed, the function $F(x, t) = x^3 + 12t$ is a solution of equation (2.3), and the resulting solution $u = 12(\log F)_{xx}$ of (1.1) exhibits this singularity at $t = 0$ and $x = 0$.

At this point we are able to explain the relevance of the work of Bowtell and Stuart (1983). See, in particular, section V of that paper. Their point of view is to use the wave–particle duality as a model to study the interaction of the solitons, with the poles representing moving particles. As they require a *faithful* representation of the two solitons by particles, they are interested only in pairs of poles. For reasons which are not clear to us, but perhaps based on their results in Bowtell and Stuart (1977) concerned with the Sine–Gordon equation, they consider only those pairs of poles which move in a perfectly horizontal fashion. As indicated in theorem 3, this can only occur in the commensurable case with $p_1$ and $p_2$ both odd. To get around this restriction, they allow complex phase shifts $x_1$ and $x_2$ in formulae (2.4) and (2.5). (In fact they set $x_1 = 0$ and allow complex $x_2$.) They prove that for all positive values of $k_1$ and $k_2$, one can find an appropriate purely imaginary phase shift $x_2$ for which there exist a pair of poles moving on the same horizontal line. As in the case of theorem 3, these poles switch affiliation between the slow and fast moving solitons during the interaction, never touching each other. Furthermore, the real part of the trailing pole is a concave function of time, and the real part of the leading pole is a convex function of time. Bowtell and Stuart interpret this as a repulsive interaction between the two solitons.

In the paper of Hodnett and Maloney (1989), an explicit decomposition of the two-soliton solution $u$ in the form $u = u_1 + u_2$ into what they term 'soliton elements' is presented. They then study the evolution of the centres of mass of the two soliton elements and, in particular, determine the point in space–time at which the two centres of mass coincide. This turns out to be exactly the interaction point found in theorem 1. Moreover, they find that the centre of mass of the slower soliton element becomes negatively infinite at this intersection point. It is interesting that their results using a spatial decomposition are parallel to what we obtain by analysis of the poles.

We next turn to the question of finite time blowup of complex-valued solutions to (1.1). As noted in the introduction, this phenomenon has already been studied by a number of authors. The point here is that the explicit two-soliton solutions of (1.1) give new examples of finite time blowup, because of the vertical movement of the poles. Let $u = 12(\log F)_{xx}$, where $F$ is given by (2.13), be the two-soliton solution of (1.1). Fix $\alpha \in \mathbb{R}$ and set

$$u_\alpha(x, t) = u(x + \mathrm{i}\alpha, t),$$

where now we consider only $x \in \mathbb{R}$. As long as $F(\cdot, t)$ has no zero with imaginary part equal to $\alpha$, then $u_\alpha(\cdot, t)$ is a smooth solution of (1.1). If $F(\cdot, t)$ has a zero which at a given time $T$ has imaginary part $\alpha$, then the solution $u_\alpha(\cdot, t)$ blows up at time $T$. Since the solution is explicit, it is straightforward to determine the asymptotic form of this blowup. In fact, the analyticity of $F$ allows us to show that this blowup is always asymptotically self-similar, but with respect to a scaling different from the scaling which leaves (1.1) invariant. More precisely, we have the following theorem.

**Theorem 4.** *Let $\alpha, T \in \mathbb{R}$ and suppose that $z(t)$ is a smooth curve of zeros of $F(\cdot, t)$ with* $\operatorname{Im} z(T) = \alpha$ *which does not have a constant imaginary part (i.e. in the commensurable case we exclude the values $(2q + 1)\lambda\pi$, where $q \in \mathbb{Z}$, as possible values of $\alpha$).*

*(i) If $T \neq 0$, or if $T = 0$ and $z(0)$ is either a slow pole or its paired fast pole, then* $\operatorname{Im} z'(T) \neq 0$. *It follows that $u_\alpha(\cdot, t)$ blows up at time $T$ at the point $x_0 = \operatorname{Re} z(T)$ and*

$$\lim_{t\to T} (T - t)^2 u_\alpha(x_0 + y(T - t), t) = \frac{-12}{(y + z'(T))^2}$$

*uniformly for $y$ contained in any fixed compact subset of $\mathbb{R}$. If $T \neq 0$, this is a single-point blowup. If $T = 0$, this is a two-point blowup, at two points symmetrically located around the origin.*

*(ii) If $T = 0$ and $z(0)$ is an unpaired fast pole, then* $\operatorname{Im} z'(0) = 0$ *and* $\operatorname{Re} z'(0) > 0$. *In this case the solution $u_\alpha(\cdot, t)$ blows up at time $t = 0$ and*

$$\lim_{t\to 0} (-t)^2 u_\alpha(x_0 + y(-t), t) = \frac{-12}{(y + \operatorname{Re} z'(0))^2}.$$

*uniformly for y contained in any fixed compact subset of* $\mathbb{R} \setminus \{-\mathrm{Re}\, z'(0)\}$. *This is a single-point blowup.*

*In all cases, the solution continues past blowup, and the limits are realized in both directions of time, as* $t \to T$ *from below and above.*

This theorem calls for several remarks. First, observe that the asymptotic profile at blowup satisfies the equation $v_{xxx} + vv_x = 0$. This is not unexpected since the rescaled solutions $(T-t)^2 u_\alpha(x_0 + y(T-t), t)$ satisfy a rescaled version of (1.1), which becomes $v_{xxx} + vv_x = 0$ as $t \to T$. Next, in situation (ii) of theorem 4, the limiting profile is itself singular. To rectify this would require a further rescaling, but the correct choice depends on the properties of the higher derivatives of $\mathrm{Im}\, z(t)$ at $t = 0$ (which we have not investigated). For example, if $\mathrm{Im}\, z''(0) \neq 0$, then

$$(-t)^4 u_\alpha(x_0 + \mathrm{Re}\, z'(0)t + y(-t)^2, t) \to \frac{-12}{(y + z''(0)/2)^2}.$$

Furthermore, it might be argued that the 'natural' rescaling for blowup is

$$(T-t)^{2/3} u_\alpha(x_0 + y(T-t)^{1/3}, t)$$

since (1.1) is invariant under the transformation $u(x,t) \to \mu^{2/3} u(\mu^{1/3} x, \mu t)$. To produce such behaviour, the imaginary part of the moving zero would have to behave like

$$\mathrm{Im}\, z(t) \sim \mathrm{Im}\, z(T) + c(T-t)^{1/3}$$

at the blowup time. This is the observed behaviour of two of the curves of zeros at the exceptional point where there is a triple zero. Unfortunately, the other curve of zeros moves in a horizontal fashion with $\alpha = (2q+1)\lambda\pi$ with $q$ integer, so the associated solution $u_\alpha(\cdot, t)$ is singular at all times $t$. These are precisely the values excluded in theorem 4. Moreover, these curves are not smooth at $T = 0$, which again means that theorem 4 does not apply.

It is interesting to observe how the maximal horizontal strip where the solution is analytic behaves as $t$ approaches a blowup time. At the two-point blowup, i.e. where a slow pole and a fast pole move vertically towards each other, ending up with the same imaginary part at $t = 0$, but symmetrically located off of the imaginary axis, this band shrinks to width 0 since a singularity approaches from above and below the solution $u_\alpha(\cdot, t)$. In the other cases of blowup, as described in theorem 4, the singularity approaches from only one side, so at blowup, the solution $u_\alpha(\cdot, t)$ is the boundary value of a solution analytic in a horizontal strip in the complex plane.

In a similar vein, it is interesting to study the behaviour as $t \to \infty$ of the largest *symmetric* strip in $\mathbb{C}$ around which a solution on $\mathbb{R}$ has an analytic extension. More precisely, if we fix $\alpha = m\pi/k_1$ with $m \in \mathbb{Z}$ odd, or $\alpha = n\pi/k_2$ with $n \in \mathbb{Z}$ odd (in the non-commensurable case, to make things simpler), then by proposition 3.4, $u_\alpha(\cdot, t) = u(x + \mathrm{i}\alpha, t)$ is a regular (complex-valued) solution of (1.1) for all $t, x \in \mathbb{R}$. Also, by theorem 2, the width of the largest symmetric strip around the real axis on which $u_\alpha(\cdot, t)$ has an analytic extension decays to 0 as $t \to \pm\infty$. The question is how fast does it shrink. By formulae (5.8), (5.9) and (5.11), which refine the asymptotics given in theorem 2, it follows that these widths all decay exponentially. On the other hand, in Bona and Grujic (2003), it is proved that the width of a symmetric strip in $\mathbb{C}$ where a *real-valued* solution of (1.1) can be analytically extended must decay no faster than an inverse power of $t$ as $t \to \infty$. This could be related to the fact that a real-valued $H^1$-solution of (1.1) is uniformly bounded for all time, while a complex-valued global solution could very well blow up as $t \to \infty$.

The plan of the rest of the paper is as follows. In section 3, the zeros $F$, given by (2.13), are studied in a stationary frame of reference, i.e. in the variables $(x, t)$. In section 4, we examine certain properties which require changing variables to a frame of reference moving with one of the two solitons. Section 5 is devoted to the completion of the proofs of theorems 1, 2 and 3. Theorem 4 is proved in section 6. Section 7 is devoted to some rather technical calculations concerning the direction of the horizontal movement of the poles. The last section contains an interpretative discussion of some of the results in this paper.

## 3. General properties of the zeros of $F(\cdot, t)$

Elucidated here are fundamental properties of the zeros of $F$ given by (2.13) that are instructive in their own right as well as important for the overall analysis.

Since

$$F(-x, -t) = \mathrm{e}^{(k_1+k_2)x-(k_1^3+k_2^3)t} F(x, t)$$

and

$$F(\overline{x}, t) = \overline{F(x, t)},$$

it follows that if $F(x, t) = 0$ then $F(-x, -t) = 0$, $F(\overline{x}, t) = 0$ and $F(-\overline{x}, -t) = 0$. It is this last identity which we will use repeatedly, since $x$ and $-\overline{x}$ have the same imaginary part, and are symmetrically located with respect to the imaginary axis. The locations of the poles are, for any $t$, symmetric about the real axis and the locations at $t$ and $-t$ are reflected in the imaginary axis. Thus, at $t = 0$, the pole structure is symmetric about both the real and imaginary axes. Of course if the solution $u(x, t)$ of (1.1) is even in $x$ at $t = 0$, which is the case here, it is automatically true that $u(x, t) = u(-x, -t)$ by the uniqueness of the $H^1$-solutions of the initial-value problem. Also, recall that in the commensurable case, if $F(x, t) = 0$, then $F(x + 2q\lambda\pi\mathrm{i}, t) = F(\overline{x} + 2q\lambda\pi\mathrm{i}, t) = 0$ for $q \in \mathbb{Z}$, where $\lambda$ is given in (2.8).

**Proposition 3.1.** *For any $t \in \mathbb{R}$, the zeros of $F(\cdot, t)$ are simple, except the following special case. If $k_1$ and $k_2$ are commensurable, and if $p_1 \in \mathbb{N}$, $p_2 \in \mathbb{N}$ and $\lambda > 0$ are given by (2.7) and (2.8), with $p_1$ odd and $p_2$ even, then there is a third order zero of $F(\cdot, 0)$ at $x = (2q + 1)\lambda\pi\mathrm{i}$, for all integers $q$.*

**Proof.** Since

$$F(x, t) = 1 + \gamma \mathrm{e}^{-k_1x+k_1^3t} + \gamma \mathrm{e}^{-k_2x+k_2^3t} + \mathrm{e}^{-(k_1+k_2)x+(k_1^3+k_2^3)t},$$

it follows that

$$F_x(x, t) = -k_1\gamma \mathrm{e}^{-k_1x+k_1^3t} - k_2\gamma \mathrm{e}^{-k_2x+k_2^3t} - (k_1 + k_2)\mathrm{e}^{-(k_1+k_2)x+(k_1^3+k_2^3)t}.$$

If $x$ is a zero of order greater than or equal to 2, then $F(x, t) = 0$ and $F_x(x, t) = 0$. For purposes of this proof only, let $X = \mathrm{e}^{-k_1x+k_1^3t}$ and $Y = \mathrm{e}^{-k_2x+k_2^3t}$. Thus, $F(x, t) = F_x(x, t) = 0$ becomes

$$1 + \gamma X + \gamma Y + XY = 0$$

and

$$\frac{k_1}{k_1 + k_2}\gamma X + \frac{k_2}{k_1 + k_2}\gamma Y + XY = 0,$$

from which it is seen that

$$\gamma Y = -\frac{k_2 + k_1}{k_1} - \frac{k_2}{k_1}\gamma X.$$

Substituting this into the first equation yields

$$1 + \gamma X - \frac{k_2 + k_1}{k_1} - \frac{k_2}{k_1}\gamma X + X\left( -\frac{k_2 - k_1}{k_1} - \frac{k_2}{k_1}X\right) = 0,$$

which reduces to $1 + 2X + X^2 = 0$, i.e. $X = -1$, since $\gamma = (k_2 + k_1)/(k_2 - k_1)$. This readily gives $Y = 1$.

Thus, if $x$ is a non-simple zero of $F(\cdot, t)$, it must be the case that $\mathrm{e}^{-k_1 x + k_1^3 t} = -1$ and $\mathrm{e}^{-k_2 x + k_2^3 t} = 1$. It must therefore be that $-k_1 x + k_1^3 t$ and $-k_2 x + k_2^3 t$ are both purely imaginary, which implies that $t = 0$ and $x$ is purely imaginary. Furthermore, $k_1 x$ must be an odd multiple of $\pi\mathrm{i}$ and $k_2 x$ must be an even multiple of $\pi\mathrm{i}$. Thus $k_2/k_1 = p_2/p_1$, where $p_1$ and $p_2$ are relatively prime positive integers with $p_1$ odd and $p_2$ even. There is precisely one such complex number $x$ in the fundamental strip $S = \{x \in \mathbb{C} : 0 < \operatorname{Im} x < 2\pi\lambda\}$, and it is $x = \pi\lambda\mathrm{i}$, where $\lambda$ is given by (2.8).

A straightforward calculation of $F_{xx}$ and $F_{xxx}$ (with $t = 0$) shows this zero to be third order, thereby concluding the proof.

In what follows, it will be convenient to express the equation $F(x, t) = 0$ in a way that emphasizes the imaginary parts of the roots. It turns out to be natural to write

$$\alpha = -\operatorname{Im} x, \tag{3.1}$$

$$A_1 = \mathrm{e}^{-k_1 \operatorname{Re} x + k_1^3 t}, \tag{3.2}$$

$$A_2 = \mathrm{e}^{-k_2 \operatorname{Re} x + k_2^3 t}. \tag{3.3}$$

With this change of variables, the equation $F(x, t) = 0$ becomes

$$1 + \gamma A_1 \mathrm{e}^{\mathrm{i}k_1\alpha} + \gamma A_2 \mathrm{e}^{\mathrm{i}k_2\alpha} + A_1 A_2 \mathrm{e}^{\mathrm{i}(k_2+k_1)\alpha} = 0. \tag{3.4}$$

Multiplying equation (3.4) by $\mathrm{e}^{-\mathrm{i}(k_2+k_1)\alpha/2}$ leads to

$$\mathrm{e}^{-\mathrm{i}(k_2+k_1)\alpha/2} + \gamma A_1 \mathrm{e}^{-\mathrm{i}(k_2-k_1)\alpha/2} + \gamma A_2 \mathrm{e}^{\mathrm{i}(k_2-k_1)\alpha/2} + A_1 A_2 \mathrm{e}^{\mathrm{i}(k_2+k_1)\alpha/2} = 0.$$

Taking the real and imaginary parts of the latter equation, there obtains the system

$$(1 + A_1 A_2) \cos\left(\frac{k_2 + k_1}{2}\right)\alpha + \gamma(A_1 + A_2) \cos\left(\frac{k_2 - k_1}{2}\right)\alpha = 0, \tag{3.5}$$

$$(-1 + A_1 A_2) \sin\left(\frac{k_2 + k_1}{2}\right)\alpha + \gamma(-A_1 + A_2) \sin\left(\frac{k_2 - k_1}{2}\right)\alpha = 0, \tag{3.6}$$

which is equivalent to the equation $F(x, t) = 0$.

Note that replacing $A_1$ and $A_2$ by $1/A_1$ and $1/A_2$ leaves system (3.5) and (3.6) invariant. Thus, if $A_1$, $A_2$ and $\alpha$ solve the system, then so do $A_1'$, $A_2'$ and $\alpha$, where $A_1' = 1/A_1$ and $A_2' = 1/A_2$. This is just another reflection of the fact that $F(x, t) = 0$ if and only if $F(-\bar{x}, -t) = 0$.

The following, somewhat technical looking, result will be useful presently.

**Lemma 3.2.** *Suppose $F(x, t) = 0$ and that either $t \neq 0$ or $A_1 \neq 1$ or $A_2 \neq 1$, where the notation is as in (3.2) and (3.3). It follows that*

*(i)* $A_1 \neq 1$, *i.e.* $\operatorname{Re} x \neq k_1^2 t$,
*(ii)* $A_2 \neq 1$, *i.e.* $\operatorname{Re} x \neq k_2^2 t$,
*(iii)* $A_1 A_2 \neq 1$,
*(iv)* $\left(\frac{A_1 + A_2}{1 + A_1 A_2}\right)^2 \neq \left(\frac{-A_1 + A_2}{-1 + A_1 A_2}\right)^2$,
*(v)* $(A_2 - 1)(A_1 - 1) > 0$, *i.e.* $\operatorname{Re} x - k_1^2 t$ *and* $\operatorname{Re} x - k_2^2 t$ *have the same sign.*

**Proof.** We prove the first four statements by contradiction.

(i) Suppose first that $A_1 = 1$. If also $A_2 = 1$, then necessarily $t = 0$, which is a contradiction. Thus, we suppose $A_2 \neq 1$. System (3.5) and (3.6) thus becomes

$$\cos\left(\frac{k_2+k_1}{2}\right)\alpha + \gamma\cos\left(\frac{k_2-k_1}{2}\right)\alpha = 0,$$

$$\sin\left(\frac{k_2+k_1}{2}\right)\alpha + \gamma\sin\left(\frac{k_2-k_1}{2}\right)\alpha = 0,$$

from which we obtain the absurdity that $\exp(\mathrm{i}\frac{k_2+k_1}{2})\alpha = -\gamma\exp(\mathrm{i}\frac{k_2-k_1}{2})\alpha$.

(ii) Suppose next that $A_2 = 1$. If also $A_1 = 1$, then necessarily $t = 0$, which is a contradiction. Thus, we suppose $A_1 \neq 1$. System (3.5) and (3.6) now becomes

$$\cos\left(\frac{k_2+k_1}{2}\right)\alpha + \gamma\cos\left(\frac{k_2-k_1}{2}\right)\alpha = 0,$$

$$\sin\left(\frac{k_2+k_1}{2}\right)\alpha - \gamma\sin\left(\frac{k_2-k_1}{2}\right)\alpha = 0,$$

from which we obtain the absurdity that $\exp(\mathrm{i}((k_2+k_1)/2))\alpha = -\gamma\exp(-\mathrm{i}((k_2-k_1)/2))\alpha$.

(iii) Now suppose that $A_1A_2 = 1$, but $A_1 \neq A_2$. In this case we have $\sin((k_2-k_1)/2)\alpha = 0$ by (3.6), and so $\cos((k_2-k_1)/2)\alpha = \pm 1$. Then (3.5) gives

$$\left|\cos\left(\frac{k_2+k_1}{2}\alpha\right)\right| = \left|\gamma\frac{A_1+A_1^{-1}}{2}\right| \geqslant \gamma > 1,$$

which is impossible. Thus $A_1 = A_2 = 1$ (since both are positive), and so $t = 0$.

(iv) Suppose next that

$$\frac{A_1+A_2}{1+A_1A_2} = \frac{-A_1+A_2}{-1+A_1A_2}.$$

A straightforward calculation shows that $A_1 = 1$, which reduces us to (i). If, on the other hand,

$$\frac{A_1+A_2}{1+A_1A_2} = -\frac{-A_1+A_2}{-1+A_1A_2},$$

then $A_2 = 1$, which reduces us to (ii).

(v) To prove the last statement, rewrite system (3.5) and (3.6) as

$$(1+A_1A_2)\cos\left(\frac{k_2+k_1}{2}\alpha\right) = -\gamma(A_1+A_2)\cos\left(\frac{k_2-k_1}{2}\alpha\right), \tag{3.7}$$

$$(-1+A_1A_2)\sin\left(\frac{k_2+k_1}{2}\alpha\right) = -\gamma(-A_1+A_2)\sin\left(\frac{k_2-k_1}{2}\alpha\right). \tag{3.8}$$

Squaring both sides of each equation gives

$$(1+A_1A_2)^2\cos^2\left(\frac{k_2+k_1}{2}\right)\alpha = \gamma^2(A_1+A_2)^2\cos^2\left(\frac{k_2-k_1}{2}\right)\alpha, \tag{3.9}$$

$$(-1+A_1A_2)^2\sin^2\left(\frac{k_2+k_1}{2}\right)\alpha = \gamma^2(-A_1+A_2)^2\sin^2\left(\frac{k_2-k_1}{2}\right)\alpha. \tag{3.10}$$

Dividing each of the equations (3.9) and (3.10) by its coefficient on the left side (recalling that $-1 + A_1A_2 \neq 0$ from part (iii)), and adding the two resulting equations, there appears

$$1 = \gamma^2 \frac{(A_1 + A_2)^2}{(1 + A_1A_2)^2} \cos^2\left(\frac{k_2 - k_1}{2}\right)\alpha + \gamma^2 \frac{(-A_1 + A_2)^2}{(-1 + A_1A_2)^2} \sin^2\left(\frac{k_2 - k_1}{2}\right)\alpha. \tag{3.11}$$

It cannot be the case that

$$\frac{(A_1 + A_2)^2}{(1 + A_1A_2)^2} \geqslant 1$$

and

$$\frac{(-A_1 + A_2)^2}{(-1 + A_1A_2)^2} \geqslant 1,$$

since that would imply $1 \geqslant \gamma^2$. On the other hand, a simple calculation shows that each of these last two inequalities is equivalent to $(A_2^2 - 1)(A_1^2 - 1) \leqslant 0$.

This concludes the proof.

**Proposition 3.3.** *Suppose $F(x, t) = 0$ and that either $t \neq 0$ or $A_1 \neq 1$ or $A_2 \neq 1$. In the commensurable case, it follows that $\alpha = -\mathrm{Im}\, x$ is uniquely determined up to sign and $2\pi\lambda$ periodicity by $A_1$ and $A_2$. In the non-commensurable case, $\alpha = -\mathrm{Im}\, x$ is uniquely determined up to sign by $A_1$ and $A_2$. In particular, given $t \neq 0$, if $F(x_1, t) = F(x_2, t) = 0$, with $\mathrm{Re}\, x_1 = \mathrm{Re}\, x_2$, then $x_1$ and $x_2$ are the same up to symmetries of the equation. The same is true for $t = 0$ if $\mathrm{Re}\, x_1 = \mathrm{Re}\, x_2 \neq 0$.*

**Proof.** It follows from (3.11) that

$$\left[\frac{(A_1 + A_2)^2}{(1 + A_1A_2)^2} - \frac{(-A_1 + A_2)^2}{(-1 + A_1A_2)^2}\right] \cos^2\left(\frac{k_2 - k_1}{2}\right)\alpha = \gamma^{-2} - \frac{(-A_1 + A_2)^2}{(-1 + A_1A_2)^2}.$$

Lemma 3.2 assures us that the coefficient on the left is non-zero, thereby yielding an explicit expression for $\cos^2((k_2 - k_1)/2)\alpha$. By an analogous calculation, dividing each of equations (3.9) and (3.10) by the coefficients on the right (if $A_1 \neq A_2$), we get a similar explicit expression for $\cos^2((k_2 + k_1)/2)\alpha$. If $A_1 = A_2$, (3.10) implies that $\sin^2((k_2 + k_1)/2)\alpha = 0$, so that $\cos^2((k_2 + k_1)/2)\alpha = 1$.

Thus, the quantities $\cos^2((k_2 + k_1)/2)\alpha$ and $\cos^2((k_2 - k_1)/2)\alpha$ are explicitly and uniquely determined. Since $\cos 2\theta$ is a polynomial in $\cos^2\theta$, it follows that $\cos(k_2 + k_1)\alpha$ and $\cos(k_2 - k_1)\alpha$ are both explicitly and uniquely determined.

Next, multiply equation (3.7) by $\cos((k_2 - k_1)/2)\alpha$ and equation (3.8) by $\sin((k_2 - k_1)/2)\alpha$. This gives $\cos((k_2 + k_1)/2)\alpha \cos((k_2 - k_1)/2)\alpha$ and $\sin((k_2 + k_1)/2)\alpha \sin((k_2 - k_1)/2)\alpha$ in terms of determined quantities. Using the formulae for the cosine of the sum and difference of two angles, we see that $\cos k_1\alpha$ and $\cos k_2\alpha$ are therefore determined explicitly.

The following quantities are therefore all determined: $\cos k_1\alpha$, $\cos k_2\alpha$, $\cos(k_2 + k_1)\alpha$, $\cos(k_2 - k_1)\alpha$. For arbitrary integers $m, n \in \mathbb{Z}$,

$$\cos[(mk_2 + nk_1)\alpha] = \cos(mk_2\alpha)\cos(nk_1\alpha) - \sin(mk_2\alpha)\sin(nk_1\alpha).$$

By the binomial theorem, $\cos(mk_2\alpha)$ and $\cos(nk_1\alpha)$ are polynomials in $\cos(k_2\alpha)$ and $\cos(k_1\alpha)$, respectively, and so both are determined. Furthermore, $\sin(mk_2\alpha)$ is the product of $\sin(k_2\alpha)$ with a polynomial in $\cos(k_2\alpha)$ and $\sin(nk_1\alpha)$ is the product of $\sin(k_1\alpha)$ with a polynomial in $\cos(k_1\alpha)$. Thus $\sin(mk_2\alpha)\sin(nk_1\alpha)$ is the product of $\sin(k_2\alpha)\sin(k_1\alpha)$ with a determined expression, i.e. polynomials in $\cos(k_2\alpha)$ and $\cos(k_1\alpha)$. Since

$$2\sin(k_2\alpha)\sin(k_1\alpha) = \cos(k_2 - k_1)\alpha - \cos(k_2 + k_1)\alpha,$$

where the right-hand side is also a determined quantity, it follows that $\sin(mk_2\alpha)\sin(nk_1\alpha)$ is determined.

Thus, $\cos[(mk_2+nk_1)\alpha]$ is determined, for all $m, n \in \mathbb{Z}$. In the non-commensurable case, the set $\{mk_2 + nk_1 : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, and so clearly $\alpha$ is determined up to sign. In the commensurable case, let $m, n \in \mathbb{Z}$ be such that $mp_2 + np_1 = 1$, where $p_1$ and $p_2$ are given by (2.7). It follows that $mk_2+nk_1 = 1/\lambda$ where $\lambda$ is given by (2.8). Thus, $\cos(\alpha/\lambda)$ is determined, which determines $\alpha$ up to sign and $2\pi\lambda$ periodicity.

**Proposition 3.4.** *Given $\alpha \in \mathbb{R}$ for which there exists a solution $F(x, t) = 0$ with $\operatorname{Im} x = -\alpha$, and such that either $\sin k_1\alpha \neq 0$ or $\sin k_2\alpha \neq 0$, then in fact neither can equal 0, and it follows there are at most two values of $x$ (i.e. $x$ and $-\bar{x}$) and of $t$ (i.e. $t$ and $-t$) such that $\operatorname{Im} x = -\alpha$ and $F(x, t) = F(-\bar{x}, -t) = 0$. In other words, on a horizontal line in the complex plane at distance $\alpha$ from the real axis, where $\sin k_1\alpha \neq 0$ or $\sin k_2\alpha \neq 0$, $F(\cdot, t)$ vanishes for at most two values of $t \in \mathbb{R}$.*

**Proof.** Multiplying (3.4) by $1 + \gamma A_1 \mathrm{e}^{-\mathrm{i}k_1\alpha}$ yields

$$|1 + \gamma A_1 \mathrm{e}^{\mathrm{i}k_1\alpha}|^2 + A_2 \mathrm{e}^{\mathrm{i}k_2\alpha}(\gamma + A_1 \mathrm{e}^{\mathrm{i}k_1\alpha} + \gamma^2 A_1 \mathrm{e}^{-\mathrm{i}k_1\alpha} + \gamma A_1^2) = 0.$$

The imaginary part of this equation is

$$A_1^2 \sin k_2\alpha + A_1\left(\frac{1}{\gamma}\sin(k_2 + k_1)\alpha + \gamma\sin(k_2 - k_1)\alpha\right) + \sin k_2\alpha = 0. \tag{3.12}$$

A similar calculation reveals that

$$A_2^2 \sin k_1\alpha + A_2\left(\frac{1}{\gamma}\sin(k_2 + k_1)\alpha - \gamma\sin(k_2 - k_1)\alpha\right) + \sin k_1\alpha = 0. \tag{3.13}$$

If $\sin k_2\alpha = 0$, then (3.12) implies that $A_1((1/\gamma) - \gamma)\sin k_1\alpha = 0$, and so $\sin k_1\alpha = 0$. Similarly, if $\sin k_1\alpha = 0$, then (3.13) implies $\sin k_2\alpha = 0$.

Thus $\sin k_1\alpha \neq 0$ and $\sin k_2\alpha \neq 0$. It then follows from (3.12) and (3.13) that $A_1$ and $A_2$ are determined up to reciprocals. By lemma 3.2, it follows that, if $t \neq 0$, they must either both be bigger than 1 or both less than 1.

This implies, by (3.2) and (3.3), that $\operatorname{Re} x - k_1^2 t$ and $\operatorname{Re} x - k_2^2 t$ are each determined up to sign, and if $t \neq 0$, then they both must have the same sign.

This concludes the proof.

**Proposition 3.5.** *Suppose $F(x, t) = 0$, and set $\operatorname{Im} x = -\alpha$. If either $\sin k_1\alpha \neq 0$ or $\sin k_2\alpha \neq 0$ (so both are non-zero by proposition 3.4), then in fact $\sin k_1\alpha$ and $\sin k_2\alpha$ have opposite signs.*

**Proof.** The real and imaginary parts of (3.4) are

$$1 + \gamma A_1 \cos k_1\alpha + \gamma A_2 \cos k_2\alpha + A_1 A_2 \cos(k_2 + k_1)\alpha = 0 \tag{3.14}$$

and

$$\gamma A_1 \sin k_1\alpha + \gamma A_2 \sin k_2\alpha + A_1 A_2 \sin(k_2 + k_1)\alpha = 0. \tag{3.15}$$

Multiplying (3.14) by $\sin(k_2 + k_1)\alpha$, and (3.15) by $\cos(k_2 + k_1)\alpha$, and subtracting the second equation from the first leads to the equation

$$\sin(k_2 + k_1)\alpha + \gamma A_1 \sin k_2\alpha + \gamma A_2 \sin k_1\alpha = 0. \tag{3.16}$$

Combining (3.15) with (3.16) yields

$$\left(A_2 - \frac{1}{A_2}\right)\sin k_1\alpha + \left(A_1 - \frac{1}{A_1}\right)\sin k_2\alpha = 0. \tag{3.17}$$

If $t \neq 0$, or if $t = 0$ and $\operatorname{Re} x \neq 0$, then proposition 3.2 implies that $A_2-(1/A_2)$ and $A_1-(1/A_1)$ are non-zero and have the same sign. Thus $\sin k_1\alpha$ and $\sin k_2\alpha$ have opposite signs.

In the case $t = 0$ and $\operatorname{Re} x = 0$, then $A_1 = A_2 = 1$. Simple trigonometric identities show that (3.15) may be written as

$$(\gamma + \cos(k_2\alpha))\sin(k_1\alpha) + (\gamma + \cos(k_1\alpha))\sin(k_2\alpha) = 0.$$

As the quanities $\gamma + \cos(k_1\alpha)$ and $\gamma + \cos(k_2\alpha)$ are both positive, the stated result is seen to be valid in this case as well.

**Proposition 3.6.** *Suppose $t = 0$ and fix integers $m < n$. The number of solutions of $F(x, 0) = 0$ with $\operatorname{Re} x = 0$ and $-\operatorname{Im} x = \alpha \in [m\pi/k_2, n\pi/k_2]$ is equal to the number of solutions of $\mathrm{e}^{\mathrm{i}k_1\alpha} + \mathrm{e}^{\mathrm{i}k_2\alpha} = 0$ in that same interval. The same is true on the interval $[m\pi/k_1, n\pi/k_1]$.*

**Proof.** We prove the first case, the second case being similar. The zeros of $F(\cdot, t)$ given by (2.13) when $t = 0$ occur when

$$1 + \gamma\mathrm{e}^{-k_1x} + \gamma\mathrm{e}^{-k_2x} + \mathrm{e}^{-(k_1+k_2)x} = 0.$$

If $x$ lies on the imaginary axis, so $x = -\mathrm{i}\alpha$, say, then,

$$1 + \gamma\mathrm{e}^{\mathrm{i}k_1\alpha} + \gamma\mathrm{e}^{\mathrm{i}k_2\alpha} + \mathrm{e}^{\mathrm{i}(k_1+k_2)\alpha} = 0.$$

This is equivalent to the condition

$$\mathrm{e}^{\mathrm{i}k_1\alpha} + \frac{1 + \gamma\mathrm{e}^{\mathrm{i}k_2\alpha}}{\gamma + \mathrm{e}^{\mathrm{i}k_2\alpha}} = 0,$$

which is equivalent to

$$\mathrm{e}^{\mathrm{i}\rho\beta} + \frac{1 + \gamma\mathrm{e}^{\mathrm{i}\beta}}{\gamma + \mathrm{e}^{\mathrm{i}\beta}} = 0,$$

where $\beta = k_2\alpha$ and $\rho = k_1/k_2$. This suggests considering the equation

$$\mathrm{e}^{\mathrm{i}\rho\beta} = \phi(\mathrm{e}^{\mathrm{i}\beta}),$$

where

$$\phi(\zeta) \equiv -\frac{1 + \gamma\zeta}{\gamma + \zeta}.$$

One easily checks that $\phi$ maps the unit circle $S^1$ into itself and that $\phi(-1) = 1$ and $\phi(1) = -1$. Moreover, $\phi$ maps the open upper and lower-half circles bijectively onto each other. Define a continuous function $\theta : [m\pi, n\pi] \to \mathbb{R}$ by

$$\mathrm{e}^{\mathrm{i}\theta(\beta)} = \phi(\mathrm{e}^{\mathrm{i}\beta}), \qquad \theta(m\pi) = (m-1)\pi.$$

It follows from the properties of $\phi$ that $\theta(n\pi) = (n-1)\pi$.

It is clear that $\mathrm{e}^{\mathrm{i}\rho\beta} = \phi(\mathrm{e}^{\mathrm{i}\beta})$ if and only if $\mathrm{e}^{\mathrm{i}\rho\beta} = \mathrm{e}^{\mathrm{i}\theta(\beta)}$. The number of solutions is the same as the number of times that

$$f(\beta) \equiv \theta(\beta) - \rho\beta$$

is an integer multiple of $2\pi$. We claim that $f$ is a (strictly) increasing function. To see this, it will be shown that $\theta'(\beta) > \rho$ except at a finite number of points. By implicit differentiation, it follows that

$$e^{i\theta(\beta)} i\theta'(\beta) = i\phi_\zeta(e^{i\beta})e^{i\beta}$$

so that

$$\theta'(\beta) = \frac{(\gamma^2 - 1)}{|\gamma + e^{i\beta}|^2} \geqslant \frac{(\gamma^2 - 1)}{(\gamma + 1)^2} = \rho,$$

where the inequality is strict except when $e^{i\beta} = 1$.

To compute the number of times $f(\beta)$ equals a multiple of $2\pi$, note that

$$f(m\pi) = (m-1)\pi - \rho m\pi,$$

$$f(n\pi) = (n-1)\pi - \rho n\pi,$$

and that

$$f(n\pi) - f(m\pi) = (1-\rho)(n-m)\pi.$$

It follows that the number of times that $f(\beta)$ equals a multiple of $2\pi$ on the interval $[m\pi, n\pi]$ is precisely equal to the number of solutions to

$$e^{i\rho\beta} + e^{i\beta} = e^{ik_1\alpha} + e^{ik_2\alpha} = 0.$$

This concludes the proof.

**Corollary 3.7.** *Let $n \in Z$ be an odd integer.*

*(i) If $m \in \mathbb{Z}$ is an odd integer such that*

$$\left|\frac{m}{k_1} - \frac{n}{k_2}\right| < \frac{1}{k_2},$$

*then there is no solution of $F(x, 0) = 0$ with $\operatorname{Re} x = 0$ and such that $-\operatorname{Im} x = \alpha$ is in the closed interval whose endpoints are $m\pi/k_1$ and $n\pi/k_2$.*

*(ii) If*

$$\left|\frac{m}{k_1} - \frac{n}{k_2}\right| > \frac{1}{k_2},$$

*for all odd integers $m \in \mathbb{Z}$, then there is precisely one solution of $F(x, 0) = 0$ with $\operatorname{Re} x = 0$ and $-\operatorname{Im} x = \alpha \in [(n-1)\pi/k_2, (n+1)\pi/k_2]$.*

**Proof.** We begin by remarking that every solution of the equation

$$e^{ik_1\alpha} + e^{ik_2\alpha} = 0$$

is of the form

$$\alpha = \frac{(2l+1)\pi}{k_2 - k_1}$$

for some integer $l$. Denote by $I$ the interval

$$I = \left[\frac{(n-1)\pi}{k_2}, \frac{(n+1)\pi}{k_2}\right].$$

(i) The assumed condition on $m$ and $n$ can be restated as

$$\frac{n-1}{k_2}\pi < \frac{m}{k_1}\pi < \frac{n+1}{k_2}\pi.$$

These inequalities imply

$$\frac{n-m-1}{k_2-k_1}\pi < \frac{n-1}{k_2}\pi < \frac{n+1}{k_2}\pi < \frac{n-m+1}{k_2-k_1}\pi.$$

Since $n-m-1$ and $n-m+1$ are two successive odd integers, this shows there are no solutions of the equation $\mathrm{e}^{\mathrm{i}k_1\alpha}+\mathrm{e}^{\mathrm{i}k_2\alpha}=0$ in $I$. Thus by proposition 3.6, there are no solutions of $F(x,0)=0$ with $\operatorname{Re} x=0$ and such that $-\operatorname{Im} x=\alpha\in I$. This proves the stated result since the interval $I$ contains the closed interval whose endpoints are $m\pi/k_1$ and $n\pi/k_2$.
(ii) The assumed condition on $n$ implies that there exists an *even* integer $m\in\mathbb{Z}$ such that

$$\frac{m-1}{k_1}\pi < \frac{n-1}{k_2}\pi < \frac{n+1}{k_2}\pi < \frac{m+1}{k_1}\pi.$$

It is straightforward to check that these inequalities imply

$$\frac{n-1}{k_2}\pi < \frac{n-m}{k_2-k_1}\pi < \frac{n+1}{k_2}\pi.$$

Since $n-m$ is odd, this shows the existence of a solution $\alpha\in I$ of the equation $\mathrm{e}^{\mathrm{i}k_1\alpha}+\mathrm{e}^{\mathrm{i}k_2\alpha}=0$. The result now follows from proposition 3.6.

**Proposition 3.8.** *Let $x(t)$ be a smooth curve such that $F(x(t),t)=0$ and $F_x(x(t),t)\neq 0$. It follows that* $\operatorname{Im} x'(t)$ *has the same sign as the quantity*

$$\left(A_1-\frac{1}{A_1}\right)\sin k_2\alpha = -\left(A_2-\frac{1}{A_2}\right)\sin k_1\alpha,$$

*where $\alpha=-\operatorname{Im} x(t)$.*

**Proof.** Let $x(t)$ be a smooth curve such that $F(x(t),t)=0$ and $F_x(x(t),t)\neq 0$. Implicit differentiation gives

$$\begin{aligned}
x'(t) &= -\frac{F_t(x(t),t)}{F_x(x(t),t)} = \frac{k_1^3\gamma A_1\mathrm{e}^{\mathrm{i}k_1\alpha}+k_2^3\gamma A_2\mathrm{e}^{\mathrm{i}k_2\alpha}+(k_1^3+k_2^3)A_1A_2\mathrm{e}^{\mathrm{i}(k_1+k_2)\alpha}}{k_1\gamma A_1\mathrm{e}^{\mathrm{i}k_1\alpha}+k_2\gamma A_2\mathrm{e}^{\mathrm{i}k_2\alpha}+(k_1+k_2)A_1A_2\mathrm{e}^{\mathrm{i}(k_1+k_2)\alpha}}\\
&= \frac{k_1^3\gamma A_1\mathrm{e}^{-\mathrm{i}k_2\alpha}+k_2^3\gamma A_2\mathrm{e}^{-\mathrm{i}k_1\alpha}+(k_1^3+k_2^3)A_1A_2}{k_1\gamma A_1\mathrm{e}^{-\mathrm{i}k_2\alpha}+k_2\gamma A_2\mathrm{e}^{-\mathrm{i}k_1\alpha}+(k_1+k_2)A_1A_2}\\
&= \frac{(k_1^3\gamma A_1\mathrm{e}^{-\mathrm{i}k_2\alpha}+k_2^3\gamma A_2\mathrm{e}^{-\mathrm{i}k_1\alpha}+(k_1^3+k_2^3)A_1A_2)(k_1\gamma A_1\mathrm{e}^{\mathrm{i}k_2\alpha}+k_2\gamma A_2\mathrm{e}^{\mathrm{i}k_1\alpha}+(k_1+k_2)A_1A_2)}{|k_1\gamma A_1\mathrm{e}^{-\mathrm{i}k_2\alpha}+k_2\gamma A_2\mathrm{e}^{-\mathrm{i}k_1\alpha}+(k_1+k_2)A_1A_2|^2}.
\end{aligned}$$

The imaginary part of the numerator in this last expression is equal to

$$\begin{aligned}
&k_1k_2(k_2^2-k_1^2)\gamma^2A_1A_2\sin(k_2-k_1)\alpha\\
&\quad+(k_1(k_1^3+k_2^3)-k_1^3(k_1+k_2))\gamma A_1^2A_2\sin k_2\alpha+(k_2(k_1^3+k_2^3)\\
&\quad-k_2^3(k_1+k_2))\gamma A_1A_2^2\sin k_1\alpha,
\end{aligned}$$

which is proportional *via* a positive real number to

$$\gamma\sin(k_2-k_1)\alpha+A_1\sin k_2\alpha-A_2\sin k_1\alpha,$$

which therefore has the same sign as $\operatorname{Im} x'(t)$. If equation (3.16) is used to eliminate $A_2\sin k_1\alpha$, the latter expression becomes

$$\gamma\sin(k_2-k_1)\alpha+A_1\sin k_2\alpha+(1/\gamma)\sin(k_2+k_1)\alpha+A_1\sin k_2\alpha=\left(A_1-\frac{1}{A_1}\right)\sin k_2\alpha,$$

where equation (3.12) has also been utilized. The other part of the conclusion now follows from (3.17).

This concludes the proof.

A more delicate issue is the sign of $\operatorname{Re} x'(t)$. Intuitively, one would expect this always to be positive, but as indicated in theorem 3, there are some exceptions. As mentioned in the introduction, we have not yet found satisfactory conditions which enable us to determine when this behaviour occurs. Here is a partial result which gives some sufficient conditions that $\operatorname{Re} x'(t) > 0$.

**Proposition 3.9.** *Let $x(\cdot)$ be a smooth curve such that $F(x(t), t) = 0$ and $F_x(x(t), t) \neq 0$. Fix $t_0$ in the domain of definition of $x(\cdot)$, and let $\alpha = -\operatorname{Im} x(t_0)$. Suppose that $\sin k_1\alpha \neq 0$ and $\sin k_2\alpha \neq 0$.*

*(i) If $\cos k_1\alpha \cos k_2\alpha > 0$, then $\operatorname{Re} x'(t_0) > 0$.*
*(ii) If $t_0 = 0$ and $\operatorname{Re} x(0) = 0$ then $\operatorname{Re} x'(0) > 0$.*

*The proof of proposition 3.9 is postponed to section 7.*

## 4. Moving frames

Some aspects of the equation $F(x, t) = 0$ are more conveniently studied in a moving frame of reference corresponding to one or the other of the solitons. Thus, define

$$z = x - k_1^2 t, \tag{4.1}$$

$$w = x - k_2^2 t, \tag{4.2}$$

$$r = \exp(k_2(k_2^2 - k_1^2)t), \tag{4.3}$$

$$s = \exp(k_1(k_2^2 - k_1^2)t), \tag{4.4}$$

so that

$$r^{k_1} = s^{k_2}. \tag{4.5}$$

If

$$G(z, r) \equiv 1 + \gamma e^{-k_1 z} + r\gamma e^{-k_2 z} + r e^{-(k_1+k_2)z} = 1 + \gamma e^{-k_1 z} + r e^{-k_2 z}(\gamma + e^{-k_1 z}) \tag{4.6}$$

and

$$H(w, s) \equiv s + \gamma e^{-k_1 w} + s\gamma e^{-k_2 w} + e^{-(k_1+k_2)w} = s(1 + \gamma e^{-k_2 w}) + e^{-k_1 w}(\gamma + e^{-k_2 w}), \tag{4.7}$$

then

$$F(x, t) = G(x - k_1^2 t, e^{k_2(k_2^2 - k_1^2)t}) = e^{-k_1(k_2^2-k_1^2)t} H(x - k_2^2 t, e^{k_1(k_2^2-k_1^2)t}). \tag{4.8}$$

and

$$G(z, r) = r^{-k_1/k_2} H\left(z - \frac{\log r}{k_2}, r^{k_1/k_2}\right), \qquad H(w, s) = sG\left(w + \frac{\log s}{k_1}, s^{k_2/k_1}\right). \tag{4.9}$$

Once the zeros of $G$ or $H$ have been identified, one then obtains the corresponding zeros of $F$ using the inverse of formulae (4.1) and (4.2), namely,

$$x = k_1^2 t + z,$$

$$x = k_2^2 t + w.$$

**Remark 4.1.** Zeros of $G$ which remain localized in the complex plane, independently of $r$ either as $r \to 0$ or as $r \to \infty$, correspond to poles associated with the slow soliton (as $t \to -\infty$ or as $t \to \infty$, respectively), and zeros of $H$ which remain localized independently of $s$ either as $s \to 0$ or as $s \to \infty$, correspond to poles associated with the fast soliton (as $t \to -\infty$ or as $t \to \infty$, respectively).

**Proposition 4.2.** *No complex number can be a root of $G(\cdot, r)$ for two different values of $r \geqslant 0$. Similarly, no complex number can be a root of $H(\cdot, s)$ for two different values of $s \geqslant 0$.*

**Proof.** The proofs for $G(\cdot, r)$ and $H(\cdot, s)$ are essentially the same. We treat the former case. If $G(z, r_2) = G(z, r_1)$ with $r_1 \neq r_2$, then $G(z, r_2) - G(z, r_1) = (r_2 - r_1)\mathrm{e}^{-k_2 z}(\gamma + \mathrm{e}^{-k_1 z}) = 0$, and so $\mathrm{e}^{-k_1 z} = -\gamma$. Substituting this into $G(z, r_1) = 0$ yields $\gamma^2 = 1$, a contradiction.

**Proposition 4.3.** *For every odd integer $m \in \mathbb{Z}$ there exists a smooth curve $z_m(r)$, defined in some interval of $r \geqslant 0$, such that $G(z_m(r), r) = 0$, and*

$$z_m(r) = \frac{1}{k_1}\log\gamma + \frac{m\pi\mathrm{i}}{k_1} - \frac{4k_2}{k_2^2 - k_1^2}\gamma^{-k_2/k_1}\mathrm{e}^{-\frac{k_2}{k_1}m\pi\mathrm{i}}r + o(r) \tag{4.10}$$

*as $r \to 0^+$. For every odd integer $n \in \mathbb{Z}$ there exists a smooth curve $w_n(s)$, with $H(w_n(s), s) = 0$, defined in some interval of $s \geqslant 0$, such that*

$$w_n(s) = -\frac{1}{k_2}\log\gamma + \frac{n\pi\mathrm{i}}{k_2} + \frac{4k_1}{k_2^2 - k_1^2}\gamma^{-k_1/k_2}\mathrm{e}^{\frac{k_1}{k_2}n\pi\mathrm{i}}s + o(s) \tag{4.11}$$

*as $s \to 0^+$.*

**Proof.** The proof begins by examining the zeros of $G(\cdot, 0)$ and $H(\cdot, 0)$. The condition $G(z, 0) = 0$, which is to say,

$$1 + \gamma\mathrm{e}^{-k_1 z} = 0,$$

is equivalent to

$$z = \frac{1}{k_1}\log\gamma + \frac{m\pi\mathrm{i}}{k_1},$$

for some odd integer $m \in \mathbb{Z}$. Similarly, the condition $H(w, 0) = 0$, i.e.

$$\gamma + \mathrm{e}^{-k_2 w} = 0,$$

is equivalent to

$$w = -\frac{1}{k_2}\log\gamma + \frac{n\pi\mathrm{i}}{k_2},$$

for some odd integer $n \in \mathbb{Z}$.

Fix a zero of $G(z, 0) = 0$, $z_0 = (1/k_1)\log\gamma + (m\pi\mathrm{i}/k_1)$ for some fixed odd $m \in \mathbb{Z}$. By the implicit function theorem, there exists a smooth curve of zeros $z(r)$, such that $G(z(r), r) = 0$ and $z(0) = z_0$. To calculate $z'(0)$, differentiate the relation $G(z(r), r) = 0$ to find that

$$(k_1\gamma\mathrm{e}^{-k_1 z} + k_2 r\gamma\mathrm{e}^{-k_2 z} + (k_1 + k_2)r\mathrm{e}^{-(k_1+k_2)z})z'(r) = \mathrm{e}^{-k_2 z}(\gamma + \mathrm{e}^{-k_1 z}),$$

where $z = z(r)$. Setting $r = 0$ and using $\mathrm{e}^{-k_1 z(0)} = -1/\gamma$ gives

$$z'(0) = -\frac{1}{k_1}\mathrm{e}^{-k_2 z(0)}(\gamma - 1/\gamma) = -\frac{4k_2}{k_2^2 - k_1^2}\gamma^{-k_2/k_1}\mathrm{e}^{-\frac{k_2}{k_1}m\pi\mathrm{i}}.$$

For future reference, note that

$$\operatorname{Re} z_m'(0) = -\frac{4k_2}{k_2^2 - k_1^2}\gamma^{-k_2/k_1}\cos\frac{k_2 m\pi}{k_1}. \tag{4.12}$$

Similarly, if we fix a zero $w_0 = -(1/k_2)\log\gamma + (n\pi\mathrm{i}/k_2)$, for some fixed odd integer $n \in \mathbb{Z}$, of $H(w, 0) = 0$, then by the implicit function theorem, there exists a smooth curve of zeros $w(s)$, such that $H(w(s), s) = 0$ and $w(0) = w_0$. Differentiating the equation $H(w(s), s) = 0$ with respect to $s$ gives

$$(k_1\gamma \mathrm{e}^{-k_1 w} + k_2 s\gamma \mathrm{e}^{-k_2 w} + (k_1 + k_2)\mathrm{e}^{-(k_1+k_2)w})w'(s) = 1 + \gamma \mathrm{e}^{-k_2 w},$$

so at $s = 0$, there obtains

$$\mathrm{e}^{-k_1 w(0)}(k_1\gamma + (k_1 + k_2)\mathrm{e}^{-k_2 w(0)})w'(0) = 1 - \gamma^2,$$

since $\mathrm{e}^{-k_2 w(0)} = -\gamma$. In consequence

$$w'(0) = \frac{(1-\gamma^2)}{-k_2\gamma}\mathrm{e}^{k_1 w(0)} = \frac{(\gamma - 1/\gamma)}{k_2}\mathrm{e}^{k_1 w(0)} = \frac{4k_1}{k_2^2 - k_1^2}\gamma^{-k_1/k_2}\mathrm{e}^{\frac{k_1}{k_2}n\pi\mathrm{i}}.$$

For future reference, note that

$$\operatorname{Re} w_n'(0) = \frac{4k_1}{k_2^2 - k_1^2}\gamma^{-k_1/k_2}\cos\frac{k_1 n\pi}{k_2}. \tag{4.13}$$

**Lemma 4.4.**

*(i) Let $p, q \in \mathbb{Z}$ be odd integers, and let $\theta, \eta \in \mathbb{R}$ be such that $|p - \theta| < |q - \eta| \leqslant 1$. It follows that $\cos\theta\pi < \cos\eta\pi$.*

*(ii) Let $p, q \in \mathbb{Z}$ be odd integers, and let $\theta, \eta \in \mathbb{R}$ be such that $0 < |p - \theta|, |q - \eta| < 1$ and $(p - \theta)(q - \eta) < 0$. It follows that $\sin\theta\pi$ and $\sin\eta\pi$ are non-zero and have opposite signs.*

**Proof.**

(i) Since

$$\cos(p - \theta)\pi = -\cos\theta\pi \qquad \text{and} \qquad \cos(q - \eta)\pi = -\cos\eta\pi,$$

it suffices to prove that

$$\cos(p - \theta)\pi > \cos(q - \eta)\pi.$$

But this is clear since both $(p - \theta)\pi$ and $(q - \eta)\pi$ are in the interval $[-\pi, \pi]$ and the cosine function is symmetrically decreasing away from 0 on this interval.

(ii) Since

$$\sin(p - \theta)\pi = \sin\theta\pi \qquad \text{and} \qquad \sin(q - \eta)\pi = \sin\eta\pi,$$

it suffices to prove that $\sin(p - \theta)\pi$ and $\sin(q - \eta)\pi$ have opposite signs. But, again, this is obvious since both $(p - \theta)\pi$ and $(q - \eta)\pi$ are in the interval $(-\pi, \pi)$ and sin is negative on $(-\pi, 0)$ and postive on $(0, \pi)$.

For every odd integer $m \in \mathbb{Z}$, define the distance

$$d_m \equiv \inf_{n\in Z, n odd}\left|\frac{m}{k_1} - \frac{n}{k_2}\right|, \tag{4.14}$$

and for every odd integer $n \in \mathbb{Z}$, define the distance

$$D_n \equiv \inf_{m\in Z, m odd}\left|\frac{m}{k_1} - \frac{n}{k_2}\right|. \tag{4.15}$$

The proof of the following proposition is elementary and will be omitted.

**Proposition 4.5.** *In the case where*

*(i)* $k_1$ *and* $k_2$ *are commensurable,*
*(ii)* $p_1$ *is odd and* $p_2$ *is even, where* $p_1 \in \mathbb{N}$, $p_2 \in \mathbb{N}$ *are given by (2.7),*
*(iii) there exists an odd integer* $q \in \mathbb{Z}$ *such that* $m = qp_1$,

*it follows that the infimum in (4.14) is realized by two odd integers,* $n = qp_2 \pm 1$. *Furthermore, in this case*

$$d_m = \left|\frac{qp_1}{k_1} - \frac{qp_2 \pm 1}{k_2}\right| = \frac{1}{k_2}.$$

*In all other cases, the infimum in (4.14) is realized by a unique odd integer* $n_m \in \mathbb{Z}$ *and*

$$d_m = \left|\frac{m}{k_1} - \frac{n_m}{k_2}\right| < \frac{1}{k_2}. \tag{4.16}$$

In what follows we will denote the minimizer in (4.14) by $n_m$. In the exceptional case where $n_m$ is not uniquely defined, $n_m$ will denote either of the two minimizers.

There is an analogue for $D_n$ of proposition 4.5, but it will not find use here. Denote by $m_n$ the (perhaps not unique) minimizer in (4.15). For the present purposes, it suffices to observe that

$$D_n \equiv \left|\frac{m_n}{k_1} - \frac{n}{k_2}\right| \leqslant \frac{1}{k_1}, \tag{4.17}$$

with equality only in the exceptional case where the minimizer is not unique. Furthermore, observe that if the odd integers $n$ and $m$ are such that $n = n_m$, then $m = m_n$ and $d_m = D_n \leqslant 1/k_2$. On the other hand, if the odd integer $n$ is such that $n \neq n_m$ for all odd integers $m$, then $D_n > 1/k_2$.

**Proposition 4.6.** *Let* $m, m', n, n'$ *all be odd integers.*

*(i) If* $d_m < d_{m'}$, *then* $\cos\frac{k_2 m\pi}{k_1} < \cos\frac{k_2 m'\pi}{k_1}$.
*(ii) If* $D_n < D_{n'}$, *then* $\cos\frac{k_1 n\pi}{k_2} < \cos\frac{k_1 n'\pi}{k_2}$.

**Proof.**

(i) By assumption,

$$\left|\frac{m}{k_1} - \frac{n_m}{k_2}\right| < \left|\frac{m'}{k_1} - \frac{n_{m'}}{k_2}\right| \leqslant \frac{1}{k_2},$$

from which it follows that

$$\left|\frac{k_2 m}{k_1} - n_m\right| < \left|\frac{k_2 m'}{k_1} - n_{m'}\right| \leqslant 1.$$

The result is thus clear after applying the first part of lemma 4.4 with $p = n_m$, $q = n_{m'}$, $\theta = k_2 m/k_1$ and $\eta = k_2 m'/k_1$.

(ii) The hypotheses imply

$$\left|m_n - \frac{k_1 n}{k_2}\right| < \left|m_{n'} - \frac{k_1 n'}{k_2}\right| \leqslant 1.$$

The result now follows from the first part of lemma 4.4 with $p = m_n, q = m_{n'}, \theta = k_1 n/k_2$ and $\eta = k_1 n'/k_2$.

**Corollary 4.7.** *Let $m, m', n, n'$ all be odd integers, and let $z_m(r)$, $z_{m'}(r)$, $w_n(s)$, $w_{n'}(s)$ be the curves of zeros constructed in proposition 4.3.*

*(i) If $d_m < d_{m'}$, then $\operatorname{Re} z_{m'}(r) < \operatorname{Re} z_m(r)$ for sufficiently small $r > 0$.*
*(ii) If $D_n < D_{n'}$, then $\operatorname{Re} w_n(s) < \operatorname{Re} w_{n'}(s)$ for sufficiently small $s > 0$.*
*(iii) If $n = n_m$ but $n'$ is not of the form $n' = n_{m'}$, then $\operatorname{Re} w_n(s) < \operatorname{Re} w_{n'}(s)$ for sufficiently small $s > 0$.*

**Proof.** Statements (i) and (ii) follow from propositions 4.3 and 4.6, along with formulae (4.12) and (4.13). Statement (iii) follows from proposition 4.6 along with the observation made just after formula (4.17).

**Proposition 4.8.** *Let $\alpha > 0$ be contained in a non-empty open interval with endpoints $m\pi/k_1$ and $n_m\pi/k_2$ for some odd integer $m \in \mathbb{N}$. Then the quantities $\sin k_1\alpha$ and $\sin k_2\alpha$ are non-zero and have opposite signs.*

**Proof.** In the first case described in proposition 4.5, the result is obvious from the specific information about $m$ and either of the two values of $n_m$. Thus, attention is turned to the case where $n_m$ is uniquely determined. Since

$$\left|\frac{m}{k_1} - \frac{n_m}{k_2}\right| < \frac{1}{k_2} < \frac{1}{k_1},$$

it follows that $\sin k_1\alpha$ and $\sin k_2\alpha$ are non-zero in the interval. To check that they have opposite signs it suffices to prove that $\sin \frac{k_2}{k_1}m\pi$ and $\sin \frac{k_1}{k_2}n_m\pi$ have opposite signs. To prove this, note that

$$0 < \left|m - \frac{k_1 n_m}{k_2}\right| < 1$$

and

$$0 < \left|\frac{k_2 m}{k_1} - n_m\right| < 1,$$

the left-hand inequality holding because the interval is assumed to be non-empty. The result now follows from the second part of lemma 4.4 with $p = m$, $q = n_m$, $\theta = k_1 n_m/k_2$ and $\eta = k_2 m/k_1$. Note that $p - \theta$ is a positive multiple of $\eta - q$, so they have the same sign.

To close this section, we study the behaviour of a curve of zeros of $F(\cdot, t)$, given by (2.13), as it approaches the triple zero described in proposition 3.1. This is more conveniently studied in one or the other of the moving frames used in this section, as will be seen in the proof of the following result.

**Proposition 4.9.** *Suppose $k_1$ and $k_2$ are commensurable, and that $p_1 \in \mathbb{N}$, $p_2 \in \mathbb{N}$, and $\lambda > 0$ are given by (2.7) and (2.8). Suppose further that $p_1$ is odd and $p_2$ is even. Let $x(t)$ be a smooth curve defined for either $t < 0$ or $t > 0$ (or both), $t$ close to 0, such that $F(x(t), t) = 0$ and $x(t) \neq \lambda\pi\mathrm{i}$, but that $x(t) \to \lambda\pi\mathrm{i}$ as $t \to 0$. It follows that*

$$\lim_{t\to 0} \frac{(x(t) - \lambda\pi\mathrm{i})^3}{t} = -12 \tag{4.18}$$

*and*

$$\lim_{t\to 0} t^2 x'(t)^3 = -\frac{4}{9}. \tag{4.19}$$

**Remark 4.10.** Interestingly, the values of these limits do not depend on the values of $k_1$ and $k_2$.

**Proof of proposition 4.9.** Let $z(r) = x(t) - k_1^2 t$ where $r$ is given by (4.3). It follows from (4.8) that $z(r)$ is a smooth curve defined for either $r < 1$ or $r > 1$ (or both), $r$ close to 1, such that $G(z(r), r) = 0$, $z(r) \neq \lambda\pi\mathrm{i}$ and $z(r) \to \lambda\pi\mathrm{i}$ as $r \to 1$. Using (4.6), rewrite the equation $G(z(r), r) = 0$ as

$$r = -\frac{\mathrm{e}^{k_2 z} + \gamma\mathrm{e}^{(k_2-k_1)z}}{\gamma + \mathrm{e}^{-k_1 z}},$$

where $z = z(r)$. Differentiating this equation with respect to $r$, there obtains

$$1 = -\frac{(\gamma + \mathrm{e}^{-k_1 z})(k_2\mathrm{e}^{k_2 z} + (k_2 - k_1)\gamma\mathrm{e}^{(k_2-k_1)z}) - (\mathrm{e}^{k_2 z} + \gamma\mathrm{e}^{(k_2-k_1)z})(-k_1\mathrm{e}^{-k_1 z})}{(\gamma + \mathrm{e}^{-k_1 z})^2} z'(r).$$

Using (2.14), this simplifies to

$$1 = -\frac{k_2\gamma\mathrm{e}^{k_2 z}(1 + \mathrm{e}^{-k_1 z})^2}{(\gamma + \mathrm{e}^{-k_1 z})^2} z'(r),$$

from which it is deduced that

$$(1 + \mathrm{e}^{-k_1 z})^2 z'(r) = -\frac{\mathrm{e}^{-k_2 z}}{k_2\gamma}(\gamma + \mathrm{e}^{-k_1 z})^2 \to -\frac{1}{k_2\gamma}(\gamma - 1)^2 = \frac{-4k_1^2}{k_2(k_2^2 - k_1^2)},$$

as $r \to 1$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}r}(1 + \mathrm{e}^{-k_1 z})^3 = 3(1 + \mathrm{e}^{-k_1 z})^2(-k_1\mathrm{e}^{-k_1 z})z'(r) \to \frac{-12k_1^3}{k_2(k_2^2 - k_1^2)}.$$

Thus, by l'Hopital's rule, it is seen that

$$\lim_{r\to 1} \frac{(1 + \mathrm{e}^{-k_1 z})^3}{r - 1} = \frac{-12k_1^3}{k_2(k_2^2 - k_1^2)}. \tag{4.20}$$

Since

$$\lim_{z\to\lambda\pi\mathrm{i}} \frac{1 + \mathrm{e}^{-k_1 z}}{z - \lambda\pi\mathrm{i}} = \lim_{z\to\lambda\pi\mathrm{i}} \frac{\mathrm{e}^{-k_1 z} - \mathrm{e}^{-p_1\pi\mathrm{i}}}{z - \lambda\pi\mathrm{i}} = \lim_{z\to\lambda\pi\mathrm{i}} \frac{\mathrm{e}^{-k_1 z} - \mathrm{e}^{-k_1\lambda\pi\mathrm{i}}}{z - \lambda\pi\mathrm{i}} = -k_1\mathrm{e}^{-k_1\lambda\pi\mathrm{i}} = k_1,$$

formula (4.20) implies that

$$\lim_{r\to 1} \frac{(z(r) - \lambda\pi\mathrm{i})^3}{r - 1} = \frac{-12}{k_2(k_2^2 - k_1^2)}. \tag{4.21}$$

Furthermore, (4.20) also implies that

$$\begin{aligned}\lim_{r\to 1}(r-1)^2 z'(r)^3 &= \lim_{r\to 1} \frac{(r-1)^2}{(1 + \mathrm{e}^{-k_1 z})^6}(1 + \mathrm{e}^{-k_1 z})^6 z'(r)^3 \\ &= \left[\frac{k_2(k_2^2 - k_1^2)}{-12k_1^3}\right]^2 \left[\frac{-4k_1^2}{k_2(k_2^2 - k_1^2)}\right]^3 = \frac{-4}{9k_2(k_2^2 - k_1^2)}.\end{aligned} \tag{4.22}$$

Translating the two limits (4.21) and (4.22) back to the curve $x(t)$ using (4.1) and (4.3), and in particular the relation $(r-1)/t \to k_2(k_2^2 - k_1^2)$ as $t \to 0$, it follows that

$$\lim_{t\to 0} \frac{(x(t) - k_1^2 t - \lambda\pi\mathrm{i})^3}{t} = -12$$

and, since $z'(r) = (x'(t) - k_1^2)/rk_2(k_2^2 - k_1^2)$,

$$\lim_{t\to 0} t^2(x'(t) - k_1^2)^3 = -\tfrac{4}{9}.$$

The desired limits now follow easily since $k_1^2 t^{2/3} \to 0$ as $t \to 0$.

## 5. Proofs of theorems 1, 2 and 3

First observe that it suffices to prove theorem 1 under condition (2.12), in which case $t_0 = x_0 = 0$. To see this, let $\tilde{x}_1$ and $\tilde{x}_2$ be arbitrary real numbers, and let $\tilde{t}_0$ and $\tilde{x}_0$ be the resulting values obtained from formulae (2.10) and (2.11) with $x_1 = \tilde{x}_1$ and $x_2 = \tilde{x}_2$. If we then define

$$\tilde{f}_1(x,t) = \mathrm{e}^{k_1\tilde{x}_1}\exp(-k_1x + k_1^3t),$$

$$\tilde{f}_2(x,t) = \mathrm{e}^{k_2\tilde{x}_2}\exp(-k_2x + k_2^3t)$$

and

$$\tilde{F}(x,t) = 1 + \tilde{f}_1(x,t) + \tilde{f}_2(x,t) + \frac{(k_2-k_1)^2}{(k_2+k_1)^2}\tilde{f}_1(x,t)\tilde{f}_2(x,t),$$

then it is easy to check that

$$\tilde{F}(x,t) = F(x-\tilde{x}_0, t-\tilde{t}_0)$$

where $F$ is given by (2.13). Thus, information about $F$, given by (2.13), easily implies corresponding information about $\tilde{F}$.

We now proceed to the proof of theorems 1, 2 and 3, where $F$ is assumed to be given by (2.13). The first point to understand is the symmetry of the solution $u(\cdot,t) = 12\ln F(\cdot,t)_{xx}$ at time $t = 0$. To see this, note that $F(-x,0) = \mathrm{e}^{(k_1+k_2)x}F(x,0)$ and so

$$\ln F(-x,0) + \frac{(k_1+k_2)(-x)}{2} = \ln F(x,0) + \frac{(k_1+k_2)x}{2}.$$

In other words, $\ln F(x,0) + (k_1+k_2)x/2$ is an even function, and so therefore must be its second derivative. Thus $u(x,0) = u(-x,0)$ for all $x \in \mathbb{C}$ except at the zeros of $F(x,t)$. Next, it is checked that there can be no other value $t = t_1$ for which the solution is symmetric about some point $x = x_1 \in \mathbb{R}$, for example. Supposing this to be the case, then the poles of $u(\cdot,t_1)$ would be symmetrically placed with respect to the vertical line $\mathrm{Re}\,x = x_1$, which is impossible by proposition 3.4.

We now turn to the detailed description of the zeros of $F(\cdot,t)$. To this end, let $J = J_1 \cup J_2$, where

$$J_1 = \{\theta \geqslant 0 : \sin k_1\theta = 0\} = \left\{\frac{m\pi}{k_1} : m = 0,1,2,3,\ldots\right\}$$

and

$$J_2 = \{\theta \geqslant 0 : \sin k_2\theta = 0\} = \left\{\frac{n\pi}{k_2} : n = 0,1,2,3,\ldots\right\}.$$

Arrange the elements of $J$ in an increasing sequence, namely,

$$J = \{\theta_0, \theta_1, \theta_2, \theta_3, \ldots\},$$

where $\theta_{j-1} < \theta_j$ for all $j = 1,2,3,\ldots$, and define the open intervals

$$I_j = (\theta_{j-1}, \theta_j)$$

for $j = 1,2,3,\ldots$.

For every odd integer $m \in \mathbb{N}$, denote by $x_{s,m}(t)$ the curve of zeros of $F(\cdot,t)$ defined for large negative $t$, whose existence was established in proposition 4.3 and which has the form

$$x_{s,m}(t) = k_1^2t + z_m(\exp(k_2(k_2^2 - k_1^2)t)). \qquad (5.1)$$

Also, for every odd integer $n \in \mathbb{N}$, denote by $x_{f,n}(t)$ the curve of zeros of $F(\cdot, t)$ defined for large negative $t$, whose existence was established in proposition 4.3 and which has the form

$$x_{f,n}(t) = k_2^2 t + w_n(\exp(k_1(k_2^2 - k_1^2)t)). \tag{5.2}$$

Note that

$$\lim_{t\to-\infty} \operatorname{Im} x_{s,m}(t) = \frac{m\pi}{k_1} \tag{5.3}$$

and

$$\lim_{t\to-\infty} \operatorname{Im} x_{f,n}(t) = \frac{n\pi}{k_2}. \tag{5.4}$$

More precisely, it follows from (4.10) and (4.11) that

$$x_{s,m}(t) = k_1^2 t + \frac{1}{k_1}\log\gamma + \frac{m\pi \mathrm{i}}{k_1} - \frac{4k_2}{k_2^2 - k_1^2}\gamma^{-k_2/k_1}\mathrm{e}^{-\frac{k_2}{k_1}m\pi\mathrm{i}}\mathrm{e}^{k_2(k_2^2-k_1^2)t} + o(\mathrm{e}^{k_2(k_2^2-k_1^2)t}) \tag{5.5}$$

and

$$x_{f,n}(t) = k_2^2 t - \frac{1}{k_2}\log\gamma + \frac{n\pi \mathrm{i}}{k_2} + \frac{4k_1}{k_2^2 - k_1^2}\gamma^{-k_1/k_2}\mathrm{e}^{\frac{k_1}{k_2}n\pi\mathrm{i}}\mathrm{e}^{k_1(k_2^2-k_1^2)t} + o(\mathrm{e}^{k_1(k_2^2-k_1^2)t}) \tag{5.6}$$

as $t \to -\infty$.

By the implicit function theorem, each such curve can be extended smoothly as a curve of zeros of $F(\cdot, t)$ as long as $x_{s,m}(t)$, respectively, $x_{f,n}(t)$, remains a simple zero, and remains in a bounded region of $\mathbb{C}$. We first note that each such curve must remain in a bounded region of $\mathbb{C}$ for any bounded region of $t$. Let $x(t)$ denote one of these curves. By proposition 3.4, $\operatorname{Im} x(t) \notin J \setminus (J_1 \cap J_2)$, and so $\operatorname{Im} x(t)$ must remain bounded. It follows that if $\operatorname{Re} x(t) \to \pm\infty$ in finite time, then the equation $F(x(t), t) = 0$ implies $1 = 0$. This contradiction shows that $x(t)$ must remain in a bounded region of $\mathbb{C}$, for any bounded region of $t$. Next it follows from proposition 3.1 that, except for one exceptional case which occurs only when $k_1$ and $k_2$ are commensurable, all the zeros of $F(\cdot, t)$ are simple. In the exceptional case, this is still true for all $t \neq 0$. Thus these curves can be smoothly extended for all $t \in \mathbb{R}$ as zeros of $F(\cdot, t)$, (for all $t < 0$ in the exceptional case). These curves of zeros of $F(\cdot, t)$ are mutually disjoint, for large negative $t$, and cannot intersect one another at the same value of $t$ as long as they are smoothly extended by the implicit function theorem, i.e. for all $t \in \mathbb{R}$, or for $t < 0$ in the exceptional case.

Henceforth, consider the curves $x_{s,m}(t)$ and $x_{f,n}(t)$ as defined for all $t \in \mathbb{R}$ ($t < 0$ in the exceptional case). It follows from (5.5) and (5.6) that

$$\operatorname{Re} x_{f,n}(t) < k_2^2 t < k_1^2 t < \operatorname{Re} x_{s,m}(t)$$

for large negative $t$, and thus by lemma 3.2 for all $t < 0$. Hence it transpires that

$$\operatorname{Re} x_{f,n}(0) \leqslant 0 \leqslant \operatorname{Re} x_{s,m}(0),$$

except of course in the one case where the curves are not defined at $t = 0$. The precise horizontal ordering among the various curves $x_{f,n}(t)$ and among the various curves $x_{s,m}(t)$ is given by corollary 4.7 and (5.1), (5.2) for large negative $t$. This ordering is preserved for all $t < 0$ thanks to proposition 3.3.

### *5.1. The commensurable case*

Let $k_1$ and $k_2$ be commensurable and let $p_1 \in \mathbb{N}$, $p_2 \in \mathbb{N}$, and $\lambda > 0$ be given by (2.7) and (2.8). In this case, $J_1 \cap J_2 = \{0, \lambda\pi, 2\lambda\pi, 3\lambda\pi, \ldots\}$, since $q\lambda\pi = qp_1\pi/k_1 = qp_2\pi/k_2 \in J_1 \cap J_2$ for all $q = 0, 1, 2, \cdots$. For future reference, note that $\lambda\pi = \theta_{p_1+p_2-1}$ and $2\lambda\pi = \theta_{2(p_1+p_2-1)}$ and that $\sin k_1\lambda\pi = \sin k_2\lambda\pi = 0$.

For each $t \in \mathbb{R}$, the function $F(\cdot, t)$ is periodic with minimal period $2\lambda\pi\mathrm{i}$ and is a polynomial function in $\mathrm{e}^{-x/\lambda}$ of degree $p_1 + p_2$. It follows that for each $t \in \mathbb{R}$, $F(\cdot, t)$ has precisely $p_1 + p_2$ zeros (counted with multiplicity) in the fundamental strip

$$S = \{x \in \mathbb{C} : 0 < \operatorname{Im} x < 2\lambda\pi\},$$

since $F(x, t) \neq 0$ for all $x \in \mathbb{R}$. Note that if $x \in S$ and $F(x, t) = 0$, then likewise $F(\overline{x} + 2\lambda\pi\mathrm{i}, t) = 0$ and $\overline{x} + 2\lambda\pi\mathrm{i} \in S$. Thus, zeros of $F(\cdot, t)$ in $S$ which do not have imaginary part precisely equal to $\lambda\pi$ necessarily come in pairs with the same real parts, and located symmetrically in $S$ in the vertical direction around the horizontal line $\{\operatorname{Im} x = \lambda\pi\}$.

Consider the curves of zeros of $F(\cdot, t)$ given by $x_{s,m}(t)$ for odd $m$ with $0 < m < 2p_1$ and $x_{f,n}(t)$ for odd $n$ in the interval $0 < n < 2p_2$. There are precisely $p_1 + p_2$ such curves, defined for all $t \in \mathbb{R}$ (for $t < 0$ in the exceptional case, which can only occur if $p_1$ is odd and $p_2$ is even, as described in proposition 3.1) and taking distinct values for any given $t$. Furthermore, for large negative $t$, these curves all lie in $S$. Since no zero can lie on the boundary of $S$, it follows by continuity that these curves remain in $S$ as long as they exist. Thus, this accounts for all zeros of $F(\cdot, t)$ in $S$, and therefore by periodicity, for all zeros of $F(\cdot, t)$ in $\mathbb{C}$, for all $t \in \mathbb{R}$ (all $t < 0$ in the exceptional case). The next step is to describe more precisely the location and movement of these zeros.

Consider the intervals $I_j$, for $1 \leqslant j \leqslant 2(p_1+p_2-1)$. By proposition 3.4, no zero of $F(\cdot, t)$ can have imaginary part equal to $\theta_j$, $1 \leqslant j \leqslant p_1+p_2-2$ or $p_1+p_2 \leqslant j \leqslant 2(p_1+p_2-2)-1$. Thus, any zero of $F(\cdot, t)$ located in $S$ must have imaginary part either equal to $\lambda\pi = \theta_{p_1+p_2-1}$ or lie in one of the open intervals $I_j$, for $1 \leqslant j \leqslant 2(p_1+p_2-1)$. Furthermore, by proposition 3.5, such zeros cannot have imaginary part in $I_j$ with $j$ odd and $1 \leqslant j \leqslant p_1 + p_2 - 1$ or with $j$ even and $p_1 + p_2 \leqslant j \leqslant 2(p_1 + p_2 - 1)$. Note that the intervals $I_{p_1+p_2-1}$ and $I_{p_1+p_2}$ are symmetrically located on either side of the shared common endpoint $\theta_{p_1+p_2-1} = \lambda\pi$.

Let $m$ be an odd integer with $0 < m < p_1$. By proposition 4.5, there is a unique odd integer $n_m$ satisfying (4.16). It follows that $0 < n_m < p_2$. Thus, $m\pi/k_1$ and $n_m\pi/k_2$ form the endpoints (in some order) of an interval $I_j$ with $j$ even and $j \leqslant p_1 + p_2 - 2$. It follows that $\operatorname{Im} x_{s,m}(t) \in I_j$ and $\operatorname{Im} x_{f,n_m}(t) \in I_j$ as long as they can be smoothly continued by the implicit function theorem. Indeed, as $t \to -\infty$ their imaginary parts converge to $\theta_{j-1}$ or $\theta_j$ (by (5.3) and (5.4)), but (for $t \in \mathbb{R}$) these imaginary parts cannot equal either $\theta_{j-1}$ or $\theta_j$ and are excluded from the neighbouring strips, with imaginary parts in $I_{j-1}$ or $I_{j+1}$ since they are odd numbered. Thus, we avoid the exceptional case, and the curves $x_{s,m}(t)$ and $x_{f,n_m}(t)$ are defined for all $t \in \mathbb{R}$. It follows that $\operatorname{Im} x_{s,m}(t) \in I_j$ and $\operatorname{Im} x_{f,n_m}(t) \in I_j$ for all $t \in \mathbb{R}$.

For all $t \in \mathbb{R}$, there are precisely two zeros of $F(\cdot, t)$ with imaginary part in $I_j$. This is true for large negative $t$, and thus for all $t \in \mathbb{R}$ by continuous dependence of the zeros of polynomials on their coefficients, and since no zeros can ever have imaginary part equal to $\theta_{j-1}$ or $\theta_j$. Since corollary 3.7 excludes the possibility of a zero of $F(\cdot, 0)$ with imaginary part in $I_j$ and real part equal to 0, we see that $\operatorname{Re} x_{f,n_m}(0) < 0 < \operatorname{Re} x_{s,m}(0)$. Since $F(x, 0) = 0$ implies $F(-\overline{x}, 0) = 0$, it follows that $x_{f,n_m}(0) = -\overline{x_{s,m}(0)}$ since otherwise there would be four zeros of $F(\cdot, 0)$ with imaginary part in $I_j$, i.e. $x_{s,m}(0), x_{f,n_m}(0), -\overline{x_{s,m}(0)}, -\overline{x_{f,n_m}(0)}$. Furthermore, since $F(x, t) = 0$ if and only if $F(-\overline{x}, -t) = 0$, it follows that $F(-\overline{x_{s,m}(t)}, -t) = F(-\overline{x_{f,n_m}(t)}, -t) = 0$ for all $t \in \mathbb{R}$, and hence that

$$x_{f,n_m}(-t) = -\overline{x_{s,m}(t)} \tag{5.7}$$

for all $t \in \mathbb{R}$. Indeed, this is true at $t = 0$, and therefore at all $t \in \mathbb{R}$ since there can be no zeros of $F(\cdot, t)$ with imaginary part in $I_j$ other than $x_{s,m}(t)$ and $x_{f,n_m}(t)$. In particular, $\operatorname{Re} x_{f,n_m}(t) < k_1^2 t < k_2^2 t < \operatorname{Re} x_{s,m}(t)$ for all $t > 0$. Since $A_1 \neq 1$ and $A_2 \neq 1$ for all $t \in \mathbb{R}$, where $A_1$ and $A_2$ are given by (3.2) and (3.3), it follows by proposition 3.8 that $\operatorname{Im} x'_{s,m}(t) \neq 0$

and $\operatorname{Im} x'_{f,n_m}(t) \neq 0$ for all $t \in \mathbb{R}$ with the two zeros moving vertically towards each other for $t < 0$, and then separating for $t > 0$. Also, notice that

$$\lim_{t\to\infty} \operatorname{Im} x_{s,m}(t) = \frac{n_m\pi}{k_2}$$

and

$$\lim_{t\to\infty} \operatorname{Im} x_{f,n_m}(t) = \frac{m\pi}{k_1}.$$

More precisely, it follows from (5.5), (5.6) and (5.7) that

$$x_{s,m}(t) = k_2^2 t + \frac{1}{k_2}\log\gamma + \frac{n_m\pi\mathrm{i}}{k_2} - \frac{4k_1}{k_2^2-k_1^2}\gamma^{-k_1/k_2}\mathrm{e}^{-\frac{k_1}{k_2}n_m\pi\mathrm{i}}\mathrm{e}^{-k_1(k_2^2-k_1^2)t} + o(\mathrm{e}^{-k_1(k_2^2-k_1^2)t}) \tag{5.8}$$

and

$$x_{f,n_m}(t) = k_1^2 t - \frac{1}{k_1}\log\gamma + \frac{m\pi\mathrm{i}}{k_1} + \frac{4k_2}{k_2^2-k_1^2}\gamma^{-k_2/k_1}\mathrm{e}^{\frac{k_2}{k_1}m\pi\mathrm{i}}\mathrm{e}^{-k_2(k_2^2-k_1^2)t} + o(\mathrm{e}^{-k_2(k_2^2-k_1^2)t}) \tag{5.9}$$

as $t \to \infty$. In particular, the curve $x_{s,m}(t)$, which was associated with a slow moving pole for large negative $t$, becomes associated with a fast moving pole for large positive $t$. The reverse switch happens for the paired curve $x_{f,n_m}(t)$. The precise horizontal ordering for $t > 0$ of these curves can be deduced from the ordering for $t < 0$ using (5.7). Alternatively, the ordering for large $t$ can be deduced from the asymptotic expressions (5.8) and (5.9) using proposition 4.6.

For information, note that since $\operatorname{Im} x'_{s,m}(t) \neq 0$ and $\operatorname{Im} x'_{f,n_m}(t) \neq 0$, with opposite signs (by proposition 3.8), since the asymptotic values of $\operatorname{Im} x_{s,m}(t)$ and $\operatorname{Im} x_{f,n_m}(t)$ are exchanged between $-\infty$ and $\infty$, and since the real parts of these curves both go from $-\infty$ to $\infty$ as $t$ goes from $-\infty$ to $\infty$, the curves $x_{s,m}(t)$ and $x_{f,n_m}(t)$ must intersect at least once, but for different values of $t$. More precisely, since $\operatorname{Re} x_{s,m}(t) < 0$ for large negative $t$ (by (5.1)) but $\operatorname{Re} x_{s,m}(0) > 0$, there exists $t_0 < 0$ such that $\operatorname{Re} x_{s,m}(t_0) = 0$. It follows that $x_{f,n_m}(-t_0) = -\overline{x_{s,m}(t_0)} = x_{s,m}(t_0)$. This is true for any $t_0$ such that $\operatorname{Re} x_{s,m}(t_0) = 0$. Since $k_2^2 t < \operatorname{Re} x_{s,m}(t)$ for all $t > 0$, any such $t_0$ must be negative. We do not have a good way in general to determine how many such $t_0$ might exist. Also, there is no other way for the curves $x_{s,m}(t)$ and $x_{f,n_m}(t)$ to intersect. Indeed if $x_{s,m}(t_1) = x_{f,n_m}(t_2)$ and $t_1 \neq -t_2$ then $x_{s,m}(-t_2) = x_{f,n_m}(-t_1)$. But, this would imply that

$$\operatorname{Im} x_{s,m}(-t_2) = \operatorname{Im} x_{f,n_m}(-t_1) = -\operatorname{Im} \overline{x_{s,m}(t_1)} = \operatorname{Im} x_{s,m}(t_1),$$

contradicting the fact that $\operatorname{Im} x'_{s,m}(t)$ has the same sign for all $t \in \mathbb{R}$. Also, if $x_{s,m}(t_0) = x_{f,n_m}(-t_0)$, then necessarily $x_{s,m}(t_0) = -\overline{x_{s,m}(t_0)}$ and so must have real part equal to 0. Finally, if $d_m \leqslant 2/k_2$, then $\cos k_1\theta < 0$ and $\cos k_2\theta < 0$ for all $\theta \in I_j$, which implies by proposition 3.9 that $\operatorname{Re} x'_{s,m}(t) > 0$ and $\operatorname{Re} x'_{f,n_m}(t) > 0$ for all $t \in \mathbb{R}$. In particular, there can only be one point of intersection of the two curves.

Next, consider a curve of zeros $x_{f,n}(t)$ where $n$ is odd, $0 < n < p_2 - 1$ and $n$ is not equal to $n_m$ for some odd integer $m$ with $0 < m < p_1$. Since $\operatorname{Im} x_{f,n}(t) \to n\pi/k_2$ as $t \to -\infty$, this curve will satisfy $\operatorname{Im} x_{f,n}(t) \in I_j$, where $I_j$ is one of two intervals with endpoint $n\pi/k_2$, the one which is even numbered. The other endpoint of the interval $I_j$ is necessarily either $(n+1)\pi/k_2$ or $(n-1)\pi/k_2$. Here again we avoid the exceptional case, since $n + 1 < p_2$, and so $x_{f,n}(t)$ is defined for all $t \in \mathbb{R}$. Furthermore, for all $t \in \mathbb{R}$ there is precisely one zero of $F(\cdot, t)$ with imaginary part in $I_j$. (This is true for large negative $t$, and so by continuous dependence of zeros of a polynomial on its coefficients, for all $t \in \mathbb{R}$.) It follows from (5.2) and proposition 4.3 that $\operatorname{Re} x_{f,n}(t) < k_2^2 t < 0$ for large negative $t$, and thus by lemma 3.2

for all $t < 0$. Furthermore, corollary 3.7 implies that there must be a zero of $F(\cdot, 0)$ with real part 0 and imaginary part in $[(n-1)\pi/k_2, (n+1)\pi/k_2]$, hence in $I_j$. We conclude that $\operatorname{Re} x_{f,n}(0) = 0$ and so $x_{f,n}(0) = -\overline{x_{f,n}(0)}$. Since $F(-\overline{x_{f,n}(t)}, -t) = F(x_{f,n}(t), t) = 0$, it transpires that

$$-\overline{x_{f,n}(t)} = x_{f,n}(-t) \tag{5.10}$$

for all $t \in \mathbb{R}$. It then follows from (5.6) that

$$x_{f,n}(t) = k_2^2 t + \frac{1}{k_2}\log\gamma + \frac{n\pi\mathrm{i}}{k_2} - \frac{4k_1}{k_2^2 - k_1^2}\gamma^{-k_1/k_2}\mathrm{e}^{-\frac{k_1}{k_2}n\pi\mathrm{i}}\mathrm{e}^{-k_1(k_2^2-k_1^2)t} + o(\mathrm{e}^{-k_1(k_2^2-k_1^2)t}) \tag{5.11}$$

as $t \to \infty$.

Corollary 4.7 tells us that for large negative $t$, the real parts of the curves $x_{f,n}(t)$, where $n$ is not of the form $n_m$ for some odd $m$, are situated between the slow poles to the right, and the paired fast poles to the left. Corollary 4.7 also gives the precise horizontal ordering of these curves. By proposition 3.3, this order is maintained for all $t < 0$. In particular, it is the case that

$$\operatorname{Re} x_{f,n_m}(t) < \operatorname{Re} x_{f,n}(t) < k_2^2 t < k_1^2 t < \operatorname{Re} x_{s,m}(t)$$

for all $t < 0$. By (5.7) and (5.10), it follows that

$$\operatorname{Re} x_{f,n_m}(t) < k_1^2 t < k_2^2 t < \operatorname{Re} x_{f,n}(t) < \operatorname{Re} x_{s,m}(t)$$

for all $t > 0$.

Finally, by proposition 3.8, $\operatorname{Im} x'_{f,n}(t) \neq 0$ for all $t \neq 0$, with $x_{f,n}(t)$ moving away from the horizontal line with imaginary part $n\pi/k_2$ for $t < 0$, and moving back towards that line asymptotically for $t > 0$. Clearly, $\operatorname{Im} x'_{f,n}(0) = 0$. By proposition 3.9, $\operatorname{Re} x'_{f,n}(0) > 0$.

*5.1.1. The commensurable case: $p_1$ and $p_2$ both odd integers.* In this case, all the zeros of $F(\cdot, t)$ are simple for all $t \in \mathbb{R}$ and all the curves of zeros can be smoothly extended for all $t \in \mathbb{R}$. Among the curves of zeros already found, there are $(p_1 - 1)/2$ curves $x_{s,m}(t)$ with $m$ odd, $0 < m < p_1$, and $(p_1 - 1)/2$ associated curves $x_{f,n_m}(t)$. There are precisely $(p_2 - p_1)/2$ remaining curves $x_{f,n}(t)$ with $n$ odd, $0 < n < p_2 - 1$. This accounts for a total of $(p_2 + p_1 - 2)/2$ zeros of $F(\cdot, t)$ in the fundamental strip $S$, given by (2.9). But all of these are in the open lower half of $S$, i.e. $\{x \in \mathbb{C} : 0 < \operatorname{Im} x < \lambda\pi\}$. Thus, each of these curves has a reflection above the line $\{x \in \mathbb{C} : \operatorname{Im} x = \lambda\pi\}$, which thus accounts for a total of $p_2 + p_1 - 2$ zeros in $S$, for each $t \in \mathbb{R}$.

In addition, there are two curves of zeros whose imaginary parts converge to $\lambda\pi$ as $t \to -\infty$, i.e. $x_{s,m}(t)$ with $m = p_1$, which we call $x_s(t)$, and the curve $x_{f,n}(t)$ with $n = p_2$, which we call $x_f(t)$. Also, in this case, the intervals $I_{p_1+p_2-1}$ and $I_{p_1+p_2}$ are, respectively, odd and even numbered, and so there cannot exist a zero with imaginary part in $I_{p_1+p_2-1}$, or in $I_{p_1+p_2}$. It follows that $\operatorname{Im} x_s(t) = \operatorname{Im} x_f(t) = \lambda\pi$ for all $t \in \mathbb{R}$. Moreover, since $d_{p_1} = D_{p_2} = 0$ (defined in (4.14) and (4.15)) it follows from corollary 4.7 that, for large negative $t$, $\operatorname{Re} x_f(t) < \operatorname{Re} x(t) < \operatorname{Re} x_s(t)$ for every other curve $x(t)$ of zeros in $S$. This is true for all $t < 0$ by proposition 3.3. Furthermore, one checks directly that, in this case, $F(\lambda\pi\mathrm{i}, 0) \neq 0$, and so $\operatorname{Re} x_f(0) < 0 < \operatorname{Re} x_s(0)$. Since these are the only two remaining zeros for $t > 0$, it must be the case that

$$x_f(-t) = -\overline{x_s(t)} \tag{5.12}$$

for all $t \in \mathbb{R}$. By proposition 3.3, it now follows that $\operatorname{Re} x_f(t) < \operatorname{Re} x(t) < \operatorname{Re} x_s(t)$ for every other curve $x(t)$ of zeros in $S$, for all $t \in \mathbb{R}$. More precisely, the ordering of the real parts of all the zeros in $S$ for large negative $t$ is given by corollary 4.7, and so, by proposition 3.3, is preserved for all $t < 0$ (all $t \in \mathbb{R}$ for the zeros not on the imaginary axis at $t = 0$). By (5.7), (5.10) and (5.12), this ordering is then continued for all $t > 0$ for all the zeros in the strip $S$.

Counting the number of zeros of $F(\cdot, 0)$ in $S$ whose real part vanishes, we see that there are precisely $p_2 - p_1$ of them, corresponding to the curves $x_{f,n}(t)$ and their reflected curves on the top half of $S$. That leaves $2p_1$ zeros of $F(\cdot, 0)$ in the strip $S$ off the imaginary axis.

Finally, direct substitution into formula (7.5), which is derived in section 7 in the proof of proposition 3.9, shows that $\operatorname{Re} x_f'(t) > 0$ and $\operatorname{Re} x_s'(t) > 0$ for all $t \in \mathbb{R}$. For this purpose, one uses $A_1 + 1/A_1 \geqslant 2$ and likewise for $A_2$, as well as the fact that $\operatorname{Im} x_f(t) = \operatorname{Im} x_s(t) = \lambda\pi$ for all $t \in \mathbb{R}$. Do not forget that $p_1$ and $p_2$ are both odd. Indeed, the quantity in (7.5), multiplied by $\gamma$, is bounded below by

$$\begin{aligned}
-k_1^4(1 - 2\gamma + \gamma^2) &- k_2^4(1 - 2\gamma + \gamma^2) + k_1k_2(k_1^2 + k_2^2)(-1 + \gamma^2) \\
&= -(k_1^4 + k_2^4)(\gamma - 1)^2 + k_1k_2(k_1^2 + k_2^2)(-1 + \gamma^2) \\
&= -(k_1^4 + k_2^4)\frac{4k_1^2}{(k_2 - k_1)^2} + k_1k_2(k_1^2 + k_2^2)\frac{4k_1k_2}{(k_2 - k_1)^2} \\
&= \frac{4k_1^2}{(k_2 - k_1)^2}[-(k_1^4 + k_2^4) + k_2^2(k_1^2 + k_2^2)] = 4k_1^4\frac{k_2^2 - k_1^2}{(k_2 - k_1)^2} > 0.
\end{aligned}$$

*5.1.2. The commensurable case. $p_1$ even and $p_2$ odd.* In this case, all the zeros of $F(\cdot, t)$ are simple for all $t \in \mathbb{R}$ and all the curves of zeros can be smoothly extended for all $t \in \mathbb{R}$. Among the curves of zeros already found, there are $p_1/2$ curves $x_{s,m}(t)$ with $m$ odd, $0 < m < p_1$, and $p_1/2$ associated curves $x_{f,n_m}(t)$. There are precisely $(p_2 - p_1 - 1)/2$ remaining curves of the form $x_{f,n}(t)$ with $n$ odd, $0 < n < p_2 - 1$. This accounts for a total of $(p_2 + p_1 - 1)/2$ zeros of $F(\cdot, t)$ in the fundamental strip $S$, given by (2.9). But all of these are in the open lower half of $S$, i.e. $\{x \in \mathbb{C} : 0 < \operatorname{Im} x < \lambda\pi\}$. Thus, each of these curves has a reflection above the line $\{x \in \mathbb{C} : \operatorname{Im} x = \lambda\pi\}$, which accounts for a total of $p_2 + p_1 - 1$ zeros in $S$, for each $t \in \mathbb{R}$. Hence, there is one zero in $S$, given by (2.9), so far unaccounted for.

There is one curve of zeros whose imaginary part converges to $\lambda\pi$ as $t \to -\infty$, i.e. the curve $x_{f,n}(t)$ with $n = p_2$, which is denoted here by $x_f(t)$. In this case the intervals $I_{p_1+p_2-1}$ and $I_{p_1+p_2}$ are, respectively, even and odd numbered, and so there might not be a zero with imaginary part in $I_{p_1+p_2-1}$ or in $I_{p_1+p_2}$. On the other hand, by symmetry, if there is a zero of $F(\cdot, t)$ whose imaginary part is in $I_{p_1+p_2-1}$, there must be one also whose imaginary part is in $I_{p_1+p_2}$. But since all but one zero of $F(\cdot, t)$ in $S$ have been accounted for (and their imaginary parts are not in $I_{p_1+p_2-1}$ or in $I_{p_1+p_2}$), it follows that $\operatorname{Im} x_f(t) = \lambda\pi$ for all $t \in \mathbb{R}$. A direct calculation shows that $F(\lambda\pi\mathrm{i}, 0) = 0$; and so $x_f(0) = \lambda\pi\mathrm{i}$, and by the reasoning already used several times,

$$x_f(-t) = -\overline{x_f(t)}. \tag{5.13}$$

Also, in this case, $D_n = 1/k_1$ where $n = p_2$ and $D_n$ is defined by (4.15). It then follows from (4.17) and propositions 4.7 and 3.3 that

$$\operatorname{Re} x_{f,n}(t) < \operatorname{Re} x_f(t) < k_2^2 t < k_1^2 t < \operatorname{Re} x_{s,m}(t)$$

for all $t < 0$, for all odd $m$ and all odd $n$, except if $x_{f,n}(t)$ is related to $x_f(t)$ by a symmetry operation. On the other hand, (5.7), (5.10) and (5.13) imply that for $t > 0$,

$$\operatorname{Re} x_{f,n_m}(t) < k_1^2 t < k_2^2 t < \operatorname{Re} x_f(t) < \operatorname{Re} x_{f,n}(t) < \operatorname{Re} x_{s,m}(t),$$

for all odd $m$ and (on the right side) all odd $n \neq p_2$ not of the form $n = n_m$ for some $m$ and not related to $x_f(t)$ by symmetries.

Counting the number of zeros of $F(\cdot, 0)$ in $S$ whose real part vanishes, we see that there are $p_2 - p_1 - 1$ of them corresponding to curves of the form $x_{f,n}(t)$ with $n$ odd not of the form $n_m$ for some $m$ odd. In addition, there is the zero $x_f(0) = \lambda\pi\mathrm{i}$. This makes a total of $p_2 - p_1$ zeros of $F(\cdot, 0)$ in the strip $S$ on the imaginary axis, and $2p_1$ zeros of $F(\cdot, 0)$ in the strip $S$ off the imaginary axis. Note that $2p_1$ is divisible by 4.

Finally, using again formula (7.5) and substituting directly yields that $\operatorname{Re} x_f'(t) > 0$ for all $t \in \mathbb{R}$. Using the fact that $p_1$ is even and $p_2$ is odd, it is seen that the expression in (7.5), multiplied by $\gamma$, equals

$$k_1^4 + k_1k_2(k_1^2 + k_2^2) + k_2^4 + [k_1^4 - k_1k_2(k_1^2 + k_2^2) + k_2^4]\gamma^2 \\ + \gamma[k_2^4(A_1 + 1/A_1) - k_1^4(A_2 + 1/A_2)].$$

However, $A_2 \to \gamma$ and $A_1 \to \infty$ as $t \to -\infty$. (This follows from (3.2), (3.3), (4.4), (5.2) and proposition 4.3.) Thus, by lemma 3.2, it follows that $A_2 > 1$ and $A_1 > 1$ for all $t < 0$. Also, we must have $A_1 > A_2 > 1$ for all $t < 0$ since it is true for large negative $t$, and if ever $A_1 = A_2$ for some $t < 0$, it follows from (3.8) that $\sin(p_1 + p_2)\pi/2 = 0$. But this is not true since $p_1 + p_2$ is odd. Thus $A_1 > A_2 > 1$ for all $t < 0$, and likewise $A_1 < A_2 < 1$ for all $t > 0$, which shows that $\operatorname{Re} x_f'(t) > 0$ for all $t \in \mathbb{R}$.

*5.1.3. The commensurable case. $p_1$ odd and $p_2$ even.* Among the curves of zeros already found, there are $(p_1 - 1)/2$ curves $x_{s,m}(t)$ with $m$ odd, $0 < m < p_1$, and $(p_1 - 1)/2$ associated curves $x_{f,n_m}(t)$. There are precisely $(p_2 - p_1 - 1)/2$ remaining curves of the form $x_{f,n}(t)$ with $n$ odd, $0 < n < p_2 - 1$. This accounts for a total of $(p_2 + p_1 - 3)/2$ zeros of $F(\cdot, t)$ in the fundamental strip $S$, given by (2.9). Again, all of these are in the open lower half of $S$, i.e. $\{x \in \mathbb{C} : 0 < \operatorname{Im} x < \lambda\pi\}$. Thus, each of these curves has a reflection above the line $\{x \in \mathbb{C} : \operatorname{Im} x = \lambda\pi\}$, which thus accounts for a total of $p_2 + p_1 - 3$ zeros in $S$, for each $t \in \mathbb{R}$. All of these curves are defined for all $t \in \mathbb{R}$. On the other hand, there are three zeros in $S$ so far unaccounted for. These are the curves $x_{s,m}(t)$ with $m = p_1$, which we henceforth refer to as $x_s(t)$, as well as the curves $x_{f,n}(t)$ with $n = p_2 \pm 1$. These two last curves are reflections of each other about the line $\{x \in \mathbb{C} : \operatorname{Im} x = \lambda\pi\}$. The curve $x_{f,n}(t)$ with $n = p_2 - 1$ is referred to here as $x_f(t)$ and the curve $x_{f,n}(t)$ with $n = p_2 + 1$ as $\tilde{x}_f(t)$.

Since $\operatorname{Im} x_f(t) \to (p_2 - 1)\pi/k_2 = \theta_{p_1+p_2-2}$ as $t \to -\infty$, and since $p_1 + p_2 - 2$ is odd, it follows that for $t < 0$, $\operatorname{Im} x_f(t)$ must lie in the interval $I_{p_1+p_2-1}$. Similarly, or by reflection, $\operatorname{Im} \tilde{x}_f(t)$ must be in $I_{p_1+p_2}$ for all $t < 0$. Indeed, if for any $t < 0$ it would happen that $\operatorname{Im} x_f(t) = \lambda\pi$, then the same would be true for $\operatorname{Im} \tilde{x}_f(t)$, thus giving a double zero. By proposition 3.1, this is impossible. Finally, in order to preserve the right number of zeros for all $t < 0$ it must be that $\operatorname{Im} x_s(t) = \lambda\pi$ for all $t < 0$. If not, there would be another zero obtained by reflection about the line $\{x \in \mathbb{C} : \operatorname{Im} x = \lambda\pi\}$, which would make too many zeros.

At $t = 0$ there is a triple zero of $F(\cdot, 0)$ at $x = \lambda\pi\mathrm{i}$. For $t \neq 0$, there are only simple zeros. By continuous dependence of zeros of a polynomial on its coefficients, we conclude that all three of the curves, $x_s(t)$, $x_f(t)$ and $\tilde{x}_f(t)$ converge as $t \to 0^-$ to $\lambda\pi\mathrm{i}$. The behaviour of these curves as $t \to 0^-$ is given by proposition 4.9. More precisely, let us first consider $x_s(t)$, so that $\operatorname{Im} x_s(t) = \lambda\pi$ for all $t < 0$. In this case $x_s(t) - \lambda\pi\mathrm{i} = \operatorname{Re} x_s(t)$ and $x_s'(t) = \operatorname{Re} x_s'(t)$. Thus, (4.18) and (4.19) imply that

$$\operatorname{Re} x_s(t) \sim -(12t)^{1/3} = (12|t|)^{1/3}$$

and

$$\operatorname{Re} x_s'(t) \sim -\frac{4}{9t^2} \tag{5.14}$$

as $t \to 0^-$. In particular, the curve $x_s(t)$ has passed through the point $\lambda\pi\mathrm{i}$ at some $t < 0$ into the right half plane, and then turned around to approach $\lambda\pi\mathrm{i}$ from the right, and at infinite speed. As for the curve $x_f(t)$, we know that $\mathrm{Im}\, x_f(t) \in I_{p_1+p_2-1}$ for $t < 0$. In particular, $\mathrm{Im}\, x_f(t) < \lambda\pi$. In this case, (4.18) and (4.19) imply that

$$x_f(t) - \lambda\pi\mathrm{i} \sim (12|t|)^{1/3}\mathrm{e}^{4\pi\mathrm{i}/3}$$

and

$$x_f'(t) \sim \left(\frac{4}{9t^2}\right)^{1/3}\mathrm{e}^{\pi\mathrm{i}/3}$$

as $t \to 0^-$. Similarly,

$$\tilde{x}_f(t) - \lambda\pi\mathrm{i} \sim (12|t|)^{1/3}\mathrm{e}^{2\pi\mathrm{i}/3}$$

and

$$\tilde{x}_f'(t) \sim \left(\frac{4}{9t^2}\right)^{1/3}\mathrm{e}^{5\pi\mathrm{i}/3}$$

as $t \to 0^-$. In view of these relations, it is natural to set, for $t > 0$,

$$x_s(t) = -\overline{x_s(-t)}, \tag{5.15}$$

$$x_f(t) = -\overline{\tilde{x}_f(-t)} \tag{5.16}$$

and

$$\tilde{x}_f(t) = -\overline{x_f(-t)}. \tag{5.17}$$

With these choices, we have the following behaviour as $t \to 0^+$:

$$\begin{aligned}
&\mathrm{Re}\, x_s(t) \sim -(12t)^{1/3},\\
&\mathrm{Re}\, x_s'(t) \sim -\frac{4}{9t^2},\\
&x_f(t) - \lambda\pi\mathrm{i} \sim (12t)^{1/3}\mathrm{e}^{\pi\mathrm{i}/3},\\
&x_f'(t) \sim \left(\frac{4}{9t^2}\right)^{1/3}\mathrm{e}^{\pi\mathrm{i}/3},\\
&\tilde{x}_f(t) - \lambda\pi\mathrm{i} \sim (12t)^{1/3}\mathrm{e}^{5\pi\mathrm{i}/3},\\
&\tilde{x}_f'(t) \sim \left(\frac{4}{9t^2}\right)^{1/3}\mathrm{e}^{5\pi\mathrm{i}/3}.
\end{aligned} \tag{5.18}$$

As for the ordering of the various curves, in this case we have $d_{p_1} = D_{p_2\pm1} = 1/k_2$. It follows from corollary 4.7 and proposition 3.3, as well as (5.15), (5.16) and (5.17), that

$$\mathrm{Re}\, x_{f,n_m}(t) < \mathrm{Re}\, x_f(t) = \mathrm{Re}\, \tilde{x}_f(t) < \mathrm{Re}\, x_{f,n}(t) < k_2^2 t < k_1^2 t < \mathrm{Re}\, x_s(t) < \mathrm{Re}\, x_{s,m}(t)$$

for all $t < 0$ and

$$\mathrm{Re}\, x_{f,n_m}(t) < \mathrm{Re}\, x_s(t) < k_1^2 t < k_2^2 t < \mathrm{Re}\, x_{f,n}(t) < \mathrm{Re}\, x_f(t) = \mathrm{Re}\, \tilde{x}_f(t) < \mathrm{Re}\, x_{s,m}(t)$$

for all $t > 0$.

Finally, if the zeros in the fundamental strip $S$ on the imaginary axis at $t = 0$ are counted, there is the triple zero at $\lambda\pi\mathrm{i}$, as well as the $p_2 - p_1 - 1$ unpaired fast poles. This makes for a total of $p_2 - p_1 + 2$ zeros counted with multiplicity, and therefore $p_2 - p_1$ counted without multiplicity, in the fundamental strip $S$ on the imaginary axis at $t = 0$, and thus $2(p_1 - 1)$ zeros in the fundamental strip $S$ lie off the imaginary axis at $t = 0$.

*5.2. The non-commensurable case*

If $k_1$ and $k_2$ are not commensurable, then $J_1 \cap J_2 = \{0\}$, and so each $\theta_j$, $j \geqslant 1$, belongs precisely to one of the sets $J_1$ or $J_2$. It follows from proposition 3.4 that no zero of $F(\cdot, t)$ can have imaginary part equal to any $\theta_j$, and by proposition 3.5 such zeros cannot have imaginary part in $I_j$ with $j$ odd. Thus, the analysis of the curves of zeros $x_{s,m}(t)$ and $x_{f,n_m}(t)$ and of the curves $x_{f,n}(t)$ when $n \neq n_m$, which was carried out in the commensurable case under the assumptions $0 < m < p_1$ and $0 < n < p_2 - 1$, can be repeated for the non-commensurable case, for any odd integers $n$ and $m$. The only difficulty is to show that no other zeros are present. In other words, since there is no polynomial function in the background, it is not immediately clear when all the zeros have been accounted for.

This can be handled by the following limiting procedure. For each $q \in \mathbb{N}$, let $k_{1,q}$ and $k_{2,q}$ be commensurable, $0 < k_{1,q} < k_{2,q}$, and such that $k_{1,q} \to k_1$ and $k_{2,q} \to k_2$. Let $p_{1,q}$, $p_{2,q}$, $\lambda_q$, $\theta_{j,q}$, $I_{j,q}$ and $F_q(x,t)$ have the obvious meanings. Clearly, $F_q \to F$ uniformly on compact subsets of $\mathbb{C} \times \mathbb{R}$. Note also that $\lambda_q \to \infty$. To see this, observe that $\lambda_q = p_{1,q}/k_{1,q} \geqslant 1/k_{1,q} \to 1/k_1$. Thus, if $\lambda_q$ has a bounded subsequence, it has a convergent subsequence, with limit $\lambda_\infty$, say, which is finite and positive. It would follow that $F(\cdot, t)$ is periodic with period $2\lambda_\infty \pi \mathrm{i}$, which is impossible. Thus, $\theta_{j,q} \to \theta_j$ and, for given $j$, we have $\theta_{j,q} < \lambda_q \pi$ for sufficiently large $q$. Since $I_{j,q} \to I_j$, the number of zeros of $F(\cdot, t)$ with imaginary part in $I_j$ must be the limit of the number of zeros of $F_q(\cdot, t)$ with imaginary part in $I_{j,q}$ by Rouché's theorem.

Also, on account of (5.14) and (5.18), it can happen that $\mathrm{Re}\, x'(t) < 0$ for some value of $t$, for some curve of zeros $x(t)$ and for some *non-commensurable* values of $k_1$ and $k_2$. Indeed, suppose $k_1$ and $k_2$ are commensurable, with $p_1$ odd and $p_2$ even, and $x_s(t)$ is the curve which verifies (5.14) and (5.18). Then by continuous dependence, if $k_1$ and $k_2$ are approached by a sequence of non-commensurable numbers $k_{1,q}$ and $k_{2,q}$, then $x_s(t)$ will be approached by a sequence of curves of zeros $x_q(t)$, which for $q$ sufficiently large must have $\mathrm{Re}\, x_q'(t) < 0$ for some values of $t$.

Next, consider the asymptotic density of zeros of $F(\cdot, 0)$ which are purely imaginary. For example, consider the number of zeros which are purely imaginary and whose real part is contained in an interval of the form $[0, n\pi/k_2]$ for some fixed $n \in \mathbb{N}$. By proposition 3.6, it follows that this is the same as the number of solutions of the equation $\mathrm{e}^{ik_1\alpha} + \mathrm{e}^{ik_2\alpha} = 0$ with $\alpha \in [0, n\pi/k_2]$. But such solutions are of the form

$$\alpha = \frac{(2l+1)\pi}{k_2 - k_1}$$

for some integer $l$. For large values of $n$ there are (up to an error of less than 1) $n(k_2 - k_1)/2k_2$ such values of $\alpha$. This give a density of

$$\frac{n(k_2 - k_1)/2k_2}{n\pi/k_2} = \frac{k_2 - k_1}{2\pi}.$$

The overall density of zeros whose imaginary parts are in a certain strip is easily found as the limit of densities in the commensurable cases, which are all easy to count.

## 6. Blowup: proof of theorem 4

Let $z(t)$ be a smooth curve of zeros of $F(\cdot, t)$. As we know, $z(t)$ is always a simple zero. (The only non-simple zero occurs where $z'(t)$ becomes infinite, and only in the commensurable case.) It follows that for each $t$, $F(\cdot, t)$ can be expressed locally near $z(t)$ by a development of the form

$$F(x,t) = c(t)(x - z(t)) + (x - z(t))^2 v(x,t)$$
$$= (x - z(t))[c(t) + (x - z(t))v(x,t)],$$

where $v(\cdot, t)$ is analytic in $x$ and $c(t) \neq 0$. It follows that

$$F_x(x,t) = c(t) + 2(x - z(t))v(x,t) + (x - z(t))^2 v_x(x,t),$$

and so locally near $z(t)$,

$$\frac{F_x(x,t)}{F(x,t)} = \frac{1}{x - z(t)} + w(x,t),$$

where $w$ is smooth. Thus $u = 12(\log F)_{xx}$ is given locally near $z(t)$ by

$$u(x,t) = \frac{-12}{(x - z(t))^2} + w_x(x,t).$$

Let $z(T) = x_0 + \mathrm{i}\alpha$, for some $x_0 \in \mathbb{R}$ and $z'(T) = a + \mathrm{i}b$, for some $a, b \in \mathbb{R}$ with $b \neq 0$. Thus, for $t$ near $T$, $z$ has the form

$$z(t) = x_0 + \mathrm{i}\alpha - (T - t)(a + \mathrm{i}b) + (T - t)^2 h(x,t),$$

for some smooth function $h(x,t)$, and so

$$u(x,t) = \frac{-12}{(x - x_0 - \mathrm{i}\alpha + (T - t)(a + \mathrm{i}b) - (T - t)^2 h(x,t))^2} + w_x(x,t),$$

or,

$$u_\alpha(x,t) = u(x + \mathrm{i}\alpha, t) = \frac{-12}{(x - x_0 + (T - t)(a + \mathrm{i}b) - (T - t)^2 h(x,t))^2} + w_x(x + \mathrm{i}\alpha, t).$$

Set $y = (x - x_0)/(T - t)$, so $x = x_0 + y(T - t)$ and

$$(T - t)^2 u_\alpha(x_0 + y(T - t), t)$$
$$= \frac{-12}{(y + (a + \mathrm{i}b) - (T - t)h(x_0 + y(T - t), t))^2}$$
$$+ (T - t)^2 w_x(x_0 + y(T - t) + \mathrm{i}\alpha, t).$$

By proposition 3.8 and lemma 3.2, if $T \neq 0$ then always $b \neq 0$. In this case, by proposition 3.4, there is only one pole with imaginary part $\alpha$ when $t = T$. This corresponds to a single-point blowup. Furthermore, by lemma 3.2 it follows that if $T = 0$ and $z(0)$ corresponds to a slow pole or its paired fast pole, then also $b \neq 0$, and in this case, there are two poles with the same imaginary part, so there is also adduced a two-point blowup. In these cases, since $b \neq 0$, it follows that

$$(T - t)^2 u_\alpha(x_0 + y(T - t), t) \to \frac{-12}{(y + (a + \mathrm{i}b))^2},$$

uniformly for $y$ contained in any fixed compact subset of $\mathbb{R}$.

Finally, if at $T = 0$, $z(0)$ corresponds to an unpaired fast pole, then $b = 0$ since $z(t) = -\overline{z(-t)}$, and so

$$\frac{z(t) - z(0)}{t} = \frac{\overline{-z(-t) + z(0)}}{t} = \frac{\overline{z(-t) - z(0)}}{-t}.$$

On the other hand, $a > 0$ by proposition 3.9. In this case it is clear that

$$(T - t)^2 u_\alpha(x_0 + y(T - t), t) \to \frac{-12}{(y + a)^2},$$

uniformly for $y$ contained in any fixed compact subset of $\mathbb{R} \setminus \{-a\}$.

## 7. Horizontal movement of poles: proof of proposition 3.9

The proof of proposition 3.9 begins with the first formula in the proof of proposition 3.8. This is a formula for $x'(t)$ where $x(t)$ is a smooth curve of zeros of $F$ as in the statement of propositions 3.8 and 3.9. The real part of $x'(t)$ has the same sign as does the quantity

$$\Gamma = k_1^4 A_1^2 |A_2 + \gamma \mathrm{e}^{\mathrm{i}k_2\alpha}|^2 + k_2^4 A_2^2 |A_1 + \gamma \mathrm{e}^{\mathrm{i}k_1\alpha}|^2 \\ + k_1 k_2 (k_1^2 + k_2^2) A_1 A_2 (A_1 A_2 + A_1 \gamma \cos k_2\alpha + A_2 \gamma \cos k_1 \alpha + \gamma^2 \cos(k_2 - k_1)\alpha). \tag{7.1}$$

Replacing $A_1$ and $A_2$ in (3.4) by $1/A_1$ and $1/A_2$, respectively (which corresponds to writing $F(-\overline{x}, -t) = 0$), we obtain the equation

$$A_2(A_1 + \gamma \mathrm{e}^{\mathrm{i}k_1\alpha}) + \gamma A_1 \mathrm{e}^{\mathrm{i}k_2\alpha} + \mathrm{e}^{\mathrm{i}(k_2+k_1)\alpha} = 0.$$

Multiplying this by $(A_1 + \gamma \mathrm{e}^{-\mathrm{i}k_1\alpha})$ yields

$$A_2 |A_1 + \gamma \mathrm{e}^{\mathrm{i}k_1\alpha}|^2 = -[A_1 \cos(k_2 + k_1)\alpha + \gamma(A_1^2 + 1)\cos k_2\alpha + \gamma^2 A_1 \cos(k_2 - k_1)\alpha]. \tag{7.2}$$

Similarly, we may derive

$$A_1 |A_2 + \gamma \mathrm{e}^{\mathrm{i}k_2\alpha}|^2 = -[A_2 \cos(k_2 + k_1)\alpha + \gamma(A_2^2 + 1)\cos k_1\alpha + \gamma^2 A_2 \cos(k_2 - k_1)\alpha]. \tag{7.3}$$

Furthermore, multiplying (3.14) by $\cos(k_2 + k_1)\alpha$ and (3.15) by $\sin(k_2 + k_1)\alpha$, and adding the two resulting equations gives

$$A_1 A_2 + A_1 \gamma \cos k_2 \alpha + A_2 \gamma \cos k_1 \alpha = -\cos(k_2 + k_1)\alpha. \tag{7.4}$$

Substituting (7.2), (7.3) and (7.4) into (7.1), the quantity $\Gamma$ in (7.1) is seen to satisfy the relation

$$\frac{\Gamma}{\gamma A_1 A_2} = -k_1^4[\gamma^{-1}\cos(k_2 + k_1)\alpha + (A_2 + 1/A_2)\cos k_1\alpha + \gamma \cos(k_2 - k_1)\alpha] \\ -k_2^4[\gamma^{-1}\cos(k_2 + k_1)\alpha + (A_1 + 1/A_1)\cos k_2\alpha + \gamma\cos(k_2 - k_1)\alpha] \\ +k_1 k_2 (k_1^2 + k_2^2)(-\gamma^{-1}\cos(k_2 + k_1)\alpha + \gamma \cos(k_2 - k_1)\alpha). \tag{7.5}$$

Observe that (7.5) has the same sign as $\mathrm{Re}\, x'(t)$ even without the assumption that $\sin k_1\alpha \neq 0$ and $\sin k_2 \alpha \neq 0$.

Now, using the hypotheses that $\sin k_1\alpha \neq 0$ and $\sin k_2\alpha \neq 0$, we simplify each line of (7.5) separately. Dividing (3.13) by $A_2 \sin k_1\alpha$ leads to

$$A_2 + 1/A_2 = -\frac{\gamma^{-1}\sin(k_2 + k_1)\alpha - \gamma \sin(k_2 - k_1)\alpha}{\sin k_1\alpha},$$

and therefore (after simplification)

$$\gamma^{-1}\cos(k_2 + k_1)\alpha + (A_2 + 1/A_2)\cos k_1\alpha + \gamma\cos(k_2 - k_1)\alpha = \left(\gamma - \frac{1}{\gamma}\right)\frac{\sin k_2\alpha}{\sin k_1\alpha}.$$

Next we treat the second line of (7.5). Dividing (3.12) by $A_1 \sin k_2\alpha$ yields

$$A_1 + 1/A_1 = -\frac{\gamma^{-1}\sin(k_2 + k_1)\alpha + \gamma\sin(k_2 - k_1)\alpha}{\sin k_2\alpha},$$

whence (again, after simplification)

$$\gamma^{-1}\cos(k_2 + k_1)\alpha + (A_1 + 1/A_1)\cos k_2\alpha + \gamma\cos(k_2 - k_1)\alpha = \left(\gamma - \frac{1}{\gamma}\right)\frac{\sin k_1\alpha}{\sin k_2\alpha}.$$

The third line of (7.5) is rewritten using trigonometric identities, namely,

$$
\begin{aligned}
&-\gamma^{-1}\cos(k_2+k_1)\alpha+\gamma\cos(k_2-k_1)\alpha\\
&\qquad=\left(\gamma-\frac{1}{\gamma}\right)\cos k_2\alpha\cos k_1\alpha+\left(\gamma+\frac{1}{\gamma}\right)\sin k_2\alpha\sin k_1\alpha.
\end{aligned}
$$

Putting this together, it is seen that the quantity in (7.5) is equal to

$$
\begin{aligned}
&-k_1^4\left(\gamma-\frac{1}{\gamma}\right)\frac{\sin k_2\alpha}{\sin k_1\alpha}-k_2^4\left(\gamma-\frac{1}{\gamma}\right)\frac{\sin k_1\alpha}{\sin k_2\alpha}\\
&\qquad+k_1k_2(k_1^2+k_2^2)\left[\left(\gamma-\frac{1}{\gamma}\right)\cos k_2\alpha\cos k_1\alpha+\left(\gamma+\frac{1}{\gamma}\right)\sin k_2\alpha\sin k_1\alpha\right].
\end{aligned}
$$

Next, divide by $k_1^4(\gamma-\frac{1}{\gamma})$ and multiply by $-\sin k_1\alpha\sin k_2\alpha$, which is positive by proposition 3.5. This gives the expression

$$
\begin{aligned}
&\sin^2 k_2\alpha+R^4\sin^2 k_1\alpha-R(1+R^2)\cos k_2\alpha\cos k_1\alpha\sin k_2\alpha\sin k_1\alpha\\
&\qquad-\left(\frac{(1+R^2)^2}{2}\right)\sin^2 k_2\alpha\sin^2 k_1\alpha,
\end{aligned}
\tag{7.6}
$$

where $R=k_2/k_1$. The quantity in (7.6) has the same sign as (7.5), which has the same sign as $\operatorname{Re} x'(t)$. Note that $A_1$ and $A_2$ are absent from (7.6), which is the main point of the above calculation. One observes that (7.6) can be re-written as

$$
\begin{aligned}
&\tfrac{1}{2}(\sin k_2\alpha\cos k_1\alpha-R^2\sin k_1\alpha\cos k_2\alpha)^2+\tfrac{1}{2}(\sin k_2\alpha+R^2\sin k_1\alpha)^2\\
&\qquad+R^2(-\sin k_2\alpha\sin k_1\alpha-\sin^2 k_2\alpha\sin^2 k_1\alpha)\\
&\qquad-R(1-R+R^2)\cos k_2\alpha\cos k_1\alpha\sin k_2\alpha\sin k_1\alpha.
\end{aligned}
\tag{7.7}
$$

The first three terms are always positive, since $\sin k_2\alpha\sin k_1\alpha<0$. If $\cos k_2\alpha\cos k_1\alpha>0$, then the last term is also positive. This proves part (i) of proposition 3.9.

To prove the second assertion, begin by remarking that since $\operatorname{Re} x(0)=0$, it follows that, at $t=0$, $A_1=A_2=1$. Substituting this into (7.1) gives

$$
\begin{aligned}
\Gamma&=k_1^4(1+2\gamma\cos k_2\alpha+\gamma^2)+k_2^4(1+2\gamma\cos k_1\alpha+\gamma^2)\\
&\qquad+k_1k_2(k_1^2+k_2^2)(1+\gamma\cos k_2\alpha+\gamma\cos k_1\alpha+\gamma^2\cos(k_2-k_1)\alpha),
\end{aligned}
$$

which needs to be shown to be positive. Next, putting $A_1=A_2=1$ in (3.4), multiplying by $\mathrm{e}^{-\mathrm{i}k_1\alpha}$, and then taking the real part, there obtains

$$\gamma^2+\gamma\cos k_2\alpha+\gamma\cos k_1\alpha+\gamma^2\cos(k_2-k_1)\alpha=0.$$

Thus, it suffices to show that

$$k_1^4(1+2\gamma\cos k_2\alpha+\gamma^2)+k_2^4(1+2\gamma\cos k_1\alpha+\gamma^2)+k_1k_2(k_1^2+k_2^2)(1-\gamma^2)>0.$$

It is straightforward to check that

$$k_1^4(1+2\gamma+\gamma^2)+k_2^4(1-2\gamma+\gamma^2)+k_1k_2(k_1^2+k_2^2)(1-\gamma^2)=0.$$

Thus, it is sufficient to show that

$$k_1^4(\cos k_2\alpha-1)+k_2^4(\cos k_1\alpha+1))>0.$$

To prove this, note that solving (3.4) for $\mathrm{e}^{\mathrm{i}k_2\alpha}$, with $A_1=A_2=1$ gives

$$\mathrm{e}^{\mathrm{i}k_2\alpha}=-\frac{1+\gamma\mathrm{e}^{\mathrm{i}k_1\alpha}}{\gamma+\mathrm{e}^{\mathrm{i}k_1\alpha}}=-\frac{(1+\gamma\mathrm{e}^{\mathrm{i}k_1\alpha})(\gamma+\mathrm{e}^{-\mathrm{i}k_1\alpha})}{|\gamma+\mathrm{e}^{\mathrm{i}k_1\alpha}|^2}.$$

Taking the real part leads to

$$\cos k_2\alpha - 1 = -\frac{(\gamma+1)^2[\cos k_1\alpha + 1]}{\gamma^2 + 2\gamma\cos k_1\alpha + 1}.$$

Since $\cos k_1\alpha + 1 > 0$ (because $\sin k_1\alpha \neq 0$), it now suffices to check that

$$-k_1^4(\gamma+1)^2 + k_2^4(\gamma^2 + 2\gamma\cos k_1\alpha + 1) > 0.$$

But, using (2.14), one concludes that

$$-k_1^4(\gamma+1)^2 + k_2^4(\gamma^2 + 2\gamma\cos k_1\alpha + 1) > -k_1^4(\gamma+1)^2 + k_2^4(\gamma-1)^2 > 0,$$

thereby completing the proof of statement (ii), and thus the proposition.

## 8. Discussion

The original idea of Kruskal (1974) was that the properties of the two solition solution of (1.1) are reflected in the dynamics of the poles of that solution in the complex plane. For large positive and negative time, the poles separate naturally into two groups which travel asymptotically at the two speeds of the independent solitons. This suggests that each set of poles represents the corresponding soliton. Since each pole can be followed individually as time evolves, one can therefore interpret this evolution as the evolution of the (real-valued) solitons. Since the poles which travel at the slow speed for large negative time all travel at the faster speed for large positive time, it seems reasonable to say that the soliton which for large negative time leads the other one and is slower, is not really *overtaken* by the fast soliton, but rather *becomes* the fast soliton during the interaction. Similarly, some of the poles which travel with the faster soliton for large negative time slow down and travel at the slower speed for large positive time. It is consistent with our interpretation to say that the solition which is faster for large negative time evolves into the slower one. The interaction consists of the re-alignment of some of the poles travelling with the trailing soliton to the leading soliton. In the authors' view, this supports the idea of energy and mass transfer between the solitons during the interaction.

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## References

Airault H, McKean H P and Moser J 1977 Rational and elliptic solutions of the Korteweg–de Vries equation and a realted many-body problem *Commun. Pure Appl. Math.* **30** 95–148

Benes N, Kasman A and Young K 2006 On decompositions of the KdV 2-soliton *J. Nonlinear Sci.* **16** 179–200

Birnir B 1987 An example of blow-up for the complex KdV equation and existence beyond the blow-up *SIAM J. Appl. Math.* **47** 710–25

Bona J L and Grujic Z 2003 Spatial analyticity properties of nonlinear waves. *Math. Models Methods Appl. Sci.* **13** 345–60

Bona J L, Grujic Z and Kalisch H 2005 Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized Korteweg–de Vries equation *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **22** 783–97

Bona J L and Weissler F B 2001 Blow-up of spatially periodic complex-valued solutions of nonlinear dispersive equations *Indiana Univ. Math. J.* **50** 759–82

Bowtell G and Stuart A E G 1977 Interacting sine–Gordon solitons and classical particles: a dynamic equivalence *Phys. Rev.* D **215** 3580–91

Bowtell G and Stuart A E G 1983 A particle representation of the Korteweg–de Vries solitons *J. Math. Phys.* **24** 969–81

Bryan A C and Stuart A E G 1992 On the dynamics of soliton interactions for the Korteweg–de Vries equation *Chaos, Solitons Fractals* **2** 487–91

Deconinck B and Segur H 2000 Pole dynamics for elliptic solutions of the Korteweg–de Vries equation *Math. Phys. Anal. Geom.* **3** 49–74

Grujic Z and Kalisch H 2002 Local well-posedness of the generalized Korteweg–de Vries equation in spaces of analytic functions *Diff. Integral Eqns* **15** 1325–34

Hodnett P F and Moloney T P 1989 On the structure during the interaction of the two-soliton solution of the Kortweg–de Vries equation *SIAM J. Math. Anal.* **49** 1174–87

Kato T and Masuda K 1986 Nonlinear evolution equations and analyticity: I. *Ann. Inst. Henri Poincaré, Anal. Nonlin.* **3** 455–67

Kruskal M D 1974 The Korteweg–de Vries equation and related evolution equations *Nonlinear Wave Motion* (*Lectures in Applied Mathematics* vol 15) ed A C Newell (Providence, RI: American Mathematical Society) pp 61–83

Lax P D 1968 Integrals of nonlinear equations of evolution and solitary waves *Commun. Pure Appl. Math.* **XXI** 467–90

LeVeque R J 1987 On the interaction of nearly equal solitons in the KdV equation *SIAM J. Appl. Math.* **47** 254–62

Li Y C 2007 Simple explicit formulae for finite time blowup solutions to the complex KdV equation *Chaos, Solitons and Fractals* at press (doi:10.1016/j.chaos.2007.04.015)

Thickstun W R 1976 A system of particles equivalent to solitons *J. Math. Anal. Appl.* **55** 335–46

Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley-Interscience)

Yuan J-M and Wu J 2005 The complex KdV equation with or without dissipation *Discrete Contin. Dyn. Syst. Ser.* B **5** 489–512

Wu J and Yuan J-M 2007 Local well-posedness and local (in space) regularity results for the complex Korteweg–de Vries equation *Proc. R. Soc. Edinb.* A **137** 203–23

Zabusky N J and Kruskal M D 1965 Interaction of ‘solitons’ in a collisionless plasma and the recurrence of intitial states *Phys. Rev. Lett.* **15** 240–3