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STABILITY OF SOLITARY-WAVE SOLUTIONS TO THE HIROTA-SATSUMA EQUATION

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(Dedicated to Roger Temam, Mentor and Friend,
on the occasion of his 70^{th} birthday)

Abstract. The evolution equation

$$u_t - u_{xxt} + u_x - uu_t + u_x \int_x^{+\infty} u_t dx' = 0, \tag{1}$$

was developed by Hirota and Satsuma as an approximate model for unidirectional propagation of long-crested water waves. It possesses solitary-wave solutions just as do the related Korteweg-de Vries and Benjamin-Bona-Mahony equations. Using the recently developed theory for the initial-value problem for (1) and an analysis of an associated Liapunov functional, nonlinear stability of these solitary waves is established.

1. Introduction.

Considered here is the evolution equation

$$u_t - u_{xxt} + u_x - uu_t + u_x \int_x^{+\infty} u_t dx' = 0, \; x \in \mathbb{R}, \; t \in \mathbb{R}_+, \tag{2}$$

derived by Hirota and Satsuma in [14] (see also [18] and [19]). The dependent variable $u = u(x,t)$ is a real-valued function of the two real variables x and t. This equation was developed as a model for the unidirectional propagation of small amplitude, long waves on the surface of an ideal fluid in a channel of constant depth. The variable x corresponds to distance in the direction of propagation, $t > 0$ is proportional to elapsed time and $u(x,t)$ is the deviation of the free surface from its rest position at the point x along the channel at time t. It has the same formal status as an approximation of the full, two-dimensional Euler equations as do the well-known Korteweg-de Vries equation (KdV-equation)

$$u_t + u_{xxx} + u_x + 2uu_x = 0 \tag{3}$$

and the Benjamin-Bona-Mahony equation (BBM-equation)

$$u_t - u_{txx} + u_x + 2uu_x = 0. \tag{4}$$

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In [15], Iorio and Pilod proved that, provided $s > \frac{1}{2}$, the initial-value problem associated to (2) is locally well-posed for initial data in the open subset

$$\Omega_s = \{\phi \in H^s(\mathbb{R}) \mid -1 \notin \sigma(-\partial_x^2 - \phi)\}, \tag{5}$$

where $\sigma(-\partial_x^2 - \phi)$ denotes the spectrum of the unbounded operator $-\partial_x^2 - \phi$ in the L^2-based Sobolev space $H^s(\mathbb{R})$. An important initial step in their analysis was to rewrite the differential equation as an integral equation, *viz.*

$$u_t = -\partial_x(1 - \partial_x^2 - u)^{-1}u, \tag{6}$$

provided $u \in \Omega_s$. One then applies a fixed-point theorem to the integral equation (6). It was also proved, taking advantage of the quantity

$$E(v) = \int_{\mathbb{R}} \left(\frac{1}{2}v(x)^2 + \frac{1}{2}v'(x)^2 - \frac{1}{6}v(x)^3\right) dx, \tag{7}$$

which is conserved by H^1-solutions of (2), that if the initial data u_0 lies in $H^1(\mathbb{R})$, then the local H^1-solution corresponding to u_0 may be extended globally in time provided the initial data satisfies the additional conditions

$$\|u_0\|_{H^1} < \|\varphi^\star\|_{H^1} \quad \text{and} \quad E(u_0) < E(\varphi^\star). \tag{8}$$

Here, $\varphi^\star$ denotes the unique, non-trivial solution of the nonlinear elliptic differential equation

$$-\varphi^{\star''} + \varphi^\star - \varphi^{\star 2} = 0 \tag{9}$$

that is bounded on all of $\mathbb{R}$. Of course, uniqueness in this context is modulo the translation-group in the underlying spatial domain.

The solitary waves of the Hirota-Satsuma equation are traveling-wave solutions of (2) of the form

$$u(x,t) = \phi(x - (1+c)t), \quad \text{with} \quad c > 0 \quad \text{and} \quad \lim_{|x|\to+\infty} \phi(x) = 0.$$

It is readily seen that ϕ must satisfy the ordinary differential equation

$$-\phi'' + \mu\phi - \phi^2 = 0, \quad \text{with} \quad \mu = \frac{c}{1+c} \in (0,1) \tag{10}$$

if u is to solve (2). Moreover, as was observed already in [15], the family of functions

$$\phi_\mu(x) = \frac{3}{2}\mu \operatorname{sech}^2\Big(\frac{\mu^{\frac{1}{2}}}{2}x\Big), \quad \mu \in (0,1), \tag{11}$$

all satisfy (10) and the additional conditions (8), so that initial data near in $H^1(\mathbb{R})$ to ϕ_μ, for some $\mu \in (0,1)$, evolve into solutions which are global in time. This fact does not mean the ϕ_μ are stable (see *e.g.* [9] for an example of solitary waves that are unstable, but whose perturbations nevertheless lead to globally defined solutions). In light of the well known stability theory for (3) and (4), it is nevertheless natural to expect these traveling-wave solutions are in fact stable.

The stability theory for the solitary waves associated to the KdV- and BBM-equations has been extensively studied in the last decades. The first rigorous stability result for KdV was proved by Benjamin [4] and Bona [6]. A principal part of the analysis, whose roots lie in the work of Boussinesq [11] in the 1870's, is to observe that the solitary waves are local minimizers of an H^1-functional over the set of all admissible functions having fixed L^2-norm, and that the H^1-functional and the L^2-norm are both conserved by the flow of the equation. The original theory of Benjamin and Bona required a full understanding of the spectrum of a linearized differential operator associated to the solitary wave.

A later method for proving stability of solitary waves, which does not rely on local analysis, was developed by Cazenave and Lions [12], [13], using Lions' method of concentration compactness (in this context, see also the results of Weinstein [21], Albert [1] and Lopes [17] and the references in these articles). The main idea is to show that the set of minimizers for a variational problem associated to the Euler-Lagrange equation satisfied by the solitary wave is not empty. One takes a minimizing sequence and uses the concentration compactness criteria to prove that, up to translations, it is precompact. A subsequence is thereby adduced which converges to a minimizer. When the method works, it implies directly the stability of the set of minimizers for the flow of the evolution equation. In general, a stability result obtained in this way is weaker than one deduced using local analysis, since one does not necessarily know if the solitary waves belong to the set of minimizers, nor do we know if the set consists of only one element, up to translations. However, in the case of the KdV-equation, it is straightforward to verify that the set of minimizers corresponding to a fixed value of the L^2-norm is exactly a solitary wave and its set of translates, so that both methods give equivalent, orbital stability results.

The present essay proposes a stability analysis of the solitary-wave solutions (11) of the Hirota-Satsuma equation. As already mentioned, one expects that these waves will be stable. On the other hand, while Hirota-Satsuma solitary waves (11) appear closely related to their BBM-counterparts, and, for small amplitudes, also with the KdV-approximation of solitary waves, the non-local character of the equation makes the question of stability less than obvious. It turns out that they are in fact stable, as our main result attests.

Theorem 1.1. *Let $c > 0$ be given and let $\mu = \frac{c}{1+c} \in (0,1)$. Then the solitary-wave solution ϕ_μ in (11) of the Hirota-Satsuma equation (2) is orbitally stable in $H^2(\mathbb{R})$. More precisely, corresponding to any $\epsilon > 0$, there is a $\delta > 0$ such that if*

$$\|u_0 - \phi_\mu\|_{H^2} < \delta,$$

then for every $t > 0$ there is a $\gamma = \gamma(t)$ such that

$$\|u(\cdot, t) - \phi_\mu(\cdot + \gamma(t))\|_{H^2} < \epsilon, \tag{12}$$

where u is the solution of (2) emanating from u_0.

There are several points worth mentioning about our analysis. The Hirota-Satsuma equation does have a Hamiltonian structure. However, the invariants of the motion, E in (7) and F, given by

$$F(v) = \int_{\mathbb{R}} \left(\frac{1}{2}v'(x)^2 + \frac{1}{2}v''(x)^2 - \frac{1}{3}v(x)^3 + \frac{1}{6}v(x)^4 - \frac{3}{2}v(x)v'(x)^2 \right) dx, \tag{13}$$

that come to the fore in our analysis are naturally defined on $H^2(\mathbb{R})$ rather than on $H^1(\mathbb{R})$ as is the case for the KdV- and BBM-equations. As a consequence, the global well-posedness theory needs to be extended to higher-order Sobolev classes.

As first suggested in [5] in the context of the Benjamin-Ono equation, a local approach could naturally proceed from a consideration of the composite functional $\Lambda := F + \mu E$. It is not difficult to see that ϕ_μ is a critical point of Λ (see Lemma 3.2 below). However, the differential operator corresponding to the quadratic form that is the second derivative of Λ at ϕ_μ is fourth order, and its spectral properties are less than transparent.

In consequence, we have elected to follow the variational method, studying the set of minimizers for the variational problem

$$(V_\lambda)\begin{cases} \text{Minimize } F(\phi) \text{ on the admissible set of functions} \\ I_\lambda = \{\phi \in H^2(\mathbb{R}) \mid E(\phi) = \lambda \text{ and } \|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}\}. \end{cases}$$

Since neither constraint functional E nor F is homogeneous, we have used the alternative ideas, developed by Lopes in [17], to prevent a minimizing sequence from dichotomizing (see Section 4). (Dichotomy is a prospect that must be eliminated in applying concentration compactness.) Indeed, it was proved in [17] that if dichotomy occurs in the minimizing sequence, then a second-order condition on the functional Λ is violated. It is this fact that is used in our analysis.

Finally, the dependence of the translation function γ on t is examined. Employing the ideas of Bona and Soyeur in [10], it is proved here that γ can be chosen to be a C^1-function whose derivative is uniformly close to the physical velocity $1+c$ of the wave ϕ_μ. This result implies that the solution generated by initial data near the solitary wave ϕ_μ is "almost" a solitary wave traveling at a speed close to that of ϕ_μ. Because the stability result is obtained via a global perspective rather than a local analysis, recent work of Albert, Bona, and Nguyen [3] has informed the approach to ensuring there is a choice of γ that has all the stated properties.

The rest of the paper is organized as follows. In Section 2, the H^2 global well-posedness theory is derived for the initial-value problem for (2). As already mentioned, this is a fundamental prerequisite for the stability result as stated in Theorem 1.1. Section 3 is devoted to the study of the variational problem (V_λ). The outcome of this study finds use in Section 4 to prove Theorem 1.1. Finally, the aforementioned refinement of the basic orbital stability result is exposed in Section 5. A short, concluding section reviews what has been accomplished and points to related lines of inquiry that would be worthwhile to pursue.

The body of the paper is followed by two appendices. The first provides details of Lopes' ideas which are used in Section 4 in the proof of the main result. The second appendix presents an ill-posedness result indicating that the restriction (5) cannot be easily discarded in the theory for (2).

2. Conservation laws and global well-posedness in higher-order Sobolev spaces.

We start with a derivation of the conservation law of the Hirota-Satsuma equation defined in (13).

Proposition 2.1. *Let u a smooth solution of (2) defined at least on the time interval $[0,T]$ and suppose that u and its first few partial derivatives all lie in $L^2(\mathbb{R})$. Then,*

$$F(u(\cdot,t)) = F(u_0), \quad \textit{for all } t \in [0,T], \tag{14}$$

where $u_0 = u(\cdot,0)$ and F is defined in (13).

Proof. Denote by $(\cdot,\cdot)_{L^2}$ the scalar product in L^2, which is to say

$$(\phi,\psi)_{L^2} = \int_{\mathbb{R}} \phi(x)\psi(x)dx,$$

when ϕ and ψ are real-valued functions in $L^2(\mathbb{R})$.

Multiply (2) by u^2 and integrate over $\mathbb{R}$ to deduce that

$$\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}} u^3(x,t)dx - (u_{xxt},u^2)_{L^2} - (uu_t,u^2)_{L^2} + (u_x\int_x^{+\infty} u_t dx', u^2)_{L^2} = 0.$$

An integration by parts reveals that

$$\big(u_x \int_x^{+\infty} u_t dx', u^2\big)_{L^2} = \frac{1}{3}(uu_t, u^2)_{L^2} = \frac{1}{12}\frac{d}{dt}\|u\|_{L^4}^4,$$

whence

$$\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}} u^3(x,t)dx + 2(u_{xt}, uu_x)_{L^2} - \frac{1}{6}\frac{d}{dt}\|u\|_{L^4}^4 = 0. \tag{15}$$

On the other hand, differentiating (2) by x, multiplying the result by u_x and then integrating over $\mathbb{R}$ leads to the formula

$$\frac{1}{2}\frac{d}{dt}\big(\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2\big) + (uu_t, u_{xx})_{L^2} - \big(u_x \int_x^{+\infty} u_t dx', u_{xx}\big)_{L^2} = 0.$$

Integrations by parts show that

$$(uu_t, u_{xx})_{L^2} = -(uu_{tx}, u_x)_{L^2} - (u_x u_t, u_x)_{L^2},$$

and

$$\big(u_x \int_x^{+\infty} u_t dx', u_{xx}\big)_{L^2} = \frac{1}{2}(u_x u_t, u_x)_{L^2}.$$

Thus, it follows that

$$\frac{1}{2}\frac{d}{dt}\big(\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2\big) - \frac{3}{2}(u_x u_t, u_x)_{L^2} - (uu_{tx}, u_x)_{L^2} = 0. \tag{16}$$

Combining (15), (16) and the fact that

$$\frac{d}{dt}(u, u_x^2)_{L^2} = (u_t, u_x^2)_{L^2} + 2(u, u_x u_{xt})_{L^2}$$

yields

$$\frac{d}{dt}\left(\frac{1}{2}\|u_x\|_{L^2}^2 + \frac{1}{2}\|u_{xx}\|_{L^2}^2 - \frac{1}{3}\int_{\mathbb{R}} u^3(x,t)dx + \frac{1}{6}\|u\|_{L^4}^4 - \frac{3}{2}(u, u_x^2)_{L^2}\right) = 0,$$

which is equivalent to (14) for smooth solutions. □

Extending this result from smooth solutions to solutions of only limited regularity can be accomplished by a suitable approximation argument. Note that just regularizing the initial data does not work in the present context since the smooth solutions emanating from the regularized data are only known to exist, as smooth solutions, on shorter and shorter time intervals as the regularization goes away (see Kato [16] and Bona and Kalisch [7]).

Instead, reason as follows. Let $u \in C(0,T;H^2(\mathbb{R}))$ be a solution as guaranteed by the local existence theory in [15]. Because of the assumptions (8) on u_0, it follows from the integral equation obtained by inverting the operator $\big(1-\partial_x^2 - u(\cdot,t)\big)$ that $u_t \in C(0,T;H^3(\mathbb{R}))$ (see Lemma 5.4 (i) below). Let $\{u_n\}_{n=1}^{\infty} \subset C^1(0,T;H^4(\mathbb{R}))$ converge to u in $C(0,T;H^2(\mathbb{R})) \cap C^1(0,T;H^3(\mathbb{R}))$. Let $\Gamma_n(x,t)$ be the residual, which is to say,

$$\partial_t u_n - \partial_x^2\partial_t u_n + \partial_x u_n - u_n\partial_t u_n + \partial_x u_n \int_x^{+\infty} \partial_t u_n dx' = \Gamma_n.$$

Then, it follows that Γ_n converges to zero in $C(0,T;H^1(\mathbb{R}))$. Moreover, we obtain, arguing as in the proof of Proposition 2.1, that

$$F(u_n(\cdot,t)) - F(u_n(\cdot,0)) = \int_0^t \big[(\partial_x\Gamma_n, \partial_x u_n)_{L_x^2} - (\Gamma_n, u_n^2)_{L_x^2}\big]d\tau \underset{n\to+\infty}{\longrightarrow} 0,$$

for all $t \in [0,T]$, and thus $F(u(\cdot,t))$ is constant on $[0,T]$.

Proposition 2.1 and Theorems 1 and 3 in [15] may be combined to prove that the initial-value problem associated to the Hirota-Satsuma equation is globally well-posed in $H^2(\mathbb{R})$ for initial data $u_0 \in H^2(\mathbb{R})$ satisfying conditions (8).

Theorem 2.2. *Let $u_0 \in H^2(\mathbb{R})$ be such that $\|u_0\|_{H^1} < \|\varphi^\star\|_{H^1}$ and $E(u_0) < E(\varphi^\star)$ as in (8). Then the local solution u to the initial-value problem*

$$\begin{cases} u_t + u_x - u_{txx} - uu_t + u_x \int_x^\infty u_t dx' = 0, \\ u(x,0) = u_0(x), \quad \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{17}$$

obtained in Theorem 1 of [15] extends uniquely to a solution $u \in C_b(\mathbb{R}; H^2(\mathbb{R}))$ that solves (2) and, additionally, has the property that

$$\|u(\cdot,t)\|_{H^1} < \|\varphi^\star\|_{H^1}, \quad \text{for all } t \in \mathbb{R}. \tag{18}$$

Proof. Let $u_0 \in H^2(\mathbb{R})$ satisfy the hypotheses, which is to say, the inequalities in (8). The initial value u_0 can be approximated in H^2 by, say, H^∞-functions $\{u_{0,i}\}_{i=1}^\infty$ that also respect (8), uniformly for $i = 1, 2, \cdots$. It follows from Theorem 1 in [15] that there exists positive times

$$T_i = T(r_0(u_{0,i}), r_1(u_{0,i}), \|u_{0,i}\|_{H^2}), \text{ with } r_j(\phi) = \|\partial_x^j(-\partial_x^2 - \phi + 1)^{-1}\|_{\mathcal{B}(L^2)}, \tag{19}$$

$j = 0, 1$, and associated solutions u_i of the Hirota-Satsuma equation such that

$$u_i \in C([0, T_i); H^2(\mathbb{R})),$$

$i = 0, 1, \cdots$. Note that T can be taken to be a continuous, positive function of its arguments. Thus, to show the solutions $u_i, i = 1, 2, \cdots$, may be extended to arbitrarily large time intervals, it suffices to derive *a priori* bounds on the quantities $r_0(u_i(\cdot,t))$, $r_1(u_i(\cdot,t))$, and $\|u_i(\cdot,t)\|_{H^2}$ and then reapply the local well-posedness result an arbitrary number of times.

But the bounds on $r_0(u(t))$ and $r_1(u(t))$ are obtained in Theorem 3 in [15] since u_0 satisfies (8), while the bound on $\|u(t)\|_{H^2}$ is a consequence of the second-order conservation law obtained in Proposition 2.1 combined with (18) and standard Sobolev embedding results. □

3. **Solitary-wave solutions and the variational problem.** Consider the solitary waves ϕ_μ in (11) which are solutions to (10), for $0 < \mu < 1$. Define $e(\mu)$ and $f(\mu)$ by

$$e(\mu) = E(\phi_\mu) \quad \text{and} \quad f(\mu) = F(\phi_\mu), \quad \text{for} \quad 0 < \mu < 1,$$

and set

$$e^\star = E(\varphi^\star) \quad \text{and} \quad f^\star = F(\varphi^\star) \quad \text{where} \quad \varphi^\star = \lim_{\mu \to 1} \phi_\mu.$$

Recall that $\varphi^\star$ is a solution to equation (9). The first step in the analysis is a technical lemma which asserts that e and f are bijections.

Lemma 3.1. *Let e and f be as defined above. Then,*

(i) the function e is a strictly increasing bijection from $(0,1)$ onto $(0, e^\star)$, and
(ii) the function f is a strictly decreasing bijection from $(0,1)$ onto $(f^\star, 0)$.

Proof. First, observe that since ϕ_μ is a solution to (10), then

$$\int_{\mathbb{R}} \left(\phi_\mu'(x)^2 + \mu\phi_\mu(x)^2\right) dx = \int_{\mathbb{R}} \phi_\mu(x)^3 dx. \tag{20}$$

From the definition of E in (7), it is seen that

$$e(\mu) = E(\phi_\mu) = \frac{1}{3}\|\phi_\mu'\|_{L^2}^2 + \left(\frac{1}{2} - \frac{\mu}{6}\right)\|\phi_\mu\|_{L^2}^2. \tag{21}$$

On the other hand, observe that $\varphi^\star(x) = \mu^{-1}\phi_\mu(\mu^{-\frac{1}{2}}x)$ does not depend on μ since it satisfies equation (9), so that (21) becomes

$$e(\mu) = \frac{1}{3}\mu^{\frac{5}{2}}\|{\varphi^\star}'\|^2_{L^2} + \big(\frac{1}{2} - \frac{\mu}{6}\big)\mu^{\frac{3}{2}}\|\varphi^\star\|^2_{L^2}.$$

It is therefore concluded that

$$e'(\mu) = \frac{5}{6}\mu^{\frac{3}{2}}\|{\varphi^\star}'\|^2_{L^2} + \big(\frac{3}{4}\mu^{\frac{1}{2}} - \frac{5}{12}\mu^{\frac{3}{2}}\big)\|\varphi^\star\|^2_{L^2} > 0, \tag{22}$$

since $\mu \in (0, 1)$, which proves (i).

Attention is now given to the proof of (ii). To obtain a helpful expression for $f(\mu)$, the following identities satisfied by ϕ_μ are useful. First, differentiate (10), multiply the result by ϕ'_μ and integrate by parts to reach the formula

$$\int_{\mathbb{R}} \big(\phi''_\mu(x)^2 + \mu\phi'_\mu(x)^2\big)\, dx = 2\int_{\mathbb{R}} \phi_\mu(x)\phi'_\mu(x)^2 dx. \tag{23}$$

Second, multiply (10) by ϕ^2_μ and integrate by parts to obtain

$$\int_{\mathbb{R}} \big(2\phi_\mu(x)\phi'_\mu(x)^2 + \mu\phi_\mu(x)^3\big)\, dx = \int_{\mathbb{R}} \phi_\mu(x)^4 dx. \tag{24}$$

Combining (13), (20), (23) and (24) yields

$$f(\mu) = F(\phi_\mu) = \int_{\mathbb{R}} \left(\frac{1-\mu}{2}\phi'_\mu(x)^2 - \frac{2-\mu}{6}\phi^3_\mu - \frac{1}{6}\phi_\mu(x)\phi'_\mu(x)^2\right) dx. \tag{25}$$

On the other hand, if (10) is multiplied by ϕ'_μ and the result integrated, there appears

$$-(\phi'_\mu)^2 + \mu\phi^2_\mu - \frac{2}{3}\phi^3_\mu = 0, \tag{26}$$

which, together with (20), gives

$$\int_{\mathbb{R}} \phi'_\mu(x)^2 dx = \frac{1}{6}\int_{\mathbb{R}} \phi_\mu(x)^3 dx. \tag{27}$$

Finally, upon gathering together (25) and (27) and performing the same change of variable as for e, it transpires that

$$f(\mu) = -\big(\frac{3}{2} - \frac{\mu}{2}\big)\mu^{\frac{5}{2}}\|{\varphi^\star}'\|^2_{L^2} - \frac{1}{6}\mu^{\frac{7}{2}}\int_{\mathbb{R}} \varphi^\star(x){\varphi^\star}'(x)^2 dx.$$

It is concluded that f and f' are strictly negative on $(0, 1)$ since $\varphi^\star$ is a positive function. $\square$

In what follows, λ connotes a fixed, but arbitrary number in the interval $(0, e^\star)$. Consider the variational problem

$$(V_\lambda)\begin{cases} \text{Minimize } F(\phi) \text{ on the admissible set of functions} \\ I_\lambda = \{\phi \in H^2(\mathbb{R}) \mid E(\phi) = \lambda \text{ and } \|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}\}. \end{cases} \tag{28}$$

For the fixed value of λ, let

$$F_\lambda = \inf\{F(\phi) \mid \phi \in I_\lambda\ \}.$$

The next result characterizes the solutions of the Euler-Lagrange equation associated to (V_λ).

Lemma 3.2. *Let $\alpha \in \mathbb{R}$ and let $\phi \in H^2(\mathbb{R})$ be such that $\|\phi\|_{H^1(\mathbb{R})} < \|\varphi^\star\|_{H^1}$. Then ϕ is a solution to the Euler-Lagrange equation associated to (V_λ) for the Lagrange multiplier α, which is to say,*

$$F'(\phi) + \alpha E'(\phi) = 0, \tag{29}$$

if and only if ϕ solves the elliptic differential equation

$$-\phi'' + \alpha\phi - \phi^2 = 0. \tag{30}$$

Proof. Let $\alpha \in \mathbb{R}$ and $\phi \in H^2(\mathbb{R})$. Define

$$\begin{aligned}\Lambda(\phi) &:= F(\phi) + \alpha E(\phi)\\ &= \int_{\mathbb{R}} \Big(\frac{1}{2}(\phi'')^2 + \frac{1}{2}(1+\alpha)(\phi')^2 + \frac{\alpha}{2}\phi^2 - \frac{2+\alpha}{6}\phi^3 + \frac{1}{6}\phi^4 - \frac{3}{2}\phi(\phi')^2 \Big)\, dx.\end{aligned}$$

A straightforward calculation reveals that

$$\begin{aligned}&\Lambda'(\phi)h\\ &= \int_{\mathbb{R}} \Big(\phi^{(4)} - (1+\alpha)\phi'' + \alpha\phi - (1+\frac{\alpha}{2})\phi^2 + \frac{2}{3}\phi^3 + \frac{3}{2}(\phi')^2 + 3\phi\phi'' \Big)\, h dx\\ &= \int_{\mathbb{R}} \Big(\big[-\frac{d^2}{dx^2} + 1 - \phi \big](-\phi'' + \alpha\phi - \phi^2) + (-\frac{1}{2}(\phi')^2 + \frac{\alpha}{2}\phi^2 - \frac{1}{3}\phi^3) \Big)\, h dx.\end{aligned} \tag{31}$$

Hence, if ϕ is a solution to (30), it is deduced from (26) and (31) that ϕ is a solution to the Euler-Lagrange equation (29).

Reciprocally, let ϕ be a solution to the Euler-Lagrange equation (29). Then, observe using (31) and integrating by parts that

$$\begin{aligned}&-\Lambda'(\phi)h'\\ &= \int_{\mathbb{R}} \Big(\frac{d}{dx}\big[-\frac{d^2}{dx^2} + 1 - \phi \big](-\phi'' + \alpha\phi - \phi^2) + \frac{d}{dx}(-\frac{1}{2}(\phi')^2 + \frac{\alpha}{2}\phi^2 - \frac{1}{3}\phi^3) \Big)\, h dx\\ &= \int_{\mathbb{R}} \Big(\big[-\frac{d^2}{dx^2} + 1 - \phi \big] \frac{d}{dx}(-\phi'' + \alpha\phi - \phi^2) \Big)\, h dx = 0,\end{aligned}$$

for any $h \in C_0^\infty(\mathbb{R})$. Since it was proved in Lemma 6 of [15] that the differential operator $-\frac{d^2}{dx^2} + 1 - \phi$ is invertible when $\|\phi\|_{H^1(\mathbb{R})} < \|\varphi^\star\|_{H^1}$, ϕ has therefore to be a solution of (30). □

Remark 3.3.

(i) Let $\phi \in H^2(\mathbb{R})$, $\phi \neq 0$, be a solution to the Euler-Lagrange equation (29). Then α has to be positive, since equation (30) does not admit non-zero bounded solutions for $\alpha \leq 0$.

(ii) If $\mu = e^{-1}(\lambda) \in (0,1)$, then the solitary-wave solution ϕ_μ defined in (11) is the unique (up to translation) non-zero solution to the Euler-Lagrange equation (29) in the admissible set I_λ.

In view of the last observation, ϕ_μ is the natural candidate to be the global minimizer for the problem (V_λ). The next result provides some analysis of the second derivative of Λ at the critical point ϕ_μ.

Proposition 3.4. *Let $\mu \in (0,1)$. Consider ϕ_μ defined in (11), the unique (up to translation) solution to the Euler-Lagrange equation (29) with Lagrange multiplier*

μ. Then, if $\Lambda = F + \mu E$, it follows that

$$\Lambda''(\phi_\mu)\left(\frac{d\phi_\mu}{d\mu}, \frac{d\phi_\mu}{d\mu}\right) < 0. \tag{32}$$

Proof. Let $\phi \in H^2(\mathbb{R})$. Straightforward calculation reveals that

$$\begin{aligned}
&\Lambda''(\phi)(h_1, h_2) \\
&= \int_{\mathbb{R}} \left(-(-\phi'' + \mu\phi - \phi^2) + \big[-\frac{d^2}{dx^2} + 1 - \phi \big] (-\frac{d^2}{dx^2} + \mu - 2\phi) \right) h_1 h_2 dx \\
&\quad + \int_{\mathbb{R}} (-\phi' h_1' h_2 + \mu\phi h_1 h_2 - \phi^2 h_1 h_2) dx.
\end{aligned}$$

Since ϕ_μ is a solution to (30), it transpires that

$$\Lambda''(\phi_\mu)(h_1, h_2) = (\mathcal{M}_\mu h_1, h_2)_{L^2}, \quad \text{where} \quad \mathcal{M}_\mu = \mathcal{H}_\mu \mathcal{L}_\mu + \mathcal{C}_\mu,$$

$$\mathcal{H}_\mu = -\frac{d^2}{dx^2} + 1 - \phi_\mu, \quad \mathcal{L}_\mu = -\frac{d^2}{dx^2} + \mu - 2\phi_\mu \quad \text{and} \quad \mathcal{C}_\mu := -\phi_\mu' \frac{d}{dx} + \phi_\mu''.$$

It is worth noting that $\mathcal{H}_\mu$ and $\mathcal{L}_\mu$ are both well-understood, self-adjoint operators. Indeed, $\mathcal{H}_\mu$ is an invertible operator since $\mu \in (0, 1)$ (see [15]) and $\mathcal{L}_\mu$ is the differential operator of (10), linearized around ϕ_μ. If (10) and (26) are differentiated with respect to μ, the identities

$$\mathcal{L}_\mu \frac{d\phi_\mu}{d\mu} = -\phi_\mu \quad \text{and} \quad -\phi_\mu' (\frac{d\phi_\mu}{d\mu})' + \mu\phi_\mu \frac{d\phi_\mu}{d\mu} - \phi_\mu^2 \frac{d\phi_\mu}{d\mu} = -\frac{1}{2}\phi_\mu^2 \tag{33}$$

appear.

Now, define $d(\mu) = \Lambda(\phi_\mu) = F(\phi_\mu) + \mu E(\phi_\mu)$. Then, since ϕ_μ is a critical point of Λ, one observes that

$$d'(\mu) = e(\mu) = E(\phi_\mu) \tag{34}$$

and

$$d''(\mu) = E'(\phi_\mu)\frac{d\phi_\mu}{d\mu} = \int_{\mathbb{R}} \big(-\phi_\mu'' + \phi_\mu - \frac{1}{2}\phi_\mu^2 \big) \frac{d\phi_\mu}{d\mu} dx. \tag{35}$$

Gathering together (10), (33), and (35), gives the result

$$\begin{aligned}
d''(\mu) &= \int_{\mathbb{R}} (\mathcal{H}_\mu \phi_\mu + \frac{1}{2}\phi_\mu^2) \frac{d\phi_\mu}{d\mu} dx \\
&= -(\mathcal{H}_\mu \mathcal{L}_\mu \frac{d\phi_\mu}{d\mu}, \frac{d\phi_\mu}{d\mu})_{L^2} + \int_{\mathbb{R}} \left(\phi_\mu' (\frac{d\phi_\mu}{d\mu})' \frac{d\phi_\mu}{d\mu} - (\mu\phi_\mu - \phi_\mu^2)(\frac{d\phi_\mu}{d\mu})^2 \right) dx \\
&= -(\mathcal{H}_\mu \mathcal{L}_\mu \frac{d\phi_\mu}{d\mu}, \frac{d\phi_\mu}{d\mu})_{L^2} + \int_{\mathbb{R}} \left(\phi_\mu' (\frac{d\phi_\mu}{d\mu})' - \phi_\mu'' \frac{d\phi_\mu}{d\mu} \right) \frac{d\phi_\mu}{d\mu} dx \\
&= -(\mathcal{M}_\mu \frac{d\phi_\mu}{d\mu}, \frac{d\phi_\mu}{d\mu})_{L^2} = -\Lambda''(\phi_\mu)\left(\frac{d\phi_\mu}{d\mu}, \frac{d\phi_\mu}{d\mu}\right).
\end{aligned}$$

On the other hand, it is already known from (22) and (34) that $d''(\mu) > 0$, which concludes the proof of Proposition 3.4. ☐

Remark 3.5. We will see that Proposition 3.4 allows us to use the upcoming Theorem 4.6, which in turn prevents minimizing sequences from dichotomizing. Moreover, the proof of this latter fact will reveal that Proposition 3.4 is equivalent to $d'' > 0$, a condition which is generally associated with stability.

4. **Existence of global minimizers and stability theory.** The main result of this section establishes the existence of global minimizers for the variational problem (V_λ). Throughout this section, $\lambda \in (0, e^\star)$ and $\mu = e^{-1}(\lambda) \in (0,1)$ are fixed, where e is as in Lemma 3.1. A sequence of functions $\{\phi_n\}$ will be called a minimizing sequence for the variational problem (V_λ) in (28) when $\{\phi_n\}$ satisfies the conditions

$$\{\phi_n\} \subset H^2(\mathbb{R}), \quad E(\phi_n) = \lambda, \quad \|\phi_n\|_{H^1} < \|\varphi^\star\|_{H^1}, \quad \text{and} \quad F(\phi_n) \underset{n\to\infty}{\to} F_\lambda. \tag{36}$$

Theorem 4.1. *If $\{\phi_n\}$ is a minimizing sequence for the variational problem (V_λ) defined in (28), then there exists a real sequence $\{c_n\}$ and a real number τ such that the sequence of translates $\{\phi_n(\cdot + c_n)\}$ has a subsequence converging strongly in $H^2(\mathbb{R})$ to $\phi_\mu(\cdot+\tau)$, where ϕ_μ is the solitary-wave solution of the Hirota-Satsuma equation defined in (11).*

Before starting with the proof of Theorem 4.1, preliminary results are enunciated and proved. The first one asserts that minimizing sequences for (V_λ) are bounded in $H^2(\mathbb{R})$.

Lemma 4.2. *Let $\{\phi_n\}$ be a minimizing sequence for (V_λ). Then, there exists a constant C such that*

$$\|\phi_n\|_{H^2} \le C, \quad \text{for all } n \in \mathbb{N}. \tag{37}$$

Proof. We already know that $\|\phi_n\|_{H^1} < \|\varphi^\star\|_{H^1}$, $n = 1, 2, \cdots$. It remains to bound $\|\phi_n''\|_{L^2}$. But, Sobolev embedding and the definition of F in (13) imply that

$$\frac{1}{2}\|\phi_n''\|_{L^2} \le 2\sup_n F(\phi_n) + c(\|\phi_n\|_{H^1}) \le C.$$

□

We will need the following result, proved as Lemma 7 in [15]. (Note that our version of the Hirota-Satsuma equation differs from that in [15] by not having a factor 2 adorning the nonlinear terms.)

Lemma 4.3. *Let $0 < \delta_0 < 1$. Assume that $\|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}$ and $E(\phi) \le (1-\delta_0)E(\varphi^\star)$. Then, there exists a δ with $0 < \delta = \delta(\delta_0) < 1$ such that*

$$\|\phi\|_{H^1} \le (1-\delta)\|\varphi^\star\|_{H^1} \quad \text{and} \quad E(\phi) \ge 0. \tag{38}$$

Corollary 4.4. *If $\{\phi_n\}$ is a minimizing sequence for (V_λ) such that $\phi_n \rightharpoonup \phi$ in $H^1(\mathbb{R})$, then $\|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}$.*

Proof. Indeed, if $\{\phi_n\}$ is a minimizing sequence for (V_λ), it follows that

$$\phi_n \in B(0, \|\varphi^\star\|_{H^1}) \quad \text{and} \quad E(\phi_n) = \lambda < e^\star = E(\varphi^\star).$$

Hence, it is deduced from Lemma 4.3 that there exists a $\delta > 0$ depending only on λ such that

$$\|\phi\|_{H^1} \le \liminf \|\phi_n\|_{H^1} \le (1-\delta)\|\varphi^\star\|_{H^1}.$$

□

The next lemma will be useful in proving the existence of a non-vanishing minimizing sequence.

Lemma 4.5. *Let $\{\phi_n\}$ a bounded sequence in $H^2(\mathbb{R})$. If for all real sequences $\{c_n\}$, the sequence $\{\phi_n(\cdot + c_n)\}$ converges to zero weakly in $H^2(\mathbb{R})$, then $\{\phi_n\}$ converges to zero strongly in $W^{1,p}(\mathbb{R})$, for any p with $2 < p \le \infty$.*

Proof. Fix $R > 0$ and $\epsilon > 0$. Denote by I_R the open interval $(-R, R)$. Then, for all $n \in \mathbb{N}$, there exists $c_n \in \mathbb{R}$ such that

$$\|\phi_n\|_{L^\infty} = \sup_{y\in\mathbb{R}} \|\phi_n\|_{L^\infty(y+I_R)} \le \|\phi_n\|_{L^\infty(c_n+I_R)} + \epsilon. \tag{39}$$

Define the translated function $\psi_n(z) = \phi_n(z + c_n)$. By hypothesis, the sequence $\{\psi_n\}$ converges to zero weakly in $H^2(\mathbb{R})$ and hence in $H^1(\mathbb{R})$. From (39) and the fact that $H^1(I_R)$ is compactly embedded in $L^\infty(I_R)$, it is deduced that

$$\limsup_{n\to+\infty} \|\phi_n\|_{L^\infty} \le \lim_{n\to+\infty} \|\psi_n\|_{L^\infty(I_R)} + \epsilon = \epsilon,$$

where $\epsilon > 0$ was arbitrary. Straightforward interpolation thus yields that for any p with $2 < p \le \infty$,

$$\|\phi_n\|_{L^p} \le \|\phi_n\|_{L^2}^{\frac{2}{p}} \|\phi_n\|_{L^\infty}^{1-\frac{2}{p}} \le C\|\phi_n\|_{L^\infty}^{1-\frac{2}{p}} \underset{n\to+\infty}{\longrightarrow} 0.$$

The same argument proves that $\{\phi_n'\}$ converges to zero strongly in $L^p(\mathbb{R})$, for any p with $2 < p \le \infty$. ☐

Here is a fundamental technical result of Lopes [17] which gives sufficient conditions preventing dichotomy to occur for minimizing sequences.

Theorem 4.6. *Assume that*

(H) if $\phi \neq 0$ is the weak limit in $H^2(\mathbb{R})$ of a minimizing sequence for (V_λ) and ϕ satisfies the Euler-Lagrange equation $\Lambda'(\phi) = F'(\phi) + \alpha E'(\phi) = 0$, then there exists an element $h \in H^2(\mathbb{R})$ such that $\Lambda''(\phi)(h, h) < 0$.

If $\{\phi_n\} \subset H^2(\mathbb{R})$ is a minimizing sequence for the variational problem (V_λ) converging weakly in $H^2(\mathbb{R})$ to some $\phi \neq 0$, then by extracting a subsequence if necessary, it follows that $\{\phi_n\}$ converges strongly to ϕ in $W^{1,p}(\mathbb{R})$ for any p with $2 < p \le \infty$, and there exists an $\alpha \in \mathbb{R}$ such that ϕ satisfies the Euler-Lagrange equation

$$F'(\phi) + \alpha E'(\phi) = 0. \tag{40}$$

The proof of Theorem 4.6 is similar to the one given by Lopes in his proof of Theorem 2.1 in [17]. However, a function ϕ in our admissible set I_λ must satisfy the condition $\|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}$ (a side condition that does not appear in Lopes' theorem). Also, hypothesis (H1) of Theorem 2.1 in [17] is not satisfied in our case, which is another cause of difficulty. For these reasons and also for the sake of completeness, we provide a sketch of the proof of Theorem 4.6 in the appendix.

A proof of Theorem 4.1 is now in sight.

Proof of Theorem 4.1. Let $\{\phi_n\}$ be a minimizing sequence for (V_λ). From Lemma 4.2, it is known that $\{\phi_n\}$ is bounded in $H^2(\mathbb{R})$.

Now, if for all real sequences $\{c_n\}$, the associated sequence $\{\phi_n(\cdot + c_n)\}$ of functions were to converge to zero weakly in $H^2(\mathbb{R})$, then Lemma 4.5 would imply that $\{\phi_n\}$ must converge to zero strongly in $W^{1,p}(\mathbb{R})$, for any p in the range $2 < p \le \infty$. Thus, it would follow that

$$F_\lambda = \lim_{n\to\infty} F(\phi_n) = \lim_{n\to\infty} \int_{\mathbb{R}} \left(\frac{1}{2}\phi_n'^2 + \frac{1}{2}\phi_n''^2 - \frac{1}{3}\phi_n^3 + \frac{1}{6}\phi_n^4 - \frac{3}{2}\phi_n\phi_n'^2 \right) dx \ge 0,$$

which is not possible since, in view of Lemma 3.1, ϕ_μ is an admissible function and $f(\mu) = F(\phi_\mu) < 0$ (here $\mu = e^{-1}(\lambda) \in (0, 1)$).

Hence, there exists a real sequence $\{c_n\}$, a subsequence of $\{\phi_n\}$ (still denoted $\{\phi_n\}$), and a $\phi \in H^2(\mathbb{R})$, $\phi \neq 0$ such that

$$\psi_n = \phi_n(\cdot + c_n) \underset{n\to+\infty}{\rightharpoonup} \phi, \quad \text{in } H^2(\mathbb{R}). \tag{41}$$

Because the functionals E and F are translation invariant, $\{\psi_n\}$ is still a minimizing sequence for (V_λ).

To verify the assumption (H) of Theorem 4.6, suppose that $\phi \in H^2(\mathbb{R})$, $\phi \neq 0$ is the weak limit of a minimizing sequence $\{\phi_n\}$, and that ϕ satisfies the Euler-Lagrange equation

$$F'(\phi) + \alpha E'(\phi) = 0, \quad \text{for some } \alpha \in \mathbb{R}.$$

Then, we deduce from Lemma 3.2 and Remark 3.3 (i) that ϕ has to be a solution to the differential equation (30) with $\alpha > 0$. Moreover, since Corollary 4.4 implies that $\|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}$, Lemma 3.1 (i) implies that $\alpha \in (0,1)$, and (H) then follows from Proposition 3.4.

Applying Theorem 4.6 to the minimizing sequence $\{\psi_n\}$, and extracting a subsequence if necessary, it transpires that

$$\psi_n \underset{n\to+\infty}{\longrightarrow} \phi \quad \text{in } W^{1,p}(\mathbb{R}) \quad \text{for } 2 < p \leq \infty, \tag{42}$$

and ϕ is a solution to the Euler-Lagrange equation (29) with $\alpha \in (0,1)$. Because of Lemma 3.2, it must be the case that $\phi = \phi_\alpha$ solves (30). From (41) and (42), it is concluded that

$$F(\phi) \leq \liminf(F(\psi_n)) = F_\lambda \quad \text{and} \quad E(\phi) \leq \liminf(E(\psi_n)) = \lambda. \tag{43}$$

Hence, Lemma 3.1 implies that $0 < \alpha \leq \mu$, since $\mu = e^{-1}(\lambda)$.

If it is supposed that $0 < \alpha < \mu$, then Lemma 3.1 (ii) would imply that $f(\alpha) = F(\phi) > F_\lambda$ which contradicts (43). Thus, we must have $\alpha = \mu$, whence ϕ is an admissible function and

$$F(\phi) = F_\lambda, \quad E(\phi) = \lambda, \quad \text{and} \quad \|\psi_n\|_{H^2} \underset{n\to+\infty}{\longrightarrow} \|\phi\|_{H^2}. \tag{44}$$

Finally, it is concluded from (41) and (44) that $\{\psi_n\}$ converge strongly to ϕ in $H^2(\mathbb{R})$. □

The set of global minimizers for (V_λ) may now be characterized.

Corollary 4.7. *Let $\lambda \in (0, e^\star)$ and let G_λ denote the set of global minimizers for (V_λ). Then,*

$$G_\lambda = \{\phi_\mu(\cdot + \tau) \mid \tau \in \mathbb{R}\},$$

where $\mu = e^{-1}(\lambda)$ and ϕ_μ is the solitary wave defined in (11).

All the elements are now in place to mount a proof of the stability result in Theorem 1.1. The proof follows the line of reasoning laid down in Lemma 2.13 in [17].

Proof of Theorem 1.1. Suppose the theorem to be false. Then, there exists a number $\epsilon > 0$, a sequence of functions $\{\psi_n\} \subset H^2(\mathbb{R})$ and a sequence of times $\{t_n\}$ such that

$$\|\psi_n\|_{H^1} < \|\varphi^\star\|_{H^1}, \quad \psi_n \underset{n\to+\infty}{\longrightarrow} \phi_\mu \text{ in } H^2(\mathbb{R}), \tag{45}$$

and

$$\inf_{\tau\in\mathbb{R}} \|u_n(\cdot, t_n) - \phi_\mu(\cdot + \tau)\|_{H^2} \geq \epsilon, \tag{46}$$

where u_n is the global H^2-solution to (2) satisfying $u_n(\cdot,0)=\psi_n$. Observe, since the functionals E and F are continuous in $H^2(\mathbb{R})$, that

$$E(\psi_n) \underset{n\to+\infty}{\longrightarrow} E(\phi_\mu)=\lambda \quad \text{and} \quad F(\psi_n) \underset{n\to+\infty}{\longrightarrow} F(\phi_\mu)=F_\lambda. \tag{47}$$

Thus, if $f_n=u_n(\cdot,t_n)$, it is deduced from (47), Theorem 2.2 and the fact that E and F are conserved quantities for (2), that the sequence $\{f_n\}$ has the properties

$$\|f_n\|_{H^1}<\|\varphi^\star\|_{H^1}, \quad E(f_n) \underset{n\to+\infty}{\longrightarrow} \lambda, \quad \text{and} \quad F(f_n) \underset{n\to+\infty}{\longrightarrow} F_\lambda. \tag{48}$$

Suppose now that for all real sequences $\{c_n\}$, the associated sequence of functions $\{f_n(\cdot+c_n)\}$ converges to zero weakly in $H^2(\mathbb{R})$. Then, it follows from Lemma 4.5 that $\{f_n\}$ converges to zero strongly in $W^{1,p}(\mathbb{R})$ for any p with $2<p<\infty$. Consequently, F_λ has to be non-negative, which is a contradiction in view of Lemma 3.1 and Corollary 4.7. Thus, it is deduced, after making suitable spatial translations and extracting a subsequence, that

$$\exists\, f\in H^2(\mathbb{R}) \quad \text{such that} \quad f\neq 0 \quad \text{and} \quad f_n \underset{n\to+\infty}{\rightharpoonup} f \quad \text{in } H^2(\mathbb{R}). \tag{49}$$

Moreover, since Lemma 4.3 implies that $\|f\|_{H^1}<\|\varphi^\star\|_{H^1}$, and the only non-zero critical points of E are $2\varphi^\star$ and its spatial translates, f is not a critical point of E, so there is a function $h\in C_c^\infty(\mathbb{R})$ satisfying

$$E'(f)h\neq 0. \tag{50}$$

Consider the polynomial

$$P_n(t)=E(f_n+th)=a_n+b_nt+c_nt^2+dt^3, \tag{51}$$

where

$$a_n=E(f_n), \quad b_n=E'(f_n)h, \tag{52}$$

$$c_n=\frac{1}{2}E''(f_n)(h,h)=\int_{\mathbb{R}}\left(\frac{1}{2}h^2+\frac{1}{2}h'^2-\frac{1}{2}f_nh^2\right)dx, \tag{53}$$

and

$$d=-\frac{1}{6}\int_{\mathbb{R}}h^3dx. \tag{54}$$

Reference to (48) assures that

$$a_n \underset{n\to\infty}{\longrightarrow} \lambda\in(0,e^\star). \tag{55}$$

If $R>0$ is chosen large enough that $\operatorname{supp} h\subset(-R,R)$, then since $H^1(-R,R)$ is compactly embedded in $L^\infty(-R,R)$, it follows that, after passing to a further subsequence if necessary,

$$b_n=E'(f_n)h=\int_{\mathbb{R}}\left(f_nh+f_n'h'-\frac{1}{2}f_n^2h\right)dx \underset{n\to+\infty}{\longrightarrow} E'(f)h\neq 0. \tag{56}$$

Similar considerations assure that

$$c_n \underset{n\to+\infty}{\longrightarrow} \frac{1}{2}E''(f)(h,h), \tag{57}$$

so that, in particular, the $\{c_n\}$ are bounded. It follows readily that for all n large enough, there exists $t_n\in\mathbb{R}$ such that

$$P_n(t_n)=\lambda \quad \textit{and} \quad t_n \underset{n\to+\infty}{\longrightarrow} 0. \tag{58}$$

Combining (48), (49) and (58) leads to the conclusion

$$E(f_n+t_nh)=\lambda \quad \text{and} \quad \lim_{n\to\infty}F(f_n+t_nh)=\lim_{n\to\infty}F(f_n)=F_\lambda, \tag{59}$$

which is to say that the sequence $\{h_n\}$, defined by $h_n = f_n + t_n h$, is a minimizing sequence for the variational problem (V_λ). Therefore, it follows from Theorem 4.1 that there is a real number $\tau \in \mathbb{R}$, a real sequence $\{c_n\}$ and a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ satisfying

$$\lim_{n\to+\infty} f_{n_k}(\cdot + c_{n_k}) = \lim_{n\to+\infty} h_{n_k}(\cdot + c_{n_k}) = \phi_\mu(\cdot + \tau) \quad \text{in } H^2(\mathbb{R}),$$

which contradicts (45)–(46). □

5. **Refinement of the stability theory.** The outcome of the ruminations in the present section, while not a result of asymptotic stability, nevertheless gives strong evidence that the solution emerging from initial data that comprises a slightly perturbed solitary wave is very nearly a solitary wave traveling at a speed close to the speed of the unperturbed solitary wave. Here is a precise statement of the result in view.

Theorem 5.1. *Let $\tilde{c} > 1$, $\mu = \mu(\tilde{c}) = \frac{\tilde{c}-1}{\tilde{c}} \in (0,1)$, and $\epsilon_0 = \|\phi_\mu'\|_{L^2}^2 / \|\phi_\mu''\|_{L^2}$. Then there exists a positive constant A depending only on $\tilde{c}$ with the following property. For every ϵ in $(0, \epsilon_0)$, there exists $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in H^2(\mathbb{R})$ with $\|u_0 - \phi_\mu\|_{H^2} < \delta$, then there exists a C^1-function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\|u(\cdot,t) - \phi_\mu(\cdot + \gamma(t))\|_{H^2} < \epsilon, \quad \text{and} \quad |\gamma'(t) + \tilde{c}| < A\epsilon, \tag{60}$$

for all $t \in \mathbb{R}$, where u is the globally defined H^2-solution of (2) satisfying $u(\cdot, 0) = u_0$.

Before proving Theorem 5.1, two technical lemmas are laid out. The first one provides a choice of γ by demanding the satisfaction of an orthogonality condition. For $\beta > 0$, define U_β, an H^2-neighborhood of the trajectory of ϕ_μ, by

$$U_\beta = \{\psi \in H^2(\mathbb{R}) \mid \inf_{\tau\in\mathbb{R}} \|\psi - \phi_\mu(\cdot + \tau)\|_{H^2} < \beta\}. \tag{61}$$

Lemma 5.2. *Fix $\mu \in (0,1)$. There exists $\beta > 0$ and a C^1-map $\gamma : U_\beta \to \mathbb{R}$ such that,*

$$\int_{\mathbb{R}} \psi(x)\phi_\mu'(x + \gamma(\psi))dx = 0, \quad \forall\, \psi \in U_\beta, \tag{62}$$

$$\gamma(\psi(\cdot + \tau)) = \gamma(\psi) + \tau, \quad \text{and} \quad \gamma(\phi_\mu) = 0. \tag{63}$$

Proof. Consider the function

$$G : H^2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R},\ (\psi, \gamma) \mapsto \int_{\mathbb{R}} \psi(x)\phi_\mu'(x+\gamma)dx;$$

clearly,

$$G(\phi_\mu, 0) = 0 \quad \text{and} \quad \partial_\gamma G(\phi_\mu, 0) = -\int_{\mathbb{R}} \phi_\mu'(x)^2 dx < 0.$$

The implicit-function theorem implies the existence of positive numbers β, η and a unique C^1-function

$$\gamma : B_\beta(\phi_\mu) = \{\psi \in H^2(\mathbb{R}) \mid \|\psi - \phi_\mu\|_{H^2} < \beta\} \to (-\eta, \eta), \tag{64}$$

such that $\gamma(\phi_\mu) = 0$ and $G(\psi, \gamma(\psi)) = 0$ for all ψ in $B_\beta(\phi_\mu)$.

To check that (63) is satisfied in $B_\beta(\phi_\mu)$, let $\psi \in B_\beta(\phi_\mu)$ and $\tau \in \mathbb{R}$ be such that $\psi(\cdot + \tau) \in B_\beta(\phi_\mu)$. Then the translation invariance of Lebesgue measure implies that

$$0 = G(\psi, \gamma(\psi)) = G(\psi(\cdot + \tau), \gamma(\psi) + \tau),$$

and because the value of $\gamma(\psi)$ is unique, it must be the case that $\gamma(\psi(\cdot+\tau)) = \gamma(\psi)+\tau$.

The mapping γ is easily extended to all of U_β, where $\beta > 0$ is the radius provided by the implicit-function theorem. If for some $\tau \in \mathbb{R}$, $\|\psi - \phi_\mu(\cdot+\tau)\|_{H^2} < \beta$, define $\gamma(\psi) = \gamma(\psi(\cdot-\tau))+\tau$. This definition makes sense. Indeed, if $\|\psi-\phi_\mu(\cdot+\tau_1)\|_{H^2} < \beta$, then both $\psi(\cdot-\tau)$ and $\psi(\cdot-\tau_1)$ belong to $B_\beta(\phi_\mu)$. Since (63) holds in $B_\beta(\phi_\mu)$, it is deduced that

$$\gamma(\psi(\cdot-\tau_1)) = \gamma(\psi(\cdot-\tau-(\tau_1-\tau))) = \gamma(\psi(\cdot-\tau)) - \tau_1 + \tau,$$

which is the same as

$$\gamma(\psi(\cdot-\tau_1)) + \tau_1 = \gamma(\psi(\cdot-\tau)) + \tau.$$

□

Remark 5.3. It follows directly from Lemma 5.2 and the definition of the extension of γ to all of U_β that for any ϵ with $\eta > \epsilon > 0$, there is a δ with $\beta > \delta > 0$ such that for all $\psi \in B_\delta(\phi_\mu(\cdot+\tau))$,

$$|\gamma(\psi) - \tau| < \epsilon. \tag{65}$$

Control of the integral operator appearing in (6) is also needed, and provided in the next lemma.

Lemma 5.4.

(i) Let $u_0 \in B(0, \|\varphi^\star\|_{H^1}) = \{\psi \in H^1(\mathbb{R}) \mid \|\psi\|_{H^1} < \|\varphi^\star\|_{H^1}\}$ be such that $E(u_0) < E(\varphi^\star)$. Then there exists a positive constant C depending only on u_0 such that, for all $t \in \mathbb{R}$, the global solution u of (2) emanating from u_0 satisfies

$$\|(1-\partial_x^2 - u(\cdot,t))^{-1}\|_{\mathcal{B}(L^2;H^2)} \le C. \tag{66}$$

(ii) Let ψ_1 and ψ_2 in $B(0,\|\varphi^\star\|_{H^1})$. Then there exists a constant C such that

$$\|(1-\partial_x^2-\psi_2)^{-1} - (1-\partial_x^2-\psi_1)^{-1}\|_{\mathcal{B}(L^2)} \le C\|\psi_2-\psi_1\|_{H^1}. \tag{67}$$

Proof. Part (i) was established as inequality (5.16) in the proof of Theorem 3 in [15].

For Part (ii), argue as follows. From Lemma 6 in [15], it is known that

$$B(0,\|\varphi^\star\|_{H^1}) \subset \Omega_1$$

(see (5) for the definition of Ω_1), so that the operators $(1-\partial_x^2-\psi_j)^{-1}$, $j = 1,\ 2$ are well defined. Then (ii) is a direct consequence of inequality (3.5) of Lemma 2 in [15], with $a = 0$, $s = 1$, and $\phi = 0$. □

The elements needed to provide a proof of Theorem 5.1 are in place.

Proof of Theorem 5.1. Fix an ϵ in the range $0 < \epsilon < \epsilon_0$. Since the mapping

$$\tau \in \mathbb{R} \mapsto \phi_\mu(\cdot+\tau) \in H^2(\mathbb{R})$$

is uniformly continuous, there exists $\eta > 0$ such that

$$|\tau_2 - \tau_1| < \eta \quad \Longrightarrow \quad \|\phi_\mu(\cdot+\tau_2) - \phi_\mu(\cdot+\tau_1)\|_{H^2} < \frac{\epsilon}{2}. \tag{68}$$

Now, choose $\beta > 0$ as in Lemma 5.2 and Remark 5.3 corresponding to the fixed value of ϵ. Thus, the associated translation function γ maps the ball $B_\beta(\phi_m u)$ to the interval $(-\epsilon,\epsilon)$, and its extended version has the property that

$$|\gamma(\psi) - \tau| < \epsilon \quad \text{if} \quad \psi \in B_\beta(\phi_\mu(\cdot+\tau)).$$

By the stability result proved in Theorem 1.1, there is a $\delta > 0$ such that when $\|u_0 - \phi_\mu\|_{H^2} < \delta$, then there exists a real function $\theta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\|u(\cdot,t) - \phi_\mu(\cdot + \theta(t))\|_{H^2} < \min(\frac{\epsilon}{2}, \beta), \tag{69}$$

where u is the solution of (2) emanating from u_0.

Since $u(\cdot,t) \in U_\beta$, the function γ is defined on $u(\cdot,t)$, and by abuse of notation, we write

$$\gamma(t) = \gamma(u(\cdot,t)). \tag{70}$$

Hence, for all t in $\mathbb{R}$, $u(\cdot,t) \in B_\beta(\phi_\mu(\cdot+\theta(t)))$, so that (65) implies $|\gamma(t)-\theta(t)| < \eta$. It is then concluded from (68) and (69) that

$$\begin{aligned} &\|u(\cdot,t) - \phi_\mu(\cdot + \gamma(t))\|_{H^2} \\ &\qquad \le \|u(\cdot,t) - \phi_\mu(\cdot + \theta(t))\|_{H^2} + \|\phi_\mu(\cdot + \gamma(t)) - \phi_\mu(\cdot + \theta(t))\|_{H^2} < \epsilon. \end{aligned} \tag{71}$$

Observe next by reference to Theorem 2.2, (6), and the fact the operator

$$\partial_x(1 - \partial_x^2 - u)^{-1} \in \mathcal{B}(L^2),$$

that $u_t \in C^0(\mathbb{R}; H^2(\mathbb{R}))$. Therefore, $u \in C^1(\mathbb{R}; H^2(\mathbb{R}))$ and hence the function $\gamma(t)$ defined in (70) is a C^1-function. Thus, we can differentiate the relation $G(u(\cdot,t),\gamma(t)) = 0$ with respect to t, thereby adducing the formula

$$\int_{\mathbb{R}} \partial_t u(x,t)\phi_\mu'(x+\gamma(t))dx + \gamma'(t)\int_{\mathbb{R}} u(x,t)\phi_\mu''(x+\gamma(t))dx = 0. \tag{72}$$

Define h by $h(x,t) = u(x,t) - \phi_\mu(x+\gamma(t))$ and use (71) and the definition of ϵ_0 to infer that

$$\int_{\mathbb{R}} u(x,t)\phi_\mu''(x+\gamma(t))dx = -\int_{\mathbb{R}} \phi_\mu'(x)^2 dx + R_1(t) < 0, \tag{73}$$

where, for all t,

$$R_1(t) = \int_{\mathbb{R}} h(x,t)\phi_\mu''(x+\gamma(t))dx \le C\epsilon \tag{74}$$

and C is a positive constant depending only on $\tilde{c}$. On the other hand, (6) yields that

$$\begin{aligned} &\int_{\mathbb{R}} \partial_t u(x,t)\phi_\mu'(x+\gamma(t))dx \\ &\qquad = -\int_{\mathbb{R}} \partial_x(1-\partial_x^2 - u(x,t))^{-1}u(x,t)\phi_\mu'(x+\gamma(t))dx \\ &\qquad = -\int_{\mathbb{R}} \partial_x(1-\partial_x^2 - \phi_\mu(x))^{-1}\phi_\mu(x)\phi_\mu'(x)dx + R_2(t), \end{aligned} \tag{75}$$

where, after integrating by parts,

$$\begin{aligned} R_2(t) = &\int_{\mathbb{R}} (1-\partial_x^2 - u(x,t))^{-1}h(x,t)\phi_\mu''(x+\gamma(t))dx \\ &+ \int_{\mathbb{R}} \Big\{ \Big([1-\partial_x^2 - u(x,t)]^{-1} - [1-\partial_x^2 - \phi_\mu(x+\gamma(t))]^{-1} \Big) \\ &\qquad \times \phi_\mu(x+\gamma(t))\phi_\mu''(x+\gamma(t)) \Big\} dx. \end{aligned} \tag{76}$$

Since $(1-\partial_x^2 - \phi_\mu)^{-1}\phi_\mu = \tilde{c}\phi_\mu$, it follows that

$$-\int_{\mathbb{R}} \partial_x(1-\partial_x^2 - \phi_\mu(x))^{-1}\phi_\mu(x)\phi_\mu'(x)dx = -\tilde{c}\int_{\mathbb{R}} \phi_\mu'(x)^2 dx. \tag{77}$$

The bounds (66) and (67), and the fact that $\|h(\cdot,t)\|_{H^2} < \epsilon$ give control of the terms on the right-hand side of (75). It is then concluded that there is a positive constant C such that, for all $t \in \mathbb{R}$,

$$R_2(t) \le C\epsilon. \tag{78}$$

Substituting formulas (73), (76) and (77) into (72) leads to the formula

$$\gamma'(t) = -\tilde{c} + \frac{R_2(t) - \tilde{c}R_1(t)}{\int_{\mathbb{R}} \phi'_\mu(x)^2 dx - R_1(t)}, \tag{79}$$

which, when combined with (74) and (78), implies the existence of a positive constant A such that, for all $t \in \mathbb{R}$,

$$|\gamma'(t) + \tilde{c}| < A\epsilon.$$

□

6. **Conclusion.**

We have considered the Hirota-Satsuma equation, which was derived as a model for surface water waves in the same small-amplitude, long-wavelength regime as was the Korteweg-de Vries and the BBM equations. The solitary-wave solutions of this equation have been shown to be orbitally stable in $H^2(\mathbb{R})$ to perturbations of this regularity. Moreover, it was demonstrated that the solution emanating from a perturbed solitary wave travels at nearly the same speed as the unperturbed solitary wave itself.

Questions that remain open, and which appear to be of some interest include the following. First, are these waves stable to rougher perturbations? It has been shown for the Korteweg-de Vries equation, for example, that solitary waves are stable even in $L^2(\mathbb{R})$ (see [20]) and the methods that come to the fore in this analysis might be applied to the present context. It is also interesting to know if they are stable in smaller spaces, such as $H^s(\mathbb{R})$ for $s > 2$. The solitary waves of the Korteweg-de Vries equation are known to be stable in $H^k(\mathbb{R})$ for $k = 2, 3, \cdots$ (see [8]) and the question of the stability of the Hirota-Satsuma solitary waves seems equally interesting.

While there are no solitary waves outside of the set Ω_1 that appears in our analysis, the question of what transpires when initial data is posited outside this set is obviously interesting. Our Appendix 2 below casts a little light on this issue, but there is clearly more to be understood.

Finally, it would be helpful to have accurate numerical simulations of solutions of the Hirota-Satsuma equation. These could help point the way toward a more detailed understanding of the long-time asymptotics of solutions.

7. **Appendix 1: Proof of Theorem 4.6.**

As in the proof of Theorem 2.1 in [17], the proof of Theorem 4.6 is split into several lemmas.

Lemma 7.1. *Let $\{\phi_n\}$ be a bounded sequence in $H^2(\mathbb{R})$ and suppose that $\{\phi_n\}$ is not precompact in $W^{1,p}(\mathbb{R})$ for some p with $2 < p \le \infty$. Then there exists a real sequence $\{c_n\}$ such that some subsequence $\{\phi_{n_k}(\cdot + c_{n_k})\}$ of the sequence of translates $\{\phi_n(\cdot + c_n)\}$ converges weakly in $H^2(\mathbb{R})$ to a non-zero function ϕ. Moreover, it must be the case that $|c_{n_k}| \to \infty$, as $k \to \infty$.*

Proof. Lemma 7.1 follows directly by applying Lemma 2.4 in [17] to the sequences $\{\phi_n\}$ and $\{\phi'_n\}$. □

Again, let λ be fixed in the interval $(0, e^\star)$. Throughout the rest of this appendix, we will say that a function $\phi \in H^2(\mathbb{R})$ is *admissible* if $\phi \in I_\lambda$, that is,

$$E(\phi) = \lambda, \quad \text{and} \quad \|\phi\|_{H^1} < \|\varphi^\star\|_{H^1}.$$

A C^2-curve $\phi : (-\delta_0, \delta_0) \to H^2(\mathbb{R})$ is *admissible* when

$$E(\phi(t)) = \lambda, \quad \text{and} \quad \|\phi(t)\|_{H^1} < \|\varphi^\star\|_{H^1}, \text{ for all } t \in (-\delta_0, \delta_0). \tag{80}$$

A sequence $\{\phi_n\}$ will be *admissible* when ϕ_n is admissible for every $n \in \mathbb{N}$. Moreover, we say that a pair of functions $(h, \widetilde{h})$ is *adapted* to an admissible function ϕ if

$$E'(\phi)h = 0 \quad \text{and} \quad E''(\phi)(h, h) + E'(\phi)\widetilde{h} = 0. \tag{81}$$

For example, differentiating the first identity in (80) twice with respect to t, it is seen that if $\phi(t)$ is an admissible curve, then the pair $(\dot{\phi}(0), \ddot{\phi}(0))$ is adapted to $\phi(0)$ (here the dots adorning ϕ connote derivatives with respect to t). This simple observation has a kind of converse, as the next lemma shows.

Lemma 7.2. *Let $\{\phi_n\}$ be an admissible sequence converging weakly in $H^2(\mathbb{R})$ to some function $\phi \neq 0$, and let $\{(h_n, \widetilde{h}_n)\}$ be a bounded sequence of pairs adapted to the sequence $\{\phi_n\}$, which is to say that, the sequences $\{h_n\}$, $\{\widetilde{h}_n\}$ are bounded in $H^2(\mathbb{R})$ and $(h_n, \widetilde{h}_n)$ is adapted to ϕ_n, for all $n \in \mathbb{N}$. Then there exist a $\delta_0 > 0$ and a sequence of bounded C^2-curves $g_n : (-\delta_0, \delta_0) \to H^2(\mathbb{R})$ such that*

(i) $g_n(0) = 0$, $\dot{g}_n(0) = h_n$ and $\ddot{g}_n(0) = \widetilde{h}_n$,
(ii) the mapping $t \in (-\delta_0, \delta_0) \mapsto \phi_n + g_n(t)$ is an admissible curve, for each n, and
(iii) the sequences $\{g_n(t)\}$, $\{\dot{g}_n(t)\}$, and $\{\ddot{g}_n(t)\}$ are equicontinuous.

Proof. Arguing as in Corollary 4.4, we deduce that the weak limit ϕ of the admissible sequence $\{\phi_n\}$ satisfies $\|\phi\|_{H^1} < \|\varphi^\star\|$. Moreover, since the only non-zero critical points of the functional E are $2\varphi^\star$ and its spatial translates, and since these do not respect the last inequality, it is clear that $E'(\phi) \neq 0$. Thus, there exists a real-valued function $\psi \in C_c^\infty(\mathbb{R})$ with compact support such that $E'(\phi)\psi \neq 0$.

For all $n \in \mathbb{N}$, consider the function

$$H_n(\sigma, t) = E\Big(\phi_n + \sigma\psi + th_n + \frac{t^2}{2}\widetilde{h}_n\Big).$$

Then, up to passage to a subsequence, it must be the case that

$$H_n(0, 0) = \lambda \quad \text{and} \quad \partial_\sigma H_n(0, 0) = E'(\phi_n)\psi \neq 0,$$

for n large enough, since $E'(\phi_n)\psi \to E'(\phi)\psi$ as $n \to \infty$. The latter convergence is assured as follows. There is an $R > 0$ such that $\operatorname{supp}\psi \subset (-R, R)$. As $H^1(-R, R)$ is compactly embedded in $L^\infty(-R, R))$, a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ converges uniformly to ϕ on $[-R, R]$. In consequence, $E'(\phi_{n_k})\psi \to E'(\phi)\psi$, whence $E'(\phi_{n_k})\psi \neq 0$ for k large.

It follows by an application of the implicit-function theorem that there exists a sequence $\{\sigma_n\}$ of functions for which

$$\sigma_n(0) = 0 \quad \text{and} \quad H_n(\sigma_n(t), t) = \lambda. \tag{82}$$

Then it is deduced, upon differentiating (82), that

$$\partial_\sigma H_n(0, 0)\dot{\sigma}(0) + E'(\phi_n)h_n = 0$$

and

$$\partial_\sigma H_n(0,0)\ddot{\sigma}(0) + \partial_\sigma^2 H_n(0,0)(\dot{\sigma}(0),\dot{\sigma}(0)) + E'(\phi_n)\widetilde{h}_n + E''(\phi_n)(h_n,h_n) = 0,$$

which imply that $\dot{\sigma}(0) = \ddot{\sigma}(0) = 0$, since $(h_n, \widetilde{h}_n)$ is adapted to ϕ_n. Clearly, the sequence $g_n(t) = \sigma_n(t)\psi + th_n + \frac{t^2}{2}\widetilde{h}_n$ satisfies the conditions (i), (ii) and (iii). Use has been made of the fact that E' is uniformly continuous on bounded sets of $H^1(\mathbb{R})$ to ensure the existence of $\delta_0 > 0$ independent of n such that σ_n is defined on $(-\delta_0, \delta_0)$ for all n and to prove the equicontinuity assertion in (iii). $\square$

The next lemma follows from standard calculus arguments.

Lemma 7.3. *Let $\{\phi_n\} \subset H^2(\mathbb{R})$ be a minimizing sequence for the problem (V_λ), converging weakly in $H^2(\mathbb{R})$ to some $\phi \neq 0$. Then,*

(i)

$$|||F'(\phi_n)||| := \sup\left\{ \frac{|F'(\phi_n)h|}{\|h\|_{H^2}} \mid h \neq 0 \text{ and } E'(\phi_n)h = 0 \right\} \underset{n\to\infty}{\longrightarrow} 0 \tag{83}$$

and

(ii) if for some $\delta_0 > 0$, $\{g_n : (-\delta_0,\delta_0) \to H^2(\mathbb{R})\}$ is a sequence of C^2-curves such that $g_n(0) = 0$, $\phi_n + g_n(t)$ is admissible, the sequences $\{\dot{g}_n(0)\}$ and $\{\ddot{g}_n(0)\}$ are bounded and the sequences of curves $\{g_n(t)\}$, $\{\dot{g}_n(t)\}$, and $\{\ddot{g}_n(t)\}$ are equicontinuous, then

$$\liminf \frac{d^2}{dt^2} F(\phi_n + g_n(t))_{|t=0} \geq 0. \tag{84}$$

The next lemma gives an expression for $|||F'(\phi_n)|||$ when ϕ is an admissible function.

Lemma 7.4. *Let ϕ be an admissible function. Then there exists a unique $\bar{h} \in H^2(\mathbb{R})$ such that*

$$|||F'(\phi)||| = F'(\phi)\bar{h}, \quad E'(\phi)\bar{h} = 0, \quad \text{and} \quad \|\bar{h}\|_{H^2} = 1, \tag{85}$$

and there exists two real constants α and ν satisfying

$$F'(\phi)h + \alpha E'(\phi)h + \nu(h,\bar{h})_{H^2} = 0, \quad \text{for all } h \in H^2(\mathbb{R}). \tag{86}$$

Moreover, if $h : (-\delta_0,\delta_0) \to H^2(\mathbb{R})$ is a C^2-curve such that $h(0) = 0$ and the mapping $t \mapsto \phi + h(t)$ is an admissible curve, then

$$\frac{d^2}{dt^2} F(\phi + h(t))_{|t=0} = \Lambda''(\phi)(\dot{h}(0),\dot{h}(0)) - \nu(\ddot{h}(0),\bar{h})_{H^2}, \tag{87}$$

where $\Lambda = F + \alpha E$.

Proof. From (83), it is clear that

$$|||F'(\phi)||| = \sup\{F'(\phi)h \mid \|h\|_{H^2} = 1 \text{ and } E'(\phi)h = 0\}. \tag{88}$$

Because $\{h \in H^2(\mathbb{R}) \mid E'(\phi)h = 0\}$ is a Hilbert space in its own right, (85) follows directly from the Riesz representation theorem and (86) is the Euler-Lagrange equation associated to (88).

Now, suppose that h is a C^2-curve as described in the second part of Lemma 7.4 and use (86) to compute

$$\begin{aligned}\frac{d^2}{dt^2} F(\phi + h(t))_{|t=0} &= F''(\phi)(\dot{h}(0),\dot{h}(0)) + F'(\phi)\ddot{h}(0)\\ &= F''(\phi)(\dot{h}(0),\dot{h}(0)) - \alpha E'(\phi)\ddot{h}(0) - \nu(\ddot{h}(0),\bar{h})_{H^2},\end{aligned}$$

which implies (87) since $(\dot{h}(0), \ddot{h}(0))$ is adapted to ϕ (see the second identity in (81)). □

In the following, let $\{\phi_n\}$ be a minimizing sequence for the variational problem (V_λ), converging weakly to some non-zero function ϕ in $H^2(\mathbb{R})$. It is deduced from Lemma 7.4 that there exists two real sequences $\{\alpha_n\}$, $\{\nu_n\}$, and a sequence $\{\bar{h}_n\}$ in $H^2(\mathbb{R})$ with $\|\bar{h}_n\|_{H^2} = 1$, $E'(\phi_n)\bar{h}_n = 0$, and $F'(\phi_n)\bar{h}_n = |||F'(\phi_n)|||$ such that

$$F'(\phi_n)h + \alpha_n E'(\phi_n)h + \nu_n (h, \bar{h}_n)_{H^2} = 0, \quad \text{for all } h \in H^2(\mathbb{R}) \text{ and } n \in \mathbb{N}. \tag{89}$$

Lemma 7.5. *The sequence $\{\nu_n\}$ converges to 0 as $n \to \infty$ and the sequence $\{\alpha_n\}$ is bounded.*

Proof. Applying (89) with $h = \bar{h}_n$ and using (83), it is ascertained that

$$\nu_n = -F'(\phi_n)\bar{h}_n = -|||F'(\phi_n)||| \underset{n\to+\infty}{\longrightarrow} 0.$$

On the other hand, if some subsequence, still denoted $\{\alpha_n\}$, of the sequence $\{\alpha_n\}$ is unbounded, then upon dividing (89) by α_n and letting n tend to $+\infty$, it would transpire that

$$E'(\phi)h = 0, \quad \text{for all } h \in H^2(\mathbb{R}),$$

which is a contradiction since $E'(\phi) \neq 0$, as was already seen in the proof of Lemma 7.2. □

From now on, it is presumed without loss of generality that $\{\alpha_n\}$ converges.

Lemma 7.6. *Let $\{\phi_n\}$, $\{\bar{h}_n\}$, $\{\alpha_n\}$ and $\{\nu_n\}$ be as above. If $\{h_n\}$ is a bounded sequence in $H^2(\mathbb{R})$ satisfying $E'(\phi_n)h_n = 0$, then*

$$\liminf \Big\{ F''(\phi_n)(h_n, h_n) + \alpha_n E''(\phi_n)(h_n, h_n) \Big\} \geq 0. \tag{90}$$

Proof. Once again, we use the fact that the non-zero weak limit ϕ of our minimizing sequence satisfies $E'(\phi) \neq 0$. Therefore, there exists $\psi \in H^2(\mathbb{R})$ with compact support such that $E'(\phi)\psi \neq 0$ and so there is a bounded real sequence $\{d_n\}$ such that at least for large n, $E''(\phi_n)(h_n, h_n) + d_n E'(\phi_n)\psi = 0$, which is to say that the pair $(h_n, d_n\psi)$ is adapted to ϕ_n. From this is deduced, via Lemma 7.2, the existence of a sequence of curves $g_n : (-\delta_0, \delta_0) \to H^2(\mathbb{R})$ satisfying (i), (ii), and (iii), and it then follows from (84) and (87) that

$$0 \leq \liminf \frac{d^2}{dt^2} F(\phi_n + g_n(t))_{|t=0} = \liminf \left\{ \Lambda_n''(\phi_n)(h_n, h_n) - d_n \nu_n (\psi, \bar{h}_n)_{H^2} \right\},$$

where $\Lambda_n = F + \alpha_n E$, which implies (90) since $\nu_n \to 0$ (see Lemma 7.5). □

Finally, here is a proof of Theorem 4.6.

Proof of Theorem 4.6. Let $\{\phi_n\}$ a minimizing sequence for (V_λ) converging weakly in $H^2(\mathbb{R})$ to a non-zero function ϕ. We will argue by contradiction and suppose that $\{\phi_n\}$ is not precompact in $W^{1,p}(\mathbb{R})$ for some p with $2 < p \leq \infty$. Then, it follows from Lemma 7.1 that, extracting a subsequence if necessary, there exist a real sequence $\{c_n\}$ and a non-zero function $\psi \in H^2(\mathbb{R})$ satisfying

$$|c_n| \underset{n\to\infty}{\longrightarrow} \infty, \quad \text{and} \quad \psi_n := \phi_n(\cdot + c_n) \underset{n\to\infty}{\rightharpoonup} \psi, \text{ in } H^2(\mathbb{R}).$$

Let $h \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} h \subset [-R, R]$, say. Taking advantage of the compact embedding of $H^2(-R, R)$ in $H^1(-R, R)$, and taking an appropriate subsequence of $\{\phi_n\}$ and $\{\psi_n\}$, it is seen from (89) that

$$F'(\phi)h + \alpha E'(\phi)h = 0 \quad \text{and} \quad F'(\psi)h + \alpha E'(\psi)h = 0.$$

As $C_c^\infty(\mathbb{R})$ is dense in $H^2(\mathbb{R})$, the latter two formulas thus hold for all $h \in H^2(\mathbb{R})$. Then, the hypothesis (H) ensures the existence of two functions h and k in $H^2(\mathbb{R})$ such that

$$(V''(\phi) + \alpha E''(\phi))\,(h, h) < 0 \quad \text{and} \quad (V''(\psi) + \alpha E''(\psi))\,(k, k) < 0. \tag{91}$$

Next, choose two real sequences $\{a_n\}$ and $\{b_n\}$ such that $h_n = a_n h + b_n k_n$ satisfies $E'(\phi_n)h_n = 0$ and $a_n^2 + b_n^2 = 1$, where $k_n = k(\cdot - c_n)$. A straightforward computation shows that

$$(F''(\phi_n) + \alpha_n E''(\phi_n))(h_n, h_n) = a_n^2(F''(\phi_n) + \alpha_n E''(\phi_n))(h, h)$$
$$+ 2a_n b_n(F''(\phi_n) + \alpha_n E''(\phi_n))(h, k_n) + b_n^2(F''(\psi_n) + \alpha_n E''(\psi_n))(k, k).$$

Extracting subsequences if necessary to ensure that $\{a_n\}$, $\{b_n\}$, and $\{\alpha_n\}$ converge to a, b, and α, respectively, it is concluded upon passing to the limit in the last equation that

$$\liminf\big\{(F''(\phi_n) + \alpha_n E''(\phi_n))(h_n, h_n)\big\}$$
$$= a^2\big(F''(\phi) + \alpha E''(\phi)\big)(h, h) + b^2\big(F''(\psi) + \alpha E''(\psi)\big)(k, k) < 0,$$

since $|c_n| \to +\infty$, which contradicts (90) in Lemma 7.6. □

8. Appendix 2: An ill-posedness result for the Hirota-Satsuma equation.

It was proved in [15] that the Hirota-Satsuma equation is locally well-posed for initial data u_0 in the open subset

$$\Omega_1 = \{\phi \in H^1(\mathbb{R}) \mid -1 \notin \sigma(-\partial_x^2 - \phi)\},$$

of $H^1(\mathbb{R})$. The main idea was to rewrite the equation in the quasi-linear form

$$u_t = -\partial_x(1 - \partial_x^2 - u)^{-1}u,$$

and apply a fixed-point argument. This argument leaves open what might transpire for initial data in the closed set

$$\mathcal{F}_1 = H^1(\mathbb{R}) \setminus \Omega_1.$$

In the following, we exhibit a function in $\mathcal{F}_1$ around which the flow map (the map that associates to initial data the solution that emanates therefrom) associated to (2) cannot be extended continuously.

Consider $\varphi^\star$, the unique, up to translation, solution to the elliptic differential equation (9). It is clear that $\varphi^\star$ belongs to $\mathcal{F}_1$; indeed, $\varphi^\star \in \partial\Omega_1$. On the other hand, it was also proved in [15] that the solutions corresponding to initial data in the subset

$$\Omega_1^0 = \{\phi \in H^1(\mathbb{R}) \mid \|\phi\|_{H^1} < \|\varphi^\star\|_{H^1} \quad \text{and} \quad E(\phi) < E(\varphi^\star)\}$$

of Ω_1 extend globally in time. Note also that $\varphi^\star \in \partial\Omega_1^0$.

Proposition 8.1. *If there exists $T > 0$ and a solution $u^\star$ of (2) in $C([0, T]; H^1(\mathbb{R}))$ satisfying $u^\star(\cdot, 0) = \varphi^\star$, then the flow map*

$$S : \Omega_1^0 \to C([0, T]; H^1(\mathbb{R})),$$

does not extend continuously to $\varphi^\star$.

Proof. The argument is made by contradiction. Suppose that S extends continuously to $\varphi^\star$. The solitary-wave solutions of (2) are

$$u_c(x,t) = \phi_\mu(x-(1+c)t), \text{ with } \mu = \frac{c}{1+c} \text{ and } \phi_\mu(z) = \mu\varphi^\star(\mu^{\frac{1}{2}}z) \in \Omega_1^0. \quad (92)$$

It is straightforward to check that

$$\phi_\mu \underset{c\to+\infty}{\longrightarrow} \varphi^\star \quad \text{in} \quad H^1(\mathbb{R}).$$

As S is assumed to be continuous, $S\phi_\mu \to S\varphi^\star = u^\star$ in $C([0,T];H^1(\mathbb{R}))$. Thus, it follows by the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, that

$$u_c = S\phi_\mu \underset{c\to+\infty}{\longrightarrow} u^\star \quad \text{in} \quad L^\infty(\mathbb{R}\times[0,T]). \quad (93)$$

On the other hand, it is deduced from (92) that for any $R>0$ and for any $\delta>0$,

$$u_c \underset{c\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad L^\infty((-R,R)\times[\delta,T]). \quad (94)$$

Therefore $u^\star(\cdot,t)=0$ if $t\in(0,T]$, and $u^\star(\cdot,0)=\varphi^\star$. This contradicts the presumption that $u^\star$ belongs to $C([0,T];H^1(\mathbb{R}))$. ☐

Contrary to what occurs for the KdV- and BBM-equations, the size of the solitary-wave solutions of (2) remains uniformly bounded even when the velocity becomes unboundedly large. The ill-posedness result subsists on this simple fact.

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