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WELL–POSEDNESS OF A MODEL FOR WATER WAVES WITH VISCOSITY

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Dedicated with great admiration to our friend Tom Beale.

Abstract. The water wave equations of ideal free–surface fluid mechanics are a fundamental model of open ocean movements with a surprisingly subtle well–posedness theory. In consequence of both theoretical and computational difficulties with the full water wave equations, various asymptotic approximations have been proposed, analyzed and used in practical situations. In this essay, we establish the well–posedness of a model system of water wave equations which is inspired by recent work of Dias, Dyachenko, and Zakharov (*Phys. Lett. A*, 372:2008). The model in question includes dissipative effects and is weakly nonlinear. The present contribution is a first step in a larger program centered around the Dias-Dychenko-Zhakharov system.

1. **Introduction.** The motion of the surface of a large body of water arises in a wide array of applications. From the interaction of waves with open–ocean oil rigs, to the formation and movement of underwater sandbars, to the generation and propagation of tsunamis, models for the "water wave problem" are of great interest to engineers and scientists alike. From a mathematical perspective, one of the historically most common and successful models of surface wave propagation (what is often termed the full water wave equations (2)) has a surprisingly subtle well–posedness theory (see [16, 17, 10, 1] and the references contained therein for a description of the current state of affairs concerning this problem).

A natural question arises: Given the centrality and importance of this model, should we expect that the fundamental task of showing solutions exist requires the delicate analytical tools evident in the preceding references? Motivated by this question, we investigate here the possibility of adding "artificial viscosity" terms to the water wave equations and establishing well-posedness of this system with viscosity by comparatively simple arguments. In the future, our goal will be to establish additional estimates which will allow the recovery of solutions of the original, non-viscous problem by taking a limit as the viscosity vanishes.

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As is not uncommon when embarking on a program such as the one just suggested, it is useful to insert viscosity into the governing equations (2) in a physically motivated fashion, rather than in some arbitrary fashion. We use as our guide in this aspect the recent work of Dias, Dyachenko, and Zakharov (DDZ henceforth) [7] who argued persuasively in favor of the model (7) below. Their system includes all the correct linear terms, but does not claim to treat nonlinear contributions adequately. They also recognize the fundamental problem with "viscous potential flow"; the modeling assumptions inherent in potential flow are incompatible with the presence of viscosity. However, they found their model useful and we suggest that theirs is a natural way to add artificial viscosity to the water wave problem with the goals we have in mind. Before leaving this point, it is worth noting that one of the authors has used the sequence of ideas set out here to produce numerically stabilized models for surface wave propagation [8].

The model analyzed here is a weakly nonlinear system which retains accurately the linear and quadratic terms of the full equations (c.f. the related work of Matsuno [11] and Choi [3] in the inviscid case). This model is augmented with viscosity in the DDZ manner. The simplified system with viscosity provides a good example on which to test the efficacy of our general line of argument. In future work, we intend to extend this framework to the full DDZ model in three spatial dimensions, and including finite depth. Our end goal will be to produce an alternate proof of well–posedness of the water wave equations using the approach of adding artificial viscosity and then sending the viscosity to zero. The necessary *a priori* estimates are beyond the scope of the present contribution, however. To be clear, in the current work, we establish estimates for the growth of solutions of our model system, and we use these estimates as the basis of a short-time well-posedness proof. We do this for any fixed, positive value of the viscosity parameter. The resulting time of existence for the solutions then depends on the viscosity parameter, and this time of existence would go to zero if we were to attempt to take the zero viscosity limit using only the current estimates. Establishing estimates which are uniform in the viscosity parameter, and which would therefore enable the zero viscosity limit to be taken, is a subject of the authors' ongoing work, for both the present system and related systems.

The paper is organized as follows. In § 2, the governing equations of free–surface, ideal fluid mechanics and the DDZ model are recalled. Convenient surface variables are introduced in § 2.1 and various analyticity results for associated non-local operators are set out in § 2.2. A suitable non–dimensionalization is provided in § 2.3 which produces small parameters that provide a formal justification of our weakly nonlinear model. In § 3 the crucial energy estimates upon which our theory hinges are discussed. Specific details regarding certain commutator estimates are given in § 3.1. These are helpful in § 3.2 where the full set of energy-type inequalities are established. In § 4 a rigorous existence theory is provided for a mollified system. The limit as the mollification parameter vanishes is studied in § 5.1. Properties of the limiting solutions, together with uniqueness and continuous dependence, are established in § 5.2 and § 5.3.

2. **Governing equations.**

Suppose that an ideal (inviscid, irrotational, incompressible) fluid occupies a semi–infinite domain bounded above by a free air–fluid interface $y = \eta = \eta(x,t)$, $x \in \mathbf{R}^{d-1}$, $d = 2, 3$. The well–known model for the motion of the fluid and the interface are the water wave equations [9]

$$\Delta\varphi = 0 \qquad y < \eta(x,t) \tag{1}$$

$$\partial_y\varphi \to 0 \qquad y \to -\infty \tag{2}$$

$$\partial_t\eta = \partial_y\varphi - \nabla_x\eta\cdot\nabla_x\varphi \qquad y = \eta(x,t) \tag{3}$$

$$\partial_t\varphi = -g\eta - \frac{1}{2}\left|\nabla_x\varphi\right|^2 - \frac{1}{2}(\partial_y\varphi)^2 \qquad y = \eta(x,t), \tag{4}$$

where φ is the velocity potential (so that the velocity field $\vec{u} = (\nabla_x\varphi, \partial_y\varphi)$) and g is the gravity constant. These equations are supplemented with initial conditions

$$\eta(x,0) = \eta_0(x), \quad \varphi(x,y,0) = \varphi_0(x,y), \tag{5}$$

and appropriate lateral boundary conditions. For simplicity we take the classical periodic boundary conditions, with period γ :

$$\varphi(x+\gamma,y,t) = \varphi(x,y,t), \quad \eta(x+\gamma,t) = \eta(x,t). \tag{6}$$

Our discussion is specialized to two–dimensional waves, but the generalization to three dimensions in the weakly nonlinear context is straightforward.

As mentioned already, the DDZ model

$$\Delta\varphi = 0 \qquad y < \eta(x,t), \tag{7a}$$

$$\partial_y\varphi \to 0 \qquad y \to -\infty, \tag{7b}$$

$$\partial_t\eta = \partial_y\varphi + 2\nu\Delta_x\eta - \nabla_x\eta\cdot\nabla_x\varphi \qquad y = \eta(x,t), \tag{7c}$$

$$\partial_t\varphi = -g\eta - 2\nu\partial_y^2\varphi - \frac{1}{2}\left|\nabla_x\varphi\right|^2 - \frac{1}{2}(\partial_y\varphi)^2 \qquad y = \eta(x,t), \tag{7d}$$

where ν is the constant of viscosity, will also be central to our work. This system was introduced by Dias, Dyachenko and Zakharov [7] to study weak viscous effects in the water wave equations. Of course, the introduction of viscosity is prohibited in ideal fluid flow (there is no longer a velocity potential, for example), but DDZ argue convincingly that their choice of viscous terms gives the correct *linear* viscous behavior. We use the DDZ model as a starting point for introducing artificial viscosity into the water wave equations.

2.1. **Surface variables.** It is convenient to follow the approach pioneered by Zakharov [18] and developed in detail by Craig & Sulem [5]. Note that to solve (7), it is sufficient to find the pair of functions $(\eta(x,t), \xi(x,t))$ where

$$\xi(x,t) := \varphi(x,\eta(x,t),t) \tag{8}$$

is the velocity potential at the free surface. Once η and ξ are known, the velocity potential in the interior of the fluid domain can be found from an appropriate integral formula (see, again, [5]).

It is clear from (7) that it will be necessary to have in hand first- and second-order derivatives of φ at the free boundary $y = \eta(x,t)$ to provide a closed system of equations for (η,ξ). The following maps will be useful in this endeavor. If v is the unique solution of the prototypical elliptic problem (c.f. (7a), (7b))

$$\Delta v = 0 \qquad y < \sigma(x), \tag{9a}$$

$$\partial_y v \to 0 \qquad y \to -\infty, \tag{9b}$$

$$v = \xi \qquad y = \sigma(x), \tag{9c}$$

$$v(x+\gamma,y) = v(x,y), \tag{9d}$$

define

$$X(\sigma)[\xi] := \partial_x v(x,\sigma), \quad Y(\sigma)[\xi] := \partial_y v(x,\sigma), \quad Z(\sigma)[\xi] := \partial_y^2 v(x,\sigma). \tag{10}$$

Here, σ and ξ are putative surface deformations and surface velocity potentials, respectively. In terms of these operators, the first of the two surface equations (7c) can be written as

$$\partial_t \eta = Y(\eta)[\xi] + 2\nu\partial_x^2\eta - (\partial_x\eta)X(\eta)[\xi].$$

Using (8) and (7d), one verifies straightforwardly that

$$\begin{aligned}
\partial_t \xi &= \partial_y \varphi(\partial_t \eta) + \partial_t \varphi \\
&= Y(\eta)[\xi]\left\{Y(\eta)[\xi] + 2\nu\partial_x^2\eta - (\partial_x\eta)X(\eta)[\xi]\right\} \\
&\quad - g\eta - 2\nu Z(\eta)[\xi] - \frac{1}{2}\left(X(\eta)[\xi]\right)^2 - \frac{1}{2}\left(Y(\eta)[\xi]\right)^2 \\
&= -g\eta - 2\nu Z(\eta)[\xi] + \frac{1}{2}\left(Y(\eta)[\xi]\right)^2 - \frac{1}{2}\left(X(\eta)[\xi]\right)^2 \\
&\quad + 2\nu(\partial_x^2\eta)Y(\eta)[\xi] - (\partial_x\eta)X(\eta)[\xi]Y(\eta)[\xi].
\end{aligned}$$

Thus, the *surface* formulation of the DDZ equations (7) (the water wave equations with viscosity) is

$$\partial_t \eta = Y(\eta)[\xi] + 2\nu\partial_x^2\eta - (\partial_x\eta)X(\eta)[\xi] \tag{11a}$$

$$\begin{aligned}
\partial_t \xi &= -g\eta - 2\nu Z(\eta)[\xi] + \frac{1}{2}\left(Y(\eta)[\xi]\right)^2 - \frac{1}{2}\left(X(\eta)[\xi]\right)^2 \\
&\quad + 2\nu(\partial_x^2\eta)Y(\eta)[\xi] - (\partial_x\eta)X(\eta)[\xi]Y(\eta)[\xi],
\end{aligned} \tag{11b}$$

supplemented with initial conditions

$$\eta(x,0) = \eta_0(x), \quad \xi(x,0) = \xi_0(x),$$

and the periodic boundary conditions

$$\eta(x+\gamma,t) = \eta(x,t), \quad \xi(x+\gamma,t) = \xi(x,t).$$

2.2. **Analytic dependence of the surface integral operators.** An important aspect of the operators X, Y, and Z defined in (10) is that they depend analytically upon the surface deformation σ. More precisely, if we set $\sigma(x) = \varepsilon f(x)$, then the series

$$X(\varepsilon f) = \sum_{n=0}^{\infty} X_n(f)\varepsilon^n, \quad Y(\varepsilon f) = \sum_{n=0}^{\infty} Y_n(f)\varepsilon^n, \quad Z(\varepsilon f) = \sum_{n=0}^{\infty} Z_n(f)\varepsilon^n, \tag{12}$$

all converge strongly in an appropriate function spaces provided that f is sufficiently smooth (see [4, 6, 14]). Our aim in this paper is to study a set of equations formally derived from (11) in the weakly nonlinear regime. The derivation depends upon having expressions for the operators X_0, X_1, Y_0, Y_1, Z_0, and Z_1.

The method of Operator Expansions [13, 5], to be described presently, is used in determining these lowest-order operators. Consider the harmonic function

$$\varphi_p(x,y) = e^{ipx+|p|y}, \quad p \in \Gamma' = \{2n\pi/\gamma : n \in \mathbb{Z}\}$$

which satisfies (9a), (9b) and (9d). Focusing upon the operator $Y(\sigma)[\xi]$, it follows from its definition in (10) that

$$Y(\sigma)\left[e^{ipx+|p|\sigma}\right] = |p|\, e^{ipx+|p|\sigma}.$$

Setting $\sigma = \varepsilon f$ and substituting in the expansion for Y in (12) and the Taylor series for the exponential, it transpires that

$$\left(\sum_{n=0}^{\infty} Y_n(f)\varepsilon^n\right)\left[e^{ipx}\sum_{n=0}^{\infty}\frac{(f(x))^n}{n!}|p|^n\varepsilon^n\right] = |p|\,e^{ipx}\sum_{n=0}^{\infty}\frac{(f(x))^n}{n!}|p|^n\varepsilon^n. \tag{13}$$

Equating the order zero terms on both sides of (13), we find

$$Y_0(f)[e^{ipx}] = |p|\,e^{ipx}$$

implying, if we use Fourier multiplier notation, that

$$Y_0(f)[e^{ipx}] = |D|\,e^{ipx},$$

where $D := (1/i)\partial_x$. Since any function of interest here can be represented via its Fourier series, it is concluded that

$$Y_0(f)[\xi(x)] = |D|\,\xi(x). \tag{14}$$

At order one in (13), we find

$$Y_1(f)[e^{ipx}] + Y_0(f)[f\,|p|\,e^{ipx}] = f\,|p|^2\,e^{ipx},$$

so that

$$Y_1(f)[e^{ipx}] + Y_0(f)[f\,|D|\,e^{ipx}] = f\,|D|^2\,e^{ipx}.$$

Again, representing a generic ξ in terms of its Fourier series, it is seen that

$$Y_1(f)[\xi(x)] = f\,|D|^2\,\xi(x) - Y_0(f)[f\,|D|\,\xi],$$

and using (14), it follow that

$$Y_1(f)[\xi(x)] = f\,|D|^2\,\xi(x) - |D|\,[f\,|D|\,\xi]. \tag{15}$$

In a similar manner, the formulas

$$X_0(f)[\xi] = iD\xi = \partial_x\xi \tag{16a}$$

$$X_1(f)[\xi] = f(iD)\,|D|\,\xi - (iD)[f\,|D|\,\xi] = f\partial_x\,|D|\,\xi - \partial_x[f\,|D|\,\xi] = (\partial_x f)(|D|\,\xi) \tag{16b}$$

$$Y_0(f)[\xi] = |D|\,\xi \tag{16c}$$

$$Y_1(f)[\xi] = f\,|D|^2\,\xi - |D|\,[f\,|D|\,\xi] \tag{16d}$$

$$Z_0(f)[\xi] = |D|^2\,\xi \tag{16e}$$

$$Z_1(f)[\xi] = f\,|D|^3\,\xi - |D|^2\,[f\,|D|\,\xi]. \tag{16f}$$

for X_0, X_1, Z_0 and Z_1 may be determined. Since the zeroth-order operator X_0 is independent of f, it is written simply X_0 rather than $X_0(f)$. The same goes for Y_0 and Z_0.

2.3. **Non-dimensionalization and the weakly nonlinear model.** The equations (11) can be non-dimensionalized using the classical scalings

$$x = \lambda x', \quad y = \lambda y', \quad t = \frac{\lambda}{\sqrt{g\lambda}}\,t', \quad \eta = a\eta', \quad \xi = a\sqrt{g\lambda}\,\xi',$$

where λ denotes a typical wavelength (which we will set to $2\pi/\gamma$), and a is a typical amplitude. Defining the nondimensional quantities

$$\alpha := \frac{a}{\lambda}, \quad \beta := \frac{\nu}{\sqrt{g\lambda^3}},$$

equation (11) in the new variables is

$$\partial_t \eta = |D|\,\xi + 2\beta\partial_x^2\eta + \alpha\left\{\eta\,|D|^2\,\xi - |D|\,[\eta\,|D|\,\xi] - (\partial_x\eta)\partial_x\xi\right\} + \mathcal{O}(\alpha^2), \tag{17a}$$

$$\begin{aligned}\partial_t \xi = -\eta - 2\beta\,|D|^2\,\xi + \alpha\Big\{&-2\beta\eta\,|D|^3\,\xi + 2\beta\,|D|^2\,[\eta\,|D|\,\xi]\\ &+\frac{1}{2}\,(|D|\,\xi)^2 - \frac{1}{2}\,(\partial_x\xi)^2 + 2\beta(\partial_x^2\eta)\,|D|\,\xi\Big\} + \mathcal{O}(\alpha^2),\end{aligned} \tag{17b}$$

where the primes have been dropped for ease of reading. If we ignore terms of order α^2 and then return to dimensional variables, we come to our weakly nonlinear model equations, *viz.*

$$\partial_t \eta = |D|\,\xi + 2\nu\partial_x^2\eta + \eta\,|D|^2\,\xi - |D|\,[\eta\,|D|\,\xi] - (\partial_x\eta)\partial_x\xi, \tag{18a}$$

$$\begin{aligned}\partial_t \xi = &-g\eta - 2\nu\,|D|^2\,\xi - 2\nu\eta\,|D|^3\,\xi + 2\nu\,|D|^2\,[\eta\,|D|\,\xi]\\ &+ \frac{1}{2}(|D|\,\xi)^2 - \frac{1}{2}(\partial_x\xi)^2 + 2\nu(\partial_x^2\eta)\,|D|\,\xi.\end{aligned} \tag{18b}$$

This system will be considered along with the initial data

$$\eta(x,0) = \eta_0(x), \qquad \xi(x,0) = \xi_0(x), \tag{19}$$

both of which are presumed to be periodic of period 2π, say.

The following is the main theorem of the paper. (Here, a period cell is denoted by X and the usual spaces of real-valued, periodic functions in the L^2-based Sobolev classes H^s are written $H^s(X)$).

Theorem 2.1. *Let s be a sufficiently large positive integer. Let $\eta_0 \in H^s(X)$ and $\xi_0(x) \in H^s(X)$ be given. Then there exists $T > 0$ and a unique solution $(\eta,\xi) \in C([0,T];H^s(X))$ to the initial-value problem (18), (19). For any $s' \in \mathbb{R}$ with $0 \leq s' < s$, the solution depends continuously on the initial data, when the norm of the solution is measured in the space $C([0,T];H^{s'}(X))$.*

Remark 1. In the above theorem, we indicate that s should be "sufficiently large." We are simply saying that there is an absolute constant $\bar{K}$ such that the theorem requires $s \geq \bar{K}$. We have not kept careful track of the minimum such value of $\bar{K}$, but a dedicated reader could deduce such a value from the proof. Certainly, $s \geq 5$ suffices to justify all the calculations that follow.

3. **Energy estimate.** Rewrite (18) using the fact that the operator $|D|$ can be written in the form

$$|D| = H\partial_x$$

where H connotes the Hilbert transform whose Fourier symbol is $\widehat{H} = -i\mathrm{sgn}(k)$. Thus, H is skew-adjoint, commutes with differentiation and, for any f with mean zero, $H^2 f = -f$. Because of these facts, (18a) can be rewritten in the form

$$\partial_t\eta = H\partial_x\xi + 2\nu\partial_x^2\eta - \eta\partial_x^2\xi - H[(\partial_x\eta)H\partial_x\xi] - H[\eta H[\partial_x^2\xi]] - (\partial_x\eta)\partial_x\xi. \tag{20}$$

Similarly, equation (18b) is equivalent to

$$\begin{aligned}\partial_t\xi = -g\eta + 2\nu\partial_x^2\xi + 2\nu\eta H[\partial_x^3\xi] - 2\nu\partial_x^2[\eta H[\partial_x\xi]] + \frac{1}{2}(H[\partial_x\xi])^2&\\ - \frac{1}{2}(\partial_x\xi)^2 + 2\nu(\partial_x^2\eta)(H[\partial_x\xi]).&\end{aligned} \tag{21}$$

This simplifies to

$$\partial_t \xi = -g\eta + 2\nu \partial_x^2 \xi - 4\nu(\partial_x \eta)(H[\partial_x^2 \xi]) + \frac{1}{2}(H[\partial_x \xi])^2 - \frac{1}{2}(\partial_x \xi)^2 \tag{22}$$

due to cancelations that occur when the term $2\nu\partial_x^2[\eta H[\partial_x \xi]]$ is worked out in detail.

We proceed now to obtain *a priori* estimates of solutions to the system (20) and (22) in the Sobolev space H^s, where s is a sufficiently large integer. It is worth noting that the upcoming energy-type inequalities apply simultaneously to η and ξ in the *same* Sobolev class. This is quite different from the situation that arises in the absence of viscosity.

The norm of a function u will be written $\|u\|_{H^s}$ or $\|u\|_{L^p}$ with the period domain X suppressed. For convenience, we will also use the notation $\|u\|_s$ for the Sobolev norm $\|u\|_{H^s}$ and $|u|_p$ for the Lebesgue norm $\|u\|_{L^p}$.

3.1. **Higher derivatives and commutators.** The calculations start by rewriting terms in the evolution equation (20) in the form

$$[A, B] = AB - BA,$$

of commutators of linear operators A and B. Notice that the final four terms in (20) are precisely the first order (in η) term in the expansion of the Dirichlet-to-Neumann operator [5] for water waves on an ocean of infinite depth, namely

$$G_1(\eta)[\xi] = -\eta \partial_x^2 \xi - H[(\partial_x \eta) H \partial_x \xi] - H[\eta H[\partial_x^2 \xi]] - (\partial_x \eta)\partial_x \xi.$$

This operator has well–known mapping properties (see, e.g., [14, 15]). Guided by what is known about this operator, we anticipate the following simplifications. Set $\zeta := \partial_x \xi$ and combine the first and last terms to reach the alternate formula

$$G_1(\eta)[\xi] = -\partial_x [\eta\zeta] - H(\partial_x \eta) H[\zeta] + \eta H[\partial_x \zeta]$$

for G_1. If the operators H and ∂_x are interchanged in the last term, one recognizes that G_1 can be further rewritten as

$$G_1(\eta)[\xi] = -\partial_x [\eta\zeta] - H\partial_x [\eta H[\zeta]] = -\partial_x \{\eta\zeta + H\eta H[\zeta]\}.$$

Introducing the commutator $[H, \eta]$, the final term may be expanded, thereby obtaining

$$\begin{aligned} G_1(\eta)[\xi] &= -\partial_x \left\{(\eta)\zeta + (\eta)H^2[\zeta] + [H,\eta](H[\zeta])\right\} \\ &= -\partial_x \{[H,\eta](H[\zeta])\}, \end{aligned}$$

where use has been made of the fact that $H^2\partial_x = -\partial_x$. Recalling the definition of ζ, we realize that the evolution equation (20) for η can be simplified to

$$\partial_t \eta = H\partial_x \xi + 2\nu\partial_x^2 \eta - \partial_x \{[H,\eta](H[\partial_x \xi])\}. \tag{23}$$

For a positive integer s, apply ∂_x^s to (23) to obtain the formula

$$\partial_t \partial_x^s \eta = H\partial_x^{s+1}\xi + 2\nu\partial_x^{s+2}\eta - \partial_x^{s+1}\{[H,\eta](H[\partial_x \xi])\}. \tag{24}$$

Similarly, apply ∂_x^s to (22) and use the product rule to derive the relation

$$\begin{aligned}\partial_t\partial_x^s\xi = &-g\partial_x^s\eta + 2\nu\partial_x^{s+2}\xi - 4\nu(\partial_x^{s+1}\eta)(H[\partial_x^2\xi]) - 4\nu(\partial_x\eta)(H[\partial_x^{s+2}\xi])\\ &- 4\nu s(\partial_x^2\eta)(H[\partial_x^{s+1}\xi]) - 4\nu\sum_{k=2}^{s-1}\binom{s}{k}(\partial_x^{k+1}\eta)(H[\partial_x^{s-k+2}\xi])\\ &+ (H[\partial_x^{s+1}\xi])(H[\partial_x\xi]) + \frac{1}{2}\sum_{k=1}^{s-1}\binom{s}{k}(H[\partial_x^{k+1}[\xi]])(H[\partial_x^{s-k+1}[\xi]])\\ &- (\partial_x^{s+1}\xi)(\partial_x\xi) - \frac{1}{2}\sum_{k=1}^{s-1}\binom{s}{k}(\partial_x^{k+1}\xi)(\partial_x^{s-k+1}\xi). \quad (25)\end{aligned}$$

If Φ is defined by

$$\begin{aligned}\Phi := -4\nu\sum_{k=2}^{s-1}\binom{s}{k}(\partial_x^{k+1}\eta)(H[\partial_x^{s-k+2}\xi]) + \frac{1}{2}\sum_{k=1}^{s-1}\binom{s}{k}(H[\partial_x^{k+1}\xi])H[\partial_x^{s-k+1}\xi]\\ - \frac{1}{2}\sum_{k=1}^{s-1}\binom{s}{k}(\partial_x^{k+1}\xi)(\partial_x^{s-k+1}\xi),\end{aligned}$$

then

$$\begin{aligned}\partial_t\partial_x^s\xi = &-g\partial_x^s\eta + 2\nu\partial_x^{s+2}\xi - 4\nu(\partial_x^{s+1}\eta)(H[\partial_x^2\xi]) - 4\nu(\partial_x\eta)(H[\partial_x^{s+2}\xi])\\ &- 4\nu s(\partial_x^2\eta)(H[\partial_x^{s+1}\xi]) + (H[\partial_x^{s+1}\xi])(H[\partial_x\xi]) - (\partial_x^{s+1}\xi)(\partial_x\xi) + \Phi. \quad (26)\end{aligned}$$

Attention is now turned to providing upper bounds on norms of G_1 and Φ. First, the term Φ is a collection of products of derivatives of η and ξ and their Hilbert transforms. Of course, for any real s, $\|Hf\|_{H^s} \le \|f\|_{H^s}$, with equality if the mean of f is zero. Also, notice that the highest derivative that appears in Φ is order s and, furthermore, it is never the case that there is a product featuring both $\partial_x^s\eta$ and $\partial_x^s\xi$, i.e., it is never the case that the highest number of derivatives occurs simultaneously on both factors. Therefore, assuming η and ξ are both in H^s, at least one of the factors in every summand lies in L^∞. It is therefore routine to derive the inequality

$$\|\Phi\|_{H^0} \le c\left(\|\eta\|_{H^s}\|\xi\|_{H^s} + \|\xi\|_{H^s}^2\right).$$

To bound G_1, as well as another commutator that will appear shortly, the following commutator estimate is helpful (c.f. [2]).

Lemma 3.1. *For any integer $s \ge 1$, if $\psi \in H^s$, then the operator $[H,\psi]$ is bounded from H^0 to H^{s-1}. If $s \ge 2$ then $[H,\psi]$ is bounded from H^{-1} to H^{s-2}. Moreover, in these cases, respectively, there are constants c_j such that*

$$\|[H,\psi]g\|_{H^{s-1+j}} \le c_j\|\psi\|_{H^s}\|g\|_{H^j},$$

for $j = 0, -1$. Furthermore, if $s \ge 3$, the same operator $[H,\psi]$ is bounded from H^{s-2} to H^s, with the corresponding estimate

$$\|[H,\psi]g\|_{H^s} \le c\|\psi\|_{H^s}\|g\|_{H^{s-2}}.$$

3.2. **The energy estimate.** For a fixed integer s, define the "energy" to be $E(t) = E_0(t) + E_s(t)$, where

$$E_k(t) = E_k^\eta(t) + E_k^\xi(t) := \frac{1}{2}\int_X (\partial_x^k \eta)^2 \, dx + \frac{1}{2}\int_X (\partial_x^k \xi)^2 \, dx$$

for $k = 0, 1, \cdots$. As before, X denotes the period cell $[0, 2\pi]$. For each fixed $t, E(t)$ is equivalent to the square of the $H^s \times H^s$-norm of $(\eta(\cdot,t), \xi(\cdot,t))$.

The aim now is to get control of the growth of E as a function of time. The strategy is to derive differential inequalities that the energy must respect and then apply a Gronwall-type argument. In working out the details, Young's Inequality

$$ab \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}, \tag{27}$$

valid for any real numbers a and b and any positive value of ε, will find frequent use.

With s chosen sufficiently large, the following differential inequality holds for E_0. By sufficiently large, we mean at the outset that the functions should have enough derivatives in L^2 to justify the formal calculations to follow, including being able to ignore boundary contributions on the basis of periodicity. The smoothness restriction can be toned down later after a suitable continuous dependence result is in hand.

Lemma 3.2. *There is a time-independent constant c such that as long as a smooth solution (η, ξ) of (18) exists, then*

$$\frac{dE_0}{dt} \le c(E + E^{3/2}). \tag{28}$$

Proof. Taking the time derivative of E_0 yields

$$\frac{dE_0}{dt} = \frac{dE_0^\eta}{dt} + \frac{dE_0^\xi}{dt} = \int_X \eta\partial_t\eta + \xi\partial_t\xi \, dx.$$

Using the equations satisfied by η and ξ leads to the formulas

$$\frac{dE_0^\eta}{dt} = \int_X \eta \left\{ H\partial_x\xi + 2\nu\partial_x^2\eta - \partial_x \left\{ [H,\eta](H[\partial_x\xi]) \right\} \right\} \, dx$$

and

$$\frac{dE_0^\xi}{dt} = \int_X \xi \left\{ -g\eta + 2\nu\partial_x^2\xi - 4\nu(\partial_x\eta)(H\partial_x^2\xi) + \frac{1}{2}(H\partial_x\xi)^2 - \frac{1}{2}(\partial_x\xi)^2 \right\} \, dx.$$

The latter two integrals are estimated using Hölder's inequality and the facts that, in one dimension and for $s > \frac{1}{2}$, H^s is embedded in L^∞ and H^s is an algebra, which is to say, there are universal constants such that

$$|f|_{L^\infty} \le c\,\|f\|_{H^s} \le c(E_0^f + E_s^f)^{\frac{1}{2}},$$
$$\|fg\|_{H^s} \le c\|f\|_{H^s}\|g\|_{H^s} \le c[(E_0^f + E_s^f)(E_0^g + E_s^g)]^{\frac{1}{2}}.$$

Turning to the terms comprising the integral equal to $\frac{dE_0^\eta}{dt}$, the first is bounded using Hölder's inequality while the second falls to an integration by parts, *viz.*

$$\int_X \eta H[\partial_x\xi] \, dx \le \|\eta\|_0\|\xi\|_1 \le E_0^{1/2}E_1^{1/2} \le E,$$

$$\int_X \eta(2\nu\partial_x^2\eta) \, dx = -2\nu\int_X (\partial_x\eta)^2 \, dx \le 0.$$

For the last term in $\frac{dE_0^\eta}{dt}$, write out the commutator in the form

$$-\partial_x \{[H,\eta](H[\partial_x\xi])\} = -\partial_x H\left[\eta H\left[\partial_x\xi\right]\right] - \partial_x(\eta\partial_x\xi)$$

and bound the contribution from the first term by using Hölder's inequality, the fact that H^s is an algebra and the unitary property $\|Hf\|_s \le \|f\|_s$ of the Hilbert transform. The upshot is the string of inequalities

$$\begin{aligned}\int_X \eta(\partial_x H[\eta H[\partial_x\xi]])\ dx \le& \|\eta\|_0\|H[\eta H[\partial_x\xi]]\|_1 \le c\|\eta\|_0\|\eta\|_1\|\xi\|_2\\ \le& cE_0^{\frac{1}{2}}E_1^{\frac{1}{2}}E_2^{\frac{1}{2}} \le cE^{\frac{3}{2}}.\end{aligned}$$

Similarly, but more simply,

$$\int_X \eta\partial_x(\eta\partial_x\xi)\ dx \le \|\eta\|_0\,\|\eta\partial_x\xi\|_1 \le \|\eta\|_0\,\|\eta\|_1\,\|\xi\|_2 \le cE^{\frac{3}{2}}.$$

As long as $s \ge 2$, it is thus concluded that

$$\frac{dE_0^\eta}{dt} \le c(E + E^{\frac{3}{2}}).$$

The same line of argument applies to $\frac{dE_0^\xi}{dt}$ with the conclusion that if $s \ge 2$, then

$$\frac{dE_0^\xi}{dt} \le c(E + E^{\frac{3}{2}}).$$

□

Lemma 3.3. *There is a time-independent constant c such that as long as a smooth solution (η,ξ) of (18) exists, then for s sufficiently large,*

$$\frac{dE_s}{dt} \le c(E + E^2). \tag{29}$$

Proof. As in Lemma 3.2, begin by computing the left-hand side of the last inequality;

$$\begin{aligned}\frac{dE_s^\eta}{dt} &= \int_X (\partial_x^s\eta)\partial_t\partial_x^s\eta\ dx\\ &= \int_X (\partial_x^s\eta)(H\partial_x^{s+1}\xi) + (\partial_x^s\eta)(2\nu\partial_x^{s+2}\eta) - (\partial_x^s\eta)(\partial_x^{s+1}\{[H,\eta](H\partial_x\xi)\})\ dx.\end{aligned}$$

Apply Young's Inequality to the first and third term on the right-hand side of the last equation and integrate the middle term by parts, leading to the inequality

$$\begin{aligned}\frac{dE_s^\eta}{dt} \le& \frac{\sigma_1}{2}\int_X(\partial_x^s\eta)^2\ dx + \frac{1}{2\sigma_1}\int_X(H\partial_x^{s+1}\xi)^2\ dx - 2\nu\int_X(\partial_x^{s+1}\eta)^2\ dx\\ &+ \frac{\sigma_2}{2}\int_X(\partial_x^s\eta)^2\ dx + \frac{1}{2\sigma_2}\int_X\left(\partial_x^{s+1}\{[H,\eta](H[\partial_x\xi])\}\right)^2\ dx,\end{aligned}$$

where the positive values of the functions $\sigma_i = \sigma_i(t)$ will be determined presently. The last term in the previous display is bounded thusly;

$$\int_X\left(\partial_x^{s+1}\{[H,\eta](H[\partial_x\xi])\}\right)^2\ dx \le \|[H,\eta](H[\partial_x\xi])\|_{H^{s+1}}^2 \le c\,\|\eta\|_{H^{s+1}}^2\,\|H\partial_x\xi\|_{H^{s-1}}^2$$

where we have used Lemma 3.1 for the final estimate. Since the H^{s+1}-norm is equivalent to the sum of the L^2 norms of the function and its $(s+1)^{st}$ derivative, another application of Young's Inequality yields

$$\begin{aligned}\int_X \left(\partial_x^{s+1}\left\{[H,\eta](H[\partial_x\xi])\right\}\right)^2\,dx &\le \tilde{c}\left(\|\eta\|_{H^0}^2+\left\|\partial_x^{s+1}\eta\right\|_{H^0}^2\right)\|\xi\|_{H^s}^2\\ &\le \tilde{c}\left(E_0+\int_X(\partial_x^{s+1}\eta)^2\,dx\right)E.\end{aligned}$$

Putting the above estimates together yields

$$\begin{aligned}\frac{dE_s^\eta}{dt} \le\ & \frac{\sigma_1}{2}E_s+\frac{1}{2\sigma_1}\int_X(\partial_x^{s+1}\xi)^2\,dx-2\nu\int_X(\partial_x^{s+1}\eta)^2\,dx\\ &+\frac{\sigma_2}{2}E_s+\frac{\tilde{c}}{2\sigma_2}\left(E_0+\int_X(\partial_x^{s+1}\eta)^2\,dx\right)E. \end{aligned}\tag{30}$$

Attention is now given to estimating the growth in time of $E_s^\xi(t)$. Proceeding as before and making routine estimates leads to

$$\begin{aligned}\frac{dE_s^\xi}{dt} &= \int_X(\partial_x^s\xi)\partial_t\partial_x^s\xi\,dx\\ &\le \frac{1}{2}\int_X(\partial_x^s\xi)^2\,dx+\frac{g^2}{2}\int_X(\partial_x^s\eta)^2\,dx-2\nu\int_X(\partial_x^{s+1}\xi)^2\,dx\\ &\quad+4\nu\left|H[\partial_x^2\xi]\right|_{L^\infty}\left(\frac{\sigma_3}{2}\int_X(\partial_x^s\xi)^2\,dx+\frac{1}{2\sigma_3}\int_X(\partial_x^{s+1}\eta)^2\,dx\right)\\ &\quad+I+4\nu s\left|\partial_x^2\eta\right|_{L^\infty}\left(\frac{\sigma_5}{2}\int_X(\partial_x^s\xi)^2\,dx+\frac{1}{2\sigma_5}\int_X(H[\partial_x^{s+1}\xi])^2\,dx\right)\\ &\quad+\left|H[\partial_x\xi]\right|_{L^\infty}\left(\frac{\sigma_6}{2}\int_X(\partial_x^s\xi)^2\,dx+\frac{1}{2\sigma_6}\int_X(H[\partial_x^{s+1}\xi])^2\,dx\right)\\ &\quad+\left|\partial_x\xi\right|_{L^\infty}\left(\frac{\sigma_7}{2}\int_X(\partial_x^s\xi)^2\,dx+\frac{1}{2\sigma_7}\int_X(\partial_x^{s+1}\xi)^2\,dx\right)\\ &\quad+\frac{1}{2}\int_X(\partial_x^s\xi)^2\,dx+\frac{1}{2}\int_X\Phi^2\,dx. \end{aligned}\tag{31}$$

The most challenging term appears to be

$$I := -4\nu\int_X(\partial_x^s\xi)(\partial_x\eta)(H[\partial_x^{s+2}\xi])\,dx.$$

To obtain control of this term, begin by interchanging a derivative with the Hilbert transform and integrating by parts to reach the inequality

$$\begin{aligned}I &= 4\nu\int_X\partial_x\left\{(\partial_x^s\xi)(\partial_x\eta)\right\}(H[\partial_x^{s+1}\xi])\,dx\\ &= 4\nu\int_X(\partial_x^s\xi)(\partial_x^2\eta)(H[\partial_x^{s+1}\xi])\,dx+4\nu\int_X(\partial_x^{s+1}\xi)(\partial_x\eta)(H[\partial_x^{s+1}\xi])\,dx\\ &\le 4\nu\left|\partial_x^2\eta\right|_{L^\infty}\left(\frac{\sigma_4}{2}\int_X(\partial_x^s\xi)^2\,dx+\frac{1}{2\sigma_4}\int_X(H[\partial_x^{s+1}\xi])^2\,dx\right)+I'\\ &\le 4\nu E_3^{1/2}\left(\frac{\sigma_4}{2}E_s+\frac{1}{2\sigma_4}\int_X(H[\partial_x^{s+1}\xi])^2\,dx\right)+I', \end{aligned}\tag{32}$$

where
$$I' = 4\nu \int_X (\partial_x^{s+1}\xi)(\partial_x\eta)(H[\partial_x^{s+1}\xi])\, dx.$$

The term I' requires further rewriting. Using the skew–adjointness of the Hilbert transform, introducing a commutator and integrating by parts leads to the formula
$$\begin{aligned}
I' &= -4\nu \int_X H\left[(\partial_x^{s+1}\xi)(\partial_x\eta)\right](\partial_x^{s+1}\xi)\, dx \\
&= -4\nu \int_X (\partial_x\eta)H\left[(\partial_x^{s+1}\xi)\right](\partial_x^{s+1}\xi)\, dx - 4\nu \int_X [H, \partial_x\eta]\left((\partial_x^{s+1}\xi)\right)(\partial_x^{s+1}\xi)\, dx \\
&= -I' + 4\nu \int_X \partial_x\left\{[H, \partial_x\eta]\left((\partial_x^{s+1}\xi)\right)\right\}(\partial_x^s\xi)\, dx.
\end{aligned}$$

Solving for I' and estimating further yields
$$\begin{aligned}
I' &= 2\nu \int_X \partial_x\left\{[H, \partial_x\eta]\left((\partial_x^{s+1}\xi)\right)\right\}(\partial_x^s\xi)\, dx \\
&\leq 2\nu\left\{\frac{1}{2}\int_X (\partial_x^s\xi)^2\, dx + \frac{1}{2}\int_X \left(\partial_x\left\{[H, \partial_x\eta]\left((\partial_x^{s+1}\xi)\right)\right\}\right)^2\, dx\right\} \\
&\leq 2\nu\left\{\frac{1}{2}E_s + \frac{1}{2}\left\|[H, \partial_x\eta]\left((\partial_x^{s+1}\xi)\right)\right\|_{H^1}^2\right\} \\
&\leq 2\nu\left\{\frac{1}{2}E_s + \frac{1}{2}c\,\|\eta\|_{H^s}^2\,\|\xi\|_{H^s}^2\right\} \\
&\leq 2\nu\left\{\frac{1}{2}E_s + \frac{1}{2}cE_s^2\right\}. \qquad (33)
\end{aligned}$$

Assembling (30), (31), (32) and (33), and using the simple inequality $E_s \leq E$, provides the differential inequality
$$\begin{aligned}
\frac{dE_s}{dt} \leq &\left(\frac{\sigma_1}{2} + \frac{\sigma_2}{2} + \frac{\tilde{c}}{2\sigma_2}E_0 + \frac{1}{2} + \frac{g^2}{2} + 4\nu E_3^{1/2}\frac{\sigma_3}{2} + 4\nu E_3^{1/2}\frac{\sigma_4}{2} + \nu + \nu c E_s\right. \\
&\quad\left. + 4\nu s E_3^{1/2}\frac{\sigma_5}{2} + E_2^{1/2}\frac{\sigma_6}{2} + E_2^{1/2}\frac{\sigma_7}{2} + \frac{1}{2} + \frac{c}{2}E_s\right)E \\
&+ \left(\frac{1}{2\sigma_1} - 2\nu + 4\nu E_3^{1/2}\frac{1}{2\sigma_4} + 4\nu s E_3^{1/2}\frac{1}{2\sigma_5} + E_2^{1/2}\frac{1}{2\sigma_6}\right. \\
&\quad\left. + E_2^{1/2}\frac{1}{2\sigma_7}\right)\int_X (\partial_x^{s+1}\xi)^2\, dx \\
&+ \left(-2\nu + \frac{\tilde{c}}{2\sigma_2}E + 4\nu E_3^{1/2}\frac{1}{2\sigma_3}\right)\int_X (\partial_x^{s+1}\eta)^2\, dx.
\end{aligned}$$

We choose the functions σ_i to be
$$\sigma_1 = \frac{5}{8\nu}, \quad \sigma_4 = \frac{5E_3^{1/2}}{2}, \quad \sigma_5 = \frac{5sE_3^{1/2}}{2}, \quad \sigma_6 = \frac{5E_2^{1/2}}{8\nu}, \quad \sigma_7 = \frac{5E_2^{1/2}}{8\nu},$$
and
$$\sigma_2 = \frac{\tilde{c}E}{4\nu}, \quad \sigma_3 = E_3^{1/2}.$$

These choices, together with the fact that $E_k \leq cE$ for all $0 \leq k \leq s$, lead to the further inequality
$$\frac{dE_s}{dt} \leq c(E + E^2) - \nu \int_X (\partial_x^{s+1}\eta)^2 + (\partial_x^{s+1}\xi)^2\, dx.$$

As the integrand in the integral is non-negative, the desired result

$$\frac{dE_s}{dt} \leq \frac{dE_s}{dt} + \nu \int_X (\partial_x^{s+1}\eta)^2 + (\partial_x^{s+1}\xi)^2 \; dx \leq c(E + E^2)$$

follows. $\square$

Remark 2. Lemma 3.2 and Lemma 3.3 combine to yield

$$\frac{dE}{dt} \leq c(E + E^2).$$

This differential inequality implies the upper bound

$$E(t) \leq \frac{E(0)e^{ct}}{1 + E(0)(1 - e^{ct})}$$

as long as $e^{ct} \leq 1$.

Remark 3. While the last results provide a bound on the growth of E in terms of E, a more precise result is available via the preceding arguments, namely

$$\frac{d}{dt}\|(\eta,\xi)\|_{H^s} + \nu \int_X (\partial_x^{s+1}\eta)^2 + (\partial_x^{s+1}\xi)^2 \; dx \leq F(\|(\eta,\xi)\|_{H^{s-1}})\|(\eta,\xi)\|_{H^s},$$

for some continuous function F. The energy inequality takes this form simply because, after any equation is differentiated, the highest derivatives appear linearly, as a consequence of the chain rule. Furthermore, integrating with respect to time, one obtains control of the solution in $L^2([0,T]; H^{s+1})$; this is the typical smoothing one would expect from these sort of damping terms.

4. **Existence for regularized equations.** To proceed from the energy estimate above to a well–posedness proof, regularize the governing equations so that a (regularized) solution is straightforwardly adduced. We use Friedrichs' mollifiers as our regularization mechanism and find existence via the classical Picard theorem for ordinary differential equations (the introduction of mollifiers will have the effect of transforming all the differential operators into bounded operators and local well-posedness then follows from a Picard iteration). Consider the evolution equation (22)

$$\partial_t\xi = -g\eta + 2\nu\partial_x^2\xi - 4\nu(\partial_x\eta)(H[\partial_x^2\xi]) + \frac{1}{2}(H[\partial_x\xi])^2 - \frac{1}{2}(\partial_x\xi)^2,$$

rewritten here for convenience, and denote by $\mathcal{S}_\varepsilon$ a smoothing operator which approaches the identity as $\varepsilon \to 0$. Various options are available; perhaps the simplest is just the truncation

$$\mathcal{S}_\varepsilon[f(x)] = \mathcal{S}_\varepsilon\left[\sum_{p=-\infty}^{\infty} \hat{f}_p e^{ipx}\right] := \sum_{|p|<1/\varepsilon} \hat{f}_p e^{ipx},$$

of a function's Fourier series. Clearly, this choice comprises a non-negative definite, self-adjoint operator with $\mathcal{S}_\varepsilon : H^s \to H^\infty$, for any s, where $H^\infty = \bigcap_{k\geq 0} H^k$. The operator $\mathcal{S}_\varepsilon$ is introduced into (22) as follows:

$$\partial_t\xi = -g\eta + 2\nu\mathcal{S}_\varepsilon^2\partial_x^2\xi - \mathcal{S}_\varepsilon\left[4\nu(\mathcal{S}_\varepsilon[\partial_x\eta])(\mathcal{S}_\varepsilon[H[\partial_x^2\xi]])\right] + \frac{1}{2}\mathcal{S}_\varepsilon\left[(\mathcal{S}_\varepsilon[H[\partial_x\xi]])^2\right] - \frac{1}{2}\mathcal{S}_\varepsilon\left[(\mathcal{S}_\varepsilon[\partial_x\xi])^2\right]. \quad (34)$$

The smoothing operator $\mathcal{S}_\varepsilon$ is similarly introduced into the evolution equation (23) for η, *viz.*

$$\partial_t \eta = H\mathcal{S}_\varepsilon \partial_x \xi + 2\nu \mathcal{S}_\varepsilon^2 \partial_x^2 \eta - \mathcal{S}_\varepsilon \partial_x \left\{ [H, \mathcal{S}_\varepsilon \eta](H[\mathcal{S}_\varepsilon \partial_x \xi]) \right\}. \tag{35}$$

The upshot of the introduction of the smoothing operators is that the system (34)–(35) has solutions corresponding to initial data in $H^s \times H^s$ which exist on a time interval $[0, T_\varepsilon)$. Denote these solutions by $\eta_\varepsilon(x,t)$ and $\xi_\varepsilon(x,t)$. We use the Continuation Theorem for Autonomous ODEs (see, for instance, Theorem 3.3 of [12]) to show that the time of existence T_ε is bounded below by a positive constant, independently of $\varepsilon > 0$. To establish this latter assertion, it suffices to show that the solutions of the approximate equations cannot blow up immediately. For this, appropriately modified versions of the foregoing energy estimates are called upon.

Generalizing the energy estimate to the regularized equations is a tedious, but relatively straightforward matter. This owes in part to the way the smoothing has been introduced. The energy E is defined just as before:

$$E(t) = E_0(t) + E_s(t), \qquad E_0(t) = \|\eta_\varepsilon\|_{L^2}^2 + \|\xi_\varepsilon\|_{L^2}^2, \qquad E_s(t) = \|\partial_x^s \eta_\varepsilon\|_{L^2}^2 + \|\partial_x^s \xi_\varepsilon\|_{L^2}^2.$$

Lemma 4.1. *For $s \geq 2$, there is a constant c independent of $\varepsilon \in (0,1]$ and $t \geq 0$ such that if $(\eta_\varepsilon, \xi_\varepsilon)$ solves the system (34)–(35), then over its time interval of existence,*

$$\frac{dE}{dt} \leq c(E + E^2).$$

Proof. As before, we take the time derivative of E_s, starting with $E_s^\eta = \|\partial_x^s \eta_\varepsilon\|_{L^2}^2$; we find the formula

$$\begin{aligned} \frac{dE_s^\eta}{dt} &= \int_X (\partial_x^s \eta_\varepsilon) \partial_t \partial_x^s \eta_\varepsilon \, dx \\ &= \int_X (\partial_x^s \eta_\varepsilon)(H\mathcal{S}_\varepsilon \partial_x^{s+1} \xi_\varepsilon) + (\partial_x^s \eta_\varepsilon)(2\nu \mathcal{S}_\varepsilon^2 \partial_x^{s+2} \eta_\varepsilon) \\ &\quad - (\partial_x^s \eta_\varepsilon)(\partial_x^{s+1} \mathcal{S}_\varepsilon \left\{ [H, \mathcal{S}_\varepsilon \eta_\varepsilon](H\mathcal{S}_\varepsilon \partial_x \xi_\varepsilon) \right\}) \, dx. \end{aligned}$$

Young's Inequality is applied to the first and third terms and the middle term is integrated by parts (using that $\mathcal{S}_\varepsilon$ is self-adjoint) to reach the inequality

$$\begin{aligned} \frac{dE_s^\eta}{dt} \leq{} & \frac{\sigma_1}{2} \int_X (\partial_x^s \eta_\varepsilon)^2 \, dx + \frac{1}{2\sigma_1} \int_X (H\mathcal{S}_\varepsilon \partial_x^{s+1} \xi_\varepsilon)^2 \, dx - 2\nu \int_X (\mathcal{S}_\varepsilon \partial_x^{s+1} \eta_\varepsilon)^2 \, dx \\ & + \frac{\sigma_2}{2} \int_X (\partial_x^s \eta_\varepsilon)^2 \, dx + \frac{1}{2\sigma_2} \int_X \left(\partial_x^{s+1} \left\{ [H, \mathcal{S}_\varepsilon \eta_\varepsilon](H[\partial_x \mathcal{S}_\varepsilon \xi_\varepsilon]) \right\} \right)^2 \, dx. \end{aligned}$$

Estimating the commutator in exactly the same way as before, it is found that

$$\begin{aligned} \frac{dE_s^\eta}{dt} \leq{} & \frac{\sigma_1}{2} E_s + \frac{1}{2\sigma_1} \int_X (\mathcal{S}_\varepsilon \partial_x^{s+1} \xi_\varepsilon)^2 \, dx - 2\nu \int_X (\mathcal{S}_\varepsilon \partial_x^{s+1} \eta_\varepsilon)^2 \, dx \\ & + \frac{\sigma_2}{2} E_s + \frac{\tilde{c}}{2\sigma_2} \left(E_0 + \int_X (\mathcal{S}_\varepsilon \partial_x^{s+1} \eta_\varepsilon)^2 \, dx \right) E_s. \end{aligned} \tag{36}$$

The various σ_i will be chosen in the same way as for the non-regularized case. As before, these choices are organized to cancel the contributions from terms involving derivatives of order $s+1$ using the helpful contribution to the energy coming from the viscous terms. As the details are largely the same as what has gone before in the earlier formal calculations, partly because of the placement of the smoothing operators in the regularization, we content ourselves with examining one interesting, but representative calculation, namely the final term on the right–hand side of (34). Define R by

$$R := -\frac{1}{2}\mathcal{S}_\varepsilon\left[(\mathcal{S}_\varepsilon[\partial_x\xi])^2\right],$$

where we have denoted ξ_ε by ξ. Differentiate R s-many times with respect to x to obtain

$$\begin{aligned}\partial_x^s R &= -\frac{1}{2}\sum_{j=0}^{s}\binom{s}{j}\mathcal{S}_\varepsilon\left[\left(\partial_x^j\mathcal{S}_\varepsilon\partial_x\xi\right)\left(\partial_x^{s-j}\mathcal{S}_\varepsilon\partial_x\xi\right)\right]\\ &= R_s - \frac{1}{2}\sum_{j=1}^{s-1}\binom{s}{j}\mathcal{S}_\varepsilon\left[\left(\partial_x^j\mathcal{S}_\varepsilon\partial_x\xi\right)\left(\partial_x^{s-j}\mathcal{S}_\varepsilon\partial_x\xi\right)\right],\end{aligned}$$

which has leading-order behavior

$$R_s = -\mathcal{S}_\varepsilon\left[(\mathcal{S}_\varepsilon[\partial_x\xi])(\mathcal{S}_\varepsilon[\partial_x^{s+1}\xi])\right].$$

Multiply the result by $\partial_x^s\xi$ and integrate over X. There obtains at leading order the formula

$$\int_X(\partial_x^s\xi)R_s\,dx = -\int_X(\partial_x^s\xi)\mathcal{S}_\varepsilon\left[(\mathcal{S}_\varepsilon[\partial_x\xi])(\mathcal{S}_\varepsilon[\partial_x^{s+1}\xi])\right]\,dx.$$

Using the fact that $\mathcal{S}_\varepsilon$ is self–adjoint, it is deduced that

$$\int_X(\partial_x^s\xi)R_s\,dx = -\int_X \mathcal{S}_\varepsilon[\partial_x^s\xi](\mathcal{S}_\varepsilon[\partial_x\xi])(\mathcal{S}_\varepsilon[\partial_x^{s+1}\xi])\,dx,$$

or, what is the same,

$$\int_X(\partial_x^s\xi)R_s\,dx = -\frac{1}{2}\int_X\left\{\partial_x[(\mathcal{S}_\varepsilon\partial_x^s\xi)^2]\right\}(\mathcal{S}_\varepsilon[\partial_x\xi])\,dx.$$

An integration by parts yields

$$\int_X(\partial_x^s\xi)R_s\,dx = \frac{1}{2}\int_X\left\{(\mathcal{S}_\varepsilon[\partial_x^s\xi])^2\right\}(\mathcal{S}_\varepsilon[\partial_x^2\xi])\,dx,$$

and thus

$$\int_X(\partial_x^s\xi)R_s\,dx \le c\left|\mathcal{S}_\varepsilon[\partial_x^2\xi]\right|_{L^\infty}\int_X(\mathcal{S}_\varepsilon[\partial_x^s\xi])^2\,dx \le C\left\|\mathcal{S}_\varepsilon[\xi]\right\|_{H^s}^3$$

provided s is large enough. Standard mollifier estimates bound this term by a constant multiplied by $\|\xi\|_{H^s}^3$ which is, in turn, bounded by a constant multiple of $E^{3/2}$. □

Remark 4. As before, the solution of this differential inequality provides a uniform bound on the solution (independent of ε) over a time interval which is independent of ε.

5. **Proof of Theorem 2.1.** We are now prepared to prove the main theorem. This is done in stages, first by establishing that the approximate solutions $(\eta_\varepsilon, \xi_\varepsilon)$ form a Cauchy sequence in the space $C([0,T]; H^{s'}(X))$, for any s' with $0 \le s' < s$. We then establish that the limit solves the initial-value problem. Next, uniqueness and continuous dependence of solutions on variations in the initial data is established. Finally, it will be shown that the limit is in fact in $C([0,T]; H^s(X))$.

5.1. **Limit as the mollifying parameter vanishes.** We will frequently use the mollifier estimate

$$\|\mathcal{S}_\varepsilon f\|_{H^r} \le c\,\|f\|_{H^r}\,,$$

where c depends on r, but not on ε. In fact, if the truncation operator defined above is used for a smoothing operator, then we may take $c = 1$ for all r.

From the Continuation Theorem [12], we know that the solutions $(\eta_\varepsilon, \xi_\varepsilon)$ of the regularized problem exist on a common time interval $[0, T]$, independent of ε, and satisfy the estimate

$$\|\eta_\varepsilon(\cdot,t)\|^2_{H^s} + \|\xi_\varepsilon(\cdot,t)\|^2_{H^s} \le K$$

for $t \in [0, T]$ and for some ε-independent constant K.

The next stage is to show that the solutions $(\eta_\varepsilon, \xi_\varepsilon)$ are Cauchy in the function class $C([0, T]; L^2 \times L^2)$ in the limit as $\varepsilon \to 0$, and thereby identify a limit (η, ξ). The functional

$$E_d(t) = \frac{1}{2}\int_X (\eta_\varepsilon - \eta_{\varepsilon'})^2 + (\xi_\varepsilon - \xi_{\varepsilon'})^2 \, dx = E_{d,\eta}(t) + E_{d,\xi}(t).$$

is used in this endeavor. Since the initial data is the same for all values of the regularization parameter ε, $E_d(0) = 0$.

Lemma 5.1. *There are constants c, independent of $t \in [0, T]$, and values of ε and ε' in $(0, 1]$, say, such that*

$$\frac{dE_d}{dt} \le cE_d + c\max\{\varepsilon, \varepsilon'\}E_d^{1/2}.$$

Proof. Differentiate E_d with respect to t, starting with $E_{d,\xi}$, to arrive at the formula

$$\frac{dE_{d,\xi}}{dt} = \int_X (\xi_\varepsilon - \xi_{\varepsilon'})(\partial_t\xi_\varepsilon - \partial_t\xi_{\varepsilon'})\, dx = J_1 + J_2 + J_3 + J_4 + J_5,$$

where each of these corresponds to one of the five terms on the right–hand side of the evolution equation (34).

We set about estimating these terms. The Cauchy-Schwarz inequality suffices for J_1, *viz.*

$$J_1 = -g\int_X (\xi_\varepsilon - \xi_{\varepsilon'})(\eta_\varepsilon - \eta_{\varepsilon'})\, dx \le \frac{g}{2}\left(\|\xi_\varepsilon - \xi_{\varepsilon'}\|_0^2 + \|\eta_\varepsilon - \eta_{\varepsilon'}\|_0^2\right) \le cE_d.$$

Write J_2 in the form

$$J_2 = 2\nu\int_X (\xi_\varepsilon - \xi_{\varepsilon'})(\mathcal{S}^2_\varepsilon\partial_x^2\xi_\varepsilon - \mathcal{S}^2_{\varepsilon'}\partial_x^2\xi_{\varepsilon'})\, dx = J_{2,A} + J_{2,B}$$

where

$$J_{2,A} = 2\nu\int_X (\xi_\varepsilon - \xi_{\varepsilon'})(\mathcal{S}^2_\varepsilon\partial_x^2\xi_\varepsilon - \mathcal{S}^2_{\varepsilon'}\partial_x^2\xi_\varepsilon)\, dx$$

and

$$J_{2,B} = 2\nu\int_X (\xi_\varepsilon - \xi_{\varepsilon'})(\mathcal{S}^2_{\varepsilon'}\partial_x^2\xi_\varepsilon - \mathcal{S}^2_{\varepsilon'}\partial_x^2\xi_{\varepsilon'})\, dx.$$

To handle $J_{2,A}$, use the standard mollifier estimate

$$\|\mathcal{S}_\varepsilon f - \mathcal{S}_{\varepsilon'} f\|_{L^2} \le c\max\{\varepsilon, \varepsilon'\}\,\|f\|_{H^1} \tag{37}$$

to obtain

$$J_{2,A} \le cE_d^{1/2}\max\{\varepsilon, \varepsilon'\}\left\|\partial_x^2\xi_\varepsilon\right\|_{H^1}.$$

Since $\left\|\partial_x^2\xi_\varepsilon\right\|_{H^1} \le K^{1/2}$, it is concluded that

$$J_{2,A} \le cE_d^{1/2}\max\{\varepsilon, \varepsilon'\}.$$

Integrating $J_{2,B}$ by parts, and using the fact that $\mathcal{S}_{\varepsilon'}$ is self-adjoint reveals that

$$J_{2,B} = -2\nu \int_X (\partial_x[\mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]])^2 \, dx.$$

The term J_3 is

$$J_3 = -4\nu \int_X (\xi_\varepsilon - \xi_{\varepsilon'}) \left(\mathcal{S}_\varepsilon[(\mathcal{S}_\varepsilon \partial_x \eta_\varepsilon)(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon)] - \mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'} H \partial_x^2 \xi_{\varepsilon'}]\right) \, dx,$$

which is rewritten by adding and subtracting various terms to obtain

$$J_3 = J_{3,A} + J_{3,B} + J_{3,C} + J_{3,D} + J_{3,E},$$

where

$$\begin{aligned}
J_{3,A} &= -4\nu \int_X (\xi_\varepsilon - \xi_{\varepsilon'}) \left(\mathcal{S}_\varepsilon[(\mathcal{S}_\varepsilon \partial_x \eta_\varepsilon)(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon)] - \mathcal{S}_{\varepsilon'}[(\mathcal{S}_\varepsilon \partial_x \eta_\varepsilon)(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon]\right) \, dx,\\
J_{3,B} &= -4\nu \int_X (\xi_\varepsilon - \xi_{\varepsilon'}) \left(\mathcal{S}_{\varepsilon'}[(\mathcal{S}_\varepsilon \partial_x \eta_\varepsilon)(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon)] - \mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_\varepsilon)(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon]\right) \, dx,\\
J_{3,C} &= -4\nu \int_X (\xi_\varepsilon - \xi_{\varepsilon'}) \left(\mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_\varepsilon)(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon)] - \mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon]\right) \, dx,\\
J_{3,D} &= -4\nu \int_X (\xi_\varepsilon - \xi_{\varepsilon'}) \left(\mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_\varepsilon H \partial_x^2 \xi_\varepsilon)] - \mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'} H \partial_x^2 \xi_\varepsilon]\right) \, dx,\\
J_{3,E} &= -4\nu \int_X (\xi_\varepsilon - \xi_{\varepsilon'}) \left(\mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'} H \partial_x^2 \xi_\varepsilon)] - \mathcal{S}_{\varepsilon'}[(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'} H \partial_x^2 \xi_{\varepsilon'}]\right) \, dx.
\end{aligned}$$

Akin to the term $J_{2,A}$, each of $J_{3,A}$, $J_{3,B}$, and $J_{3,D}$ includes an instance of the operator $\mathcal{S}_\varepsilon - \mathcal{S}_{\varepsilon'}$ acting on a function which does not feature a difference such as $\eta_\varepsilon - \xi_{\varepsilon'}$ or the like. It follows as for $J_{2,A}$ that

$$J_{3,A} \le cE_d^{1/2} \max\{\varepsilon, \varepsilon'\}, \qquad J_{3,B} \le cE_d^{1/2} \max\{\varepsilon, \varepsilon'\}, \qquad J_{3,D} \le cE_d^{1/2} \max\{\varepsilon, \varepsilon'\}.$$

We use the self-adjointness of $\mathcal{S}_{\varepsilon'}$ to write $J_{3,E}$ as

$$J_{3,E} = -4\nu \int_X (\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'}(\xi_\varepsilon - \xi_{\varepsilon'})) H \partial_x^2 (\mathcal{S}_{\varepsilon'}(\xi_\varepsilon - \xi_{\varepsilon'})) \, dx$$

and integrate by parts once to come to $J_{3,E} = J_{3,F} + J_{3,G}$, with

$$J_{3,F} = 4\nu \int_X (\mathcal{S}_{\varepsilon'} \partial_x^2 \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'}(\xi_\varepsilon - \xi_{\varepsilon'})) H \partial_x (\mathcal{S}_{\varepsilon'}(\xi_\varepsilon - \xi_{\varepsilon'})) \, dx$$

and

$$J_{3,G} = 4\nu \int_X (\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'}))(H \mathcal{S}_{\varepsilon'} \partial_x (\xi_{\varepsilon'} - \xi_{\varepsilon'})) \, dx.$$

As H is skew-adjoint,

$$J_{3,G} = -4\nu \int_X \{H[(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'}))]\} \{\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})\} \, dx.$$

After pulling $\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'}$ outside the Hilbert transform (at the cost of a commutator), this becomes

$$\begin{aligned}
J_{3,G} &= -4\nu \int_X \{(\mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'}) H [\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})]\} \{\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})\} \, dx\\
&\quad - 4\nu \int_X \{[H, \mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'}] [\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})]\} \{\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})\} \, dx\\
&= -J_{3,G} - 4\nu \int_X \{[H, \mathcal{S}_{\varepsilon'} \partial_x \eta_{\varepsilon'}] [\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})]\} \{\mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})\} \, dx.
\end{aligned}$$

In consequence,

$$J_{3,G} = -2\nu \int_X \{[H, \mathcal{S}_{\varepsilon'}\partial_x\eta_{\varepsilon'}]\,[\mathcal{S}_{\varepsilon'}\partial_x(\xi_\varepsilon - \xi_{\varepsilon'})]\}\,\{\mathcal{S}_{\varepsilon'}\partial_x(\xi_\varepsilon - \xi_{\varepsilon'})\}\ dx,$$

and, after an integration by parts, we find

$$J_{3,G} = 2\nu \int_X (\mathcal{S}_{\varepsilon'}(\xi_\varepsilon - \xi_{\varepsilon'}))\partial_x \{[H, \mathcal{S}_{\varepsilon'}\partial_x\eta_{\varepsilon'}]\,[\mathcal{S}_{\varepsilon'}\partial_x(\xi_\varepsilon - \xi_{\varepsilon'})]\}\ dx.$$

It follows readily that

$$J_{3,G} \le cE_d.$$

Write $J_{3,C}$ as

$$J_{3,C} = -4\nu \int_X (\mathcal{S}_\varepsilon H\partial_x^2\xi_\varepsilon)(\mathcal{S}_\varepsilon(\xi_\varepsilon - \xi_{\varepsilon'}))(\mathcal{S}_{\varepsilon'}\partial_x(\eta_\varepsilon - \eta_{\varepsilon'}))\ dx.$$

Notice that by Sobolev embedding, we can write

$$|\mathcal{S}_\varepsilon H\partial_x^2\xi_\varepsilon|_{L^\infty} \le c\|\mathcal{S}_\varepsilon H\partial_x^2\xi_\varepsilon\|_{H^1} \le c\|\xi_\varepsilon\|_{H^3} \le K^{1/2}.$$

Now, use Young's inequality to deduce that

$$J_{3,C} \le \frac{4\nu K^{1/2}\sigma_1}{2}\|\xi_\varepsilon - \xi_{\varepsilon'}\|_{L^2}^2 + \frac{4\nu K^{1/2}}{2\sigma_1}\|\mathcal{S}_{\varepsilon'}\partial_x[\eta_\varepsilon - \eta_{\varepsilon'}]\|_{L^2}^2$$

and

$$J_{3,F} \le \frac{4\nu K^{1/2}\sigma_2}{2}\|\xi_\varepsilon - \xi_{\varepsilon'}\|_{L^2}^2 + \frac{4\nu K^{1/2}}{2\sigma_2}\|\mathcal{S}_{\varepsilon'}\partial_x[\xi_\varepsilon - \xi_{\varepsilon'}]\|_{L^2}^2.$$

If we choose $\sigma_1 = \sigma_2 = 4(1 + K^{1/2})$, then

$$J_{3,C} \le 8\nu K^{1/2}(1 + K^{1/2})\|\xi_\varepsilon - \xi_{\varepsilon'}\|_{L^2}^2 + \frac{\nu}{2}\|\mathcal{S}_{\varepsilon'}\partial_x[\eta_\varepsilon - \eta_{\varepsilon'}]\|_{L^2}^2,$$

$$J_{3,F} \le 8\nu K^{1/2}(1 + K^{1/2})\|\xi_\varepsilon - \xi_{\varepsilon'}\|_{L^2}^2 + \frac{\nu}{2}\|\mathcal{S}_{\varepsilon'}\partial_x[\xi_\varepsilon - \xi_{\varepsilon'}]\|_{L^2}^2.$$

Split the term

$$J_4 = \frac{1}{2}\int_X (\xi_\varepsilon - \xi_{\varepsilon'})\left\{\mathcal{S}_\varepsilon([H[\mathcal{S}_\varepsilon[\partial_x\xi_\varepsilon]]])^2 - \mathcal{S}_{\varepsilon'}(H[\mathcal{S}_{\varepsilon'}[\partial_x\xi_{\varepsilon'}]])^2\right\}\ dx,$$

as $J_4 = J_{4,A} + J_{4,B}$, where

$$J_{4,A} = \frac{1}{2}\int_X (\xi_\varepsilon - \xi_{\varepsilon'})\left\{\mathcal{S}_\varepsilon(H[\mathcal{S}_\varepsilon[\partial_x\xi_\varepsilon]])^2 - \mathcal{S}_{\varepsilon'}(H[\mathcal{S}_\varepsilon[\partial_x\xi_\varepsilon]])^2\right\}\ dx$$

and

$$J_{4,B} = \frac{1}{2}\int_X (\xi_\varepsilon - \xi_{\varepsilon'})\left\{\mathcal{S}_{\varepsilon'}(H[\mathcal{S}_\varepsilon[\partial_x\xi_\varepsilon]])^2 - \mathcal{S}_{\varepsilon'}(H[\mathcal{S}_{\varepsilon'}[\partial_x\xi_{\varepsilon'}]])^2\right\}\ dx.$$

As for the estimate of $J_{2,A}$ above,

$$J_{4,A} \le cE_d^{1/2}\max\{\varepsilon, \varepsilon'\}.$$

For $J_{4,B}$, use the self-adjointness of $\mathcal{S}_{\varepsilon'}$ to write

$$\begin{aligned}
J_{4,B} &= \frac{1}{2}\int_X \mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]\left\{(H[\mathcal{S}_\varepsilon[\partial_x\xi_\varepsilon]])^2 - (H[\mathcal{S}_{\varepsilon'}[\partial_x\xi_{\varepsilon'}]])^2\right\}\ dx \\
&= \frac{1}{2}\int_X \mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]\left\{H[\mathcal{S}_\varepsilon[\partial_x\xi_\varepsilon]] + H[\mathcal{S}_{\varepsilon'}[\partial_x\xi_{\varepsilon'}]]\right\}\cdot \\
&\qquad \cdot\left\{H[\mathcal{S}_\varepsilon[\partial_x\xi_\varepsilon]] - H[\mathcal{S}_{\varepsilon'}[\partial_x\xi_{\varepsilon'}]]\right\}\ dx. \qquad (38)
\end{aligned}$$

After adding and subtracting the term

$$\mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]\,\{H[\mathcal{S}_\varepsilon[\partial_x \xi_\varepsilon]] + H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]\}\,H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_\varepsilon]],$$

there results $J_{4,B} = J_{4,C} + J_{4,D}$ where

$$J_{4,C} = \frac{1}{2}\int_X \mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]\,\{H[\mathcal{S}_\varepsilon[\partial_x \xi_\varepsilon]] + H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]\}\cdot \\ \cdot\{H[\mathcal{S}_\varepsilon[\partial_x \xi_\varepsilon]] - H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_\varepsilon]]\}\ dx, \quad (39)$$

$$J_{4,D} = \frac{1}{2}\int_X \mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]\,\{H[\mathcal{S}_\varepsilon[\partial_x \xi_\varepsilon]] + H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]\}\cdot \\ \cdot\{H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_\varepsilon]] - H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]\}\ dx. \quad (40)$$

Since $J_{4,C}$ possesses a difference $\mathcal{S}_\varepsilon - \mathcal{S}_{\varepsilon'}$ of mollifiers, (37) and the energy estimate imply that

$$J_{4,C} \le cE_d^{1/2}\max\{\varepsilon,\varepsilon'\}.$$

Notice that the middle factor in $J_{4,D}$ can be bounded as follows:

$$\begin{aligned} |H[\mathcal{S}_\varepsilon[\partial_x \xi_\varepsilon]] + H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]|_{L^\infty} &\le \|H[\mathcal{S}_\varepsilon[\partial_x \xi_\varepsilon]] + H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]\|_{H^1} \\ &\le \|\xi_\varepsilon\|_{H^2} + \|\xi_{\varepsilon'}\|_{H^2} \\ &\le 2K^{1/2}. \end{aligned}$$

Consequently, it transpires that

$$J_{4,D} \le K^{1/2}\int_X |\mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]|\,|H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_\varepsilon]] - H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]|\ dx.$$

Young's inequality, with parameter $\sigma_3 = (1+K)^{1/2}/\nu$ gives

$$J_{4,D} \le \frac{K^{1/2}(1+K^{1/2})}{2\nu}\int_X |\mathcal{S}_{\varepsilon'}[(\xi_\varepsilon - \xi_{\varepsilon'})]|^2\ \mathrm{d}x \\ + \frac{K^{1/2}\nu}{2(1+K^{1/2})}\int_X |H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_\varepsilon]] - H[\mathcal{S}_{\varepsilon'}[\partial_x \xi_{\varepsilon'}]]|^2\ dx.$$

This clearly implies that

$$J_{4,D} \le cE_d + \frac{\nu}{2}\int_X (\partial_x \mathcal{S}_{\varepsilon'}[\xi_\varepsilon - \xi_{\varepsilon'}])^2\ dx.$$

The last term on the right–hand side will cancel with part of J_2, just as in the previous, unmollified energy estimate. We now consider

$$J_5 = -\frac{1}{2}\int_X (\xi_\varepsilon - \xi_{\varepsilon'})\left(\mathcal{S}_\varepsilon(\mathcal{S}_\varepsilon \partial_x \xi_\varepsilon)^2 - \mathcal{S}_{\varepsilon'}(\mathcal{S}_{\varepsilon'}\partial_x \xi_{\varepsilon'})^2\right)\ dx$$

which is estimated in a manner almost identical to that of J_4. Add and subtract as usual, finding $J_5 = J_{5,A} + J_{5,B} + J_{5,C}$, with

$$J_{5,A} = -\frac{1}{2}\int_X (\xi_\varepsilon - \xi_{\varepsilon'})(\mathcal{S}_\varepsilon - \mathcal{S}_{\varepsilon'})[(\mathcal{S}_\varepsilon \partial_x \xi_\varepsilon)^2]\ dx \le c\max\{\varepsilon,\varepsilon'\}E_d^{1/2},$$

$$J_{5,B} = -\frac{1}{2}\int_X (\mathcal{S}_{\varepsilon'}[\xi_\varepsilon - \xi_{\varepsilon'}])((\mathcal{S}_\varepsilon - \mathcal{S}_{\varepsilon'})[\partial_x \xi_\varepsilon])((\mathcal{S}_\varepsilon + \mathcal{S}_{\varepsilon'})[\partial_x \xi_\varepsilon])\ dx \le c\max\{\varepsilon,\varepsilon'\}E_d^{1/2},$$

$$J_{5,C} = -\frac{1}{2}\int_X (\mathcal{S}_{\varepsilon'}[\xi_\varepsilon - \xi_{\varepsilon'}])(\mathcal{S}_{\varepsilon'}\partial_x[\xi_\varepsilon - \xi_{\varepsilon'}])(\mathcal{S}_{\varepsilon'}\partial_x[\xi_\varepsilon + \xi_{\varepsilon'}])\ dx.$$

We recognize an exact derivative in $J_{5,C}$ and integrate by parts, thereby coming to

$$J_{5,C} = \frac{1}{4} \int_X (\mathcal{S}_{\varepsilon'}[\xi_\varepsilon - \xi_{\varepsilon'}])^2 (\mathcal{S}_{\varepsilon'} \partial_x^2 [\xi_\varepsilon + \xi_{\varepsilon'}]) \, dx \le c E_d.$$

Consider now $dE_{d,\eta}/dt$ and write it as

$$\frac{dE_{d,\eta}}{dt} = \int_X (\eta_\varepsilon - \eta_{\varepsilon'}) \partial_t (\eta_\varepsilon - \eta_{\varepsilon'}) \, dx = J_6 + J_7 + J_8,$$

where each of these three terms correspond to a term on the right-hand side of the evolution equation (35) for η. Add and subtract $\mathcal{S}_{\varepsilon'} H \partial_x \xi_\varepsilon$ in

$$J_6 = \int_X (\eta_\varepsilon - \eta_{\varepsilon'})(\mathcal{S}_\varepsilon H \partial_x \xi_\varepsilon - \mathcal{S}_{\varepsilon'} H \partial_x \xi_{\varepsilon'}) \, dx$$

to obtain $J_6 = J_{6,A} + J_{6,B}$, where

$$J_{6,A} = \int_X (\eta_\varepsilon - \eta_{\varepsilon'})(\mathcal{S}_\varepsilon H \partial_x \xi_\varepsilon - \mathcal{S}_{\varepsilon'} H \partial_x \xi_\varepsilon) \, dx,$$

$$J_{6,B} = \int_X (\eta_\varepsilon - \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'} H \partial_x \xi_\varepsilon - \mathcal{S}_{\varepsilon'} H \partial_x \xi_{\varepsilon'}) \, dx.$$

Since $J_{6,A}$ has a difference of $\mathcal{S}_\varepsilon$ and $\mathcal{S}_{\varepsilon'}$, we have as before that

$$J_{6,A} \le c \max\{\varepsilon, \varepsilon'\} E_d^{1/2}.$$

Rewrite $J_{6,B}$ as

$$J_{6,B} = \int_X (\eta_\varepsilon - \eta_{\varepsilon'}) \mathcal{S}_{\varepsilon'} [H \partial_x (\xi_\varepsilon - \xi_{\varepsilon'})] \, dx$$

and apply Young's Inequality with a parameter $\sigma = \sigma_4 = \frac{2}{\nu}$ to derive

$$J_{6,B} \le c E_d + \nu \| \mathcal{S}_{\varepsilon'} \partial_x (\xi_\varepsilon - \xi_{\varepsilon'}) \|_{L^2}^2.$$

For

$$J_7 = 2\nu \int_X (\eta_\varepsilon - \eta_{\varepsilon'})(\mathcal{S}_\varepsilon^2 \partial_x^2 \eta_\varepsilon - \mathcal{S}_{\varepsilon'}^2 \partial_x^2 \eta_{\varepsilon'}) \, dx.$$

Add and subtract $\mathcal{S}_{\varepsilon'}^2 \partial_x^2 \eta_\varepsilon$ to express J_7 as the sum $J_7 = J_{7,A} + J_{7,B}$ with

$$J_{7,A} = 2\nu \int_X (\eta_\varepsilon - \eta_{\varepsilon'})(\mathcal{S}_\varepsilon^2 \partial_x^2 \eta_\varepsilon - \mathcal{S}_{\varepsilon'}^2 \partial_x^2 \eta_\varepsilon) \, dx$$

and

$$J_{7,B} = 2\nu \int_X (\eta_\varepsilon - \eta_{\varepsilon'})(\mathcal{S}_{\varepsilon'}^2 \partial_x^2 \eta_\varepsilon - \mathcal{S}_{\varepsilon'}^2 \partial_x^2 \eta_{\varepsilon'}) \, dx.$$

Since $J_{7,A}$ contains the operator $\mathcal{S}_\varepsilon - \mathcal{S}_{\varepsilon'}$, we have

$$J_{7,A} \le c \max\{\varepsilon, \varepsilon'\} E_d^{1/2}.$$

And, because $\mathcal{S}_\varepsilon$ is self-adjoint, an integration by parts provides the formula

$$J_{7,B} = -2\nu \int_X (\mathcal{S}_{\varepsilon'} \partial_x (\eta_\varepsilon - \eta_{\varepsilon'}))^2 \, dx.$$

Finally, consider

$$J_8 = -\int_X (\eta_\varepsilon - \eta_{\varepsilon'}) \, \partial_x \Big(\mathcal{S}_\varepsilon [H, \mathcal{S}_\varepsilon \eta_\varepsilon](H \mathcal{S}_\varepsilon \partial_x \xi_\varepsilon) - \mathcal{S}_{\varepsilon'} [H, \mathcal{S}_{\varepsilon'} \eta_{\varepsilon'}](H \mathcal{S}_{\varepsilon'} \partial_x \xi_{\varepsilon'}) \Big) \, dx.$$

The formula $[H,a]f-[H,b]f=[H,a-b]f$, the self–adjointness of $\mathcal{S}_\varepsilon$ and judicious adding and subtracting leads to $J_8 = J_{8,A}+J_{8,B}+J_{8,C}+J_{8,D}+J_{8,E}$ with

$$J_{8,A} = -\int_X \mathcal{S}_\varepsilon[\eta_\varepsilon-\eta_{\varepsilon'}]\partial_x\Big([H,\mathcal{S}_\varepsilon\eta_\varepsilon](H\mathcal{S}_\varepsilon\partial_x\xi_\varepsilon)-[H,\mathcal{S}_{\varepsilon'}\eta_\varepsilon](H\mathcal{S}_\varepsilon\partial_x\xi_\varepsilon)\Big)\,dx,$$

$$J_{8,B} = -\int_X \mathcal{S}_\varepsilon[\eta_\varepsilon-\eta_{\varepsilon'}]\partial_x\Big([H,\mathcal{S}_{\varepsilon'}\eta_\varepsilon](H\mathcal{S}_\varepsilon\partial_x\xi_\varepsilon)-[H,\mathcal{S}_{\varepsilon'}\eta_{\varepsilon'}](H\mathcal{S}_\varepsilon\partial_x\xi_\varepsilon)\Big)\,dx,$$

$$J_{8,C} = -\int_X \mathcal{S}_\varepsilon[\eta_\varepsilon-\eta_{\varepsilon'}]\partial_x\Big([H,\mathcal{S}_{\varepsilon'}\eta_{\varepsilon'}](H\mathcal{S}_\varepsilon\partial_x\xi_\varepsilon)-[H,\mathcal{S}_{\varepsilon'}\eta_{\varepsilon'}](H\mathcal{S}_{\varepsilon'}\partial_x\xi_\varepsilon)\Big)\,dx,$$

$$J_{8,D} = -\int_X \mathcal{S}_\varepsilon[\eta_\varepsilon-\eta_{\varepsilon'}]\partial_x\Big([H,\mathcal{S}_{\varepsilon'}\eta_{\varepsilon'}](H\mathcal{S}_{\varepsilon'}\partial_x\xi_\varepsilon)-[H,\mathcal{S}_{\varepsilon'}\eta_{\varepsilon'}](H\mathcal{S}_{\varepsilon'}\partial_x\xi_{\varepsilon'})\Big)\,dx,$$

$$J_{8,E} = -\int_X (\eta_\varepsilon-\eta_{\varepsilon'})(\mathcal{S}_\varepsilon-\mathcal{S}_{\varepsilon'})\partial_x\left\{[H,\mathcal{S}_{\varepsilon'}\eta_{\varepsilon'}](H\mathcal{S}_{\varepsilon'}\partial_x\xi_{\varepsilon'})\right\}\,dx.$$

Each of $J_{8,A}$, $J_{8,C}$, and $J_{8,E}$ have a difference $\mathcal{S}_\varepsilon-\mathcal{S}_{\varepsilon'}$, and so by estimates that are by now familiar,

$$J_{8,A}\le c\max\{\varepsilon,\varepsilon'\}E_d^{1/2},\qquad J_{8,C}\le c\max\{\varepsilon,\varepsilon'\}E_d^{1/2},\qquad J_{8,E}\le c\max\{\varepsilon,\varepsilon'\}E_d^{1/2}.$$

Rewrite $J_{8,B}$ as

$$J_{8,B} = -\int_X \mathcal{S}_\varepsilon[\eta_\varepsilon-\eta_{\varepsilon'}]\partial_x\Big([H,\mathcal{S}_{\varepsilon'}(\eta_\varepsilon-\eta_{\varepsilon'})](H\mathcal{S}_\varepsilon\partial_x\xi_\varepsilon)\Big)\,dx.$$

This commutator is treated differently than in our previous machinations. The commutator is of the form $[H,\psi]g$ where we currently have sufficient regularity on $g = H\mathcal{S}_\varepsilon\partial_x\xi_\varepsilon$ so that it is not necessary to use the smoothing effects of such commutators. Indeed, the simple inequality $\|[H,\psi]g\|_{H^1}\le c\|\psi\|_{H^1}\|g\|_{H^1}$ implies that

$$J_{8,B}\le cE_d^{1/2}K^{1/2}\|\mathcal{S}_{\varepsilon'}[\eta_\varepsilon-\eta_{\varepsilon'}]\|_{H_1}\le cE_d + cE_d^{1/2}K^{1/2}\|\mathcal{S}_{\varepsilon'}[\partial_x(\eta_\varepsilon-\eta_{\varepsilon'})]\|_{L^2}.$$

Young's Inequality gives the further bound

$$J_{8,B}\le cE_d+\frac{cE_dK\sigma_5}{2}+\frac{1}{2\sigma_5}\|\mathcal{S}_{\varepsilon'}[\partial_x(\eta_\varepsilon-\eta_{\varepsilon'})]\|_{L^2}^2,$$

and upon choosing $\sigma_5=\frac{2}{\nu}$, it is found that

$$J_{8,B}\le cE_d+\nu\|\mathcal{S}_{\varepsilon'}[\partial_x(\eta_\varepsilon-\eta_{\varepsilon'})]\|_{L^2}^2.$$

A straightforward application of Lemma 3.1 yields

$$J_{8,D} = -\int_X \mathcal{S}_\varepsilon[\eta_\varepsilon-\eta_{\varepsilon'}]\partial_x\Big([H,\mathcal{S}_{\varepsilon'}\eta_{\varepsilon'}](H\mathcal{S}_{\varepsilon'}\partial_x[\xi_\varepsilon-\xi_{\varepsilon'}])\Big)\,dx\le cE_d.$$

Adding the estimates just obtained for the various J's, we find only terms proportional to E_d and $\max\{\varepsilon,\varepsilon'\}E_d^{1/2}$. All of the other terms (e.g., $\|\mathcal{S}_{\varepsilon'}\partial_x(\eta_\varepsilon-\eta_{\varepsilon'})\|$ or $\|\mathcal{S}_{\varepsilon'}\partial_x(\xi_\varepsilon-\xi_{\varepsilon'}\|$) cancel because of the careful choice of the σ_i's. In consequence, we have

$$\frac{dE_d}{dt}\le cE_d+c\max\{\varepsilon,\varepsilon'\}E_d^{1/2},$$

as advertised. The proof is complete. □

Remark 5. Writing $E_d = (E_d^{1/2})^2$, the inequality in the last Lemma is equivalent to

$$\frac{dE_d^{1/2}}{dt} \le cE_d^{1/2} + c\max\{\varepsilon, \varepsilon'\}.$$

Since $E_d(0) = 0$, a Gronwall-type argument yields

$$E_d^{1/2}(t) \le \max\{\varepsilon, \varepsilon'\}(e^{ct} - 1).$$

Thus $\{(\eta_\varepsilon, \xi_\varepsilon)\}$ is Cauchy in $C([0,T]; L^2 \times L^2)$, and hence converges as $\varepsilon \to 0$ in this space to a limit (η, ξ) as $\varepsilon \to 0$.

A straightforward application of the elementary Sobolev interpolation theorem (Lemma 5.2 below) together with the uniform bound in H^s demonstrates that $\{(\eta_\varepsilon, \xi_\varepsilon)\}$ is, for any $s' < s$, Cauchy in $C([0,T]; H^{s'} \times H^{s'})$. Hence, (η, ξ) lies in this latter space. The further conclusion $(\eta, \xi) \in C([0,T]; H^s \times H^s)$ will be dealt with in Section 5.4.

Lemma 5.2. *Let $0 < m < s$ be given, with $f \in H^s$. Then*

$$\|f\|_{H^m} \le c\|f\|_{H^s}^{m/s}\|f\|_{H^0}^{1-m/s}.$$

Further properties of the limit (η, ξ) are the subject of the remainder of this section.

5.2. **The limit solution solves (18).** The one-parameter family of mollified PDEs has the form $\partial_t u_\varepsilon = G_\varepsilon(u_\varepsilon)$ where $u_\varepsilon = (\eta_\varepsilon, \xi_\varepsilon)$ and G_ε is a non-local, second-order, nonlinear operator given by the right-hand sides of equations (34) and (35). Integrating in time, we find that

$$u_\varepsilon(\cdot, t) = u_\varepsilon(\cdot, 0) + \int_0^t G_\varepsilon(u_\varepsilon(\cdot, \tau))\, d\tau.$$

Since $u_\varepsilon \to u$ in $C([0,T]; H^{s'})$ for s' sufficiently large, we can pass to the limit in all of these terms. In particular, it is straightforward to conclude that $G_\varepsilon(u_\varepsilon) \to G(u)$ in $C([0,T]; H^{s'-2})$, where G is the operator given by the right-hand sides of (22) and (23). This in turn implies

$$u(\cdot, t) = u(\cdot, 0) + \int_0^t G(u(\cdot, \tau))\, d\tau.$$

Differentiating with respect to time shows that the original, unregularized system (22)–(23) is satisfied by the limit.

5.3. **Uniqueness and continuous dependence.** We now address uniqueness and continuous dependence on the data. Each of these is established by estimates which are similar to ones already in hand. In fact, if we have two solutions (η, ξ) and $(\tilde{\eta}, \tilde{\xi})$, perhaps with different data, an estimate for the growth of $\tilde{E}_d = \|(\eta - \tilde{\eta}, \xi - \tilde{\xi})\|_{L^2}$ would be helpful. This is essentially the same as the estimate for E_d, but in the simpler case $\varepsilon = \varepsilon' = 0$. It is thus concluded that

$$\frac{d\tilde{E}_d}{dt} \le c\tilde{E}_d,$$

which implies that $\tilde{E}_d$ grows at most exponentially. If the initial condition is $\tilde{E}_d(0) = 0$ (which corresponds to having the same data), we see that $\tilde{E}_d(t)$ remains zero. Uniqueness of solutions is therefore established. Furthermore, if the two initial conditions are not the same, the norm of the difference only grows exponentially;

the difference at time t can be made small, uniformly for $t \in [0,T]$, by taking the initial value $\tilde{E}_d(0)$ small. This establishes continuous dependence on the initial data in L^2. Continuous dependence in higher Sobolev norms then follows by applying Lemma 5.2.

5.4. **The highest regularity.** Here, it is shown that each of η and ξ actually lie in $C([0,T];H^s(X))$. The argument parallels a proof of regularity of strong solutions of the Navier-Stokes equations (see Chapter 3 of [12]). This argument requires several steps, the first of which is to show H^s regularity, pointwise in time. The second step is to show weak continuity in time. Then, continuity of the H^s norm is shown. Together, these steps establish that η and ξ are indeed in $C([0,T];H^s(X))$.

We begin with the regularity, pointwise in time. For any fixed $t \in [0,T]$, $\eta_\varepsilon(\cdot,t)$ and $\xi_\varepsilon(\cdot,t)$ are uniformly bounded in H^s. Since the unit ball of a Hilbert space is weakly compact, $\eta_\varepsilon(\cdot,t)$ and $\xi_\varepsilon(\cdot,t)$ converge as $\varepsilon \to 0$, weakly in H^s. Clearly, these weak limits must be $\eta(\cdot,t)$ and $\xi(\cdot,t)$. So, for each t, $\eta(\cdot,t)$ and $\xi(\cdot,t)$ lie in H^s and possess the same uniform bound satisfied by the approximating sequences.

Next, it is shown that each of η and ξ are in $C_W([0,T];H^s)$, which is to say η and ξ are continuous in time with values in H^s endowed with its weak topology. Attention is given to η, but there is no difference between η and ξ in the present context. Recall that for any s' with $0 \le s' < s$, it is known that $\eta_\varepsilon \to \eta \in C([0,T];H^{s'})$. Therefore, for any $\phi \in H^{-s'}$, the duality pairing $\langle \phi, \eta_\varepsilon\rangle \to \langle \phi, \eta\rangle$ as $\varepsilon \to 0$, uniformly on $[0,T]$. Since $s' < s$, it is certainly true that $-s < -s'$, and therefore $H^{-s'}$ is dense in H^{-s}. Recall the already established uniform bound $\|\eta_\varepsilon\|_{H^s} \le K$. For $\psi \in H^{-s}$ and $\delta > 0$ given, choose $\psi_\delta \in H^{-s'}$ so that $\|\psi_\delta - \psi\|_{H^{-s}} < \frac{\delta}{3(1+K)}$. Let ε be small enough that $|\langle \psi_\delta, \eta_\varepsilon - \eta\rangle| < \frac{\delta}{3}$. With these restrictions, it follows that

$$|\langle \psi, \eta_\varepsilon - \eta\rangle| \le |\langle \psi - \psi_\delta, \eta_\varepsilon - \eta\rangle| + |\langle \psi_\delta, \eta_\varepsilon - \eta\rangle| < \delta.$$

Notice that these choices can be made independently of t. This is enough to conclude that η (and similarly ξ) lie in $C_W([0,T];H^s)$.

Next, we show strong right-continuity in time of the solutions at $t = 0$. Since weak convergence and convergence of the norm imply strong convergence, it is only necessary to show that $\|\eta(\cdot,t)\|_{H^s} \to \|\eta(\cdot,0)\|_{H^s}$ and $\|\xi(\cdot,t)\|_{H^s} \to \|\xi(\cdot,0)\|_{H^s}$ as $t \to 0^+$. By Fatou's Lemma, it must be the case that

$$\|\eta(\cdot,0)\|^2_{H^s} \le \liminf_{t\to 0^+} \|\eta(\cdot,t)\|^2_{H^s}, \tag{41}$$

$$\|\xi(\cdot,0)\|^2_{H^s} \le \liminf_{t\to 0^+} \|\xi(\cdot,t)\|^2_{H^s}. \tag{42}$$

Adding (41) and (42) shows that

$$E(0) \le \liminf_{t\to 0^+} E(t).$$

But, the energy inequality reveals that

$$\limsup_{t\to 0^+} E(t) \le E(0),$$

and so the energy is right-continuous at $t = 0$. The inequalities (41) and (42) imply that $\|\eta\|_{H^s}$ and $\|\xi\|_{H^s}$ are right lower semi-continuous at $t = 0$. As their sum is right-continuous, it follows that they are each right-continuous at $t = 0$.

Finally, we prove strong continuity of solutions on $(0,T]$. The strategy is to use the smoothing that derives from the viscosity. As the energy estimates demonstrate (see Remark 3), the quantity

$$\nu \int_0^T \int_X (\mathcal{S}_\varepsilon \partial_x^{s+1} \eta_\varepsilon)^2 + (\mathcal{S}_\varepsilon \partial_x^{s+1} \xi_\varepsilon)^2 \; dxdt$$

is bounded, independently of $\varepsilon \in (0,1]$, say. It follows immediately that $\mathcal{S}_\varepsilon \eta_\varepsilon$ and $\mathcal{S}_\varepsilon \xi_\varepsilon$ have weak limits in $L^2([0,T]; H^{s+1})$. As before, elementary considerations show these limits must be η and ξ. This means that at almost every time, η and ξ are members of H^{s+1}.

Given $\delta > 0$, there must be times T_0 with $0 < T_0 < \delta$, such that $\eta(\cdot, T_0), \xi(\cdot, T_0) \in H^{s+1}$. Using T_0 as an initial time, and repeating the entire existence theory (now with H^{s+1} data) leads to existence of a solution on the interval $[T_0, \tilde{T}]$, which lies in $C([T_0, \tilde{T}]; H^{\tilde{s}})$ for any $\tilde{s}$ with $0 \le \tilde{s} < s+1$. By uniqueness, this solution and the previously found solution are the same. Taking $\tilde{s} = s$ shows that (η, ξ) lies in $C([T_0, \tilde{T}]; H^s \times H^s)$.

The size of $\tilde{T}$ depends on the interval on which we can make estimates of the H^{s+1} norm of the solution. Performing an H^{s+1} estimate for the system, and as in Remark 3, we derive

$$\frac{d}{dt}\left(\|\eta\|_{H^{s+1}}^2 + \|\xi\|_{H^{s+1}}^2\right) \le F(\|\eta\|_{H^s}, \|\xi\|_{H^s})\left(\|\eta\|_{H^{s+1}}^2 + \|\xi\|_{H^{s+1}}^2\right),$$

for some continuous function F. These H^{s+1} norms are therefore uniformly bounded as long as the H^s norms are under control. As these are already known to be bounded on $[0,T]$, we may take $\tilde{T} = T$. It is concluded that η and ξ are each in $C([T_0, T]; H^s)$. Since δ was an arbitrary positive number, we see that in fact η and ξ both lie in $C((0,T]; H^s)$.

Combining this result with the right-continuity at $t = 0$ implies that η and ξ are in $C([0,T]; H^s)$. This finishes the proof of Theorem 2.1.

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