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WELL-POSEDNESS FOR THE BBM-EQUATION IN A QUARTER PLANE

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Abstract. The so-called wave-maker problem for the BBM-equation is studied on the half-line. Improving on earlier results, global well-posedness is established for square-integrable initial data and boundary data that is only assumed to be locally bounded.

1. **Introduction.** Boundary-value problems for nonlinear, dispersive wave equations of Boussinesq-Korteweg-de Vries type were introduced in [4] in the context of some laboratory experiments being conducted at the time in the Fluid Mechanics Research Institute at the University of Essex. Comparisons between experimentally obtained data and theory based upon an initial-boundary-value problem for the regularized long-wave, or BBM, equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.1}$$

appeared later in [10] (and see also the earlier, related work of Peregrine [26] and Hammack [23]).

Such evolution equations have appeared as models in a variety of other physical systems (see, for example, [1], [5] and the references in these articles for an indication of the range of applicability of equations like (1.1)). In most cases, the independent variable x characterizes position in the medium of propagation whilst t is proportional to elapsed time. The dependent variable u may be an amplitude, a pressure, a velocity or other measurable quantity, depending upon the physical system and the modeling stance taken.

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Since the early foray in [4], many works have appeared devoted to initial-boundary-value problems for nonlinear, dispersive wave equations, including the Korteweg-de Vries equation and some of its generalizations, generalized versions of the BBM equation and the cubic nonlinear Schrödiner equation. We mention as a sample [8], [9], [11], [12] [13], [15], [16], [17], [24] and [25], but point out that there are many further references in the works just cited.

The present essay adds to this general discussion. In a recent paper [14], it was shown that the pure initial-value problem for the *BBM*–equation with initial data given in all of $\mathbb{R}$ features global well-posedness in the L_2–based Sobolev classes $H^s(\mathbb{R})$ provided only that $s \geq 0$ (see also the related work on generalized *BBM*-equations in [6]). It was also shown in [14] that in a certain sense, this result is best possible. This well-posedness result improved upon the original work in [2] which showed global well-posedness in H^s for integer values of $s \geq 1$ for the pure initial-value problem for (1.1) with both localized and bore-like initial data. For the initial-boundary-value problem

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, \\ u(x,0) = g(x), \\ u(0,t) = h(t) \end{cases} \tag{1.2}$$

with $x, t \geq 0$, local well-posedness is known in quite general circumstances. For example, the problem is known to be locally well posed if g is simply assumed to be a bounded, continuous function on $\mathbb{R}^+$, h is assumed to be continuous and the compatibility condition $g(0) = h(0)$ holds (see [4], [7], [9]). For global well-posedness, the principal topic of the present essay, the initial datum has been assumed to decay to zero as $x \to \infty$. More precisely, the theory to date is based on the conditions

$$\begin{aligned} &g(x) \in H^1(\mathbb{R}^+) \cap C_b^2(\mathbb{R}^+), \\ &h(t) \in C^1(\mathbb{R}^+), \end{aligned} \tag{1.3}$$

together with the consistency condition $h(0) = g(0)$. Under these assumptions, a globally defined, classical solution to the problem (1.2) obtains (see [4]).

It is the aim here to bring the theory for (1.2) more closely into line with that appearing in [14] for the pure initial-value problem. The next section introduces an integral equation equivalent to (1.2) and provides a detailed statement of the results in view. Analysis leading to the conclusions advertised in the main result is presented in Sections 3 and 4. Section 3 is a short proof of local well-posedness, while Section 4 provides a proof of global well-posedness. The latter proceeds via a decompostion, analogous to that used in [14], of the initial data into a small, rough part and a smooth portion along with an associated decomposition of the boundary data into a small part and a locally bounded remainder. The function-class notation is mostly standard, but it is reviewed for the reader's convenience in Appendix A while the detailed construction of the decomposition used in Section 4 is indicated in Appendix B.

2. **Integral equation formulation and statement of the main result.** The problem under consideration is the initial-boundary-value problem (1.2) posed in the quarter-plane $\{(x,t) : x,\, t \geq 0\}$. The principal issue dealt with here is global well-posedness of this quarter-plane problem. As mentioned, this system is a model for long water waves of small but finite amplitude, generated by a wave-maker at the left-hand end and propagating to the right in a uniform, open channel. Variable

coefficient versions of this problem have also been used to model long-crested, near-shore zone waves incoming from deep water (see, for example, [3] and [18]). The channel is taken to be infinitely long to avoid dealing with a boundary condition at the right. Indeed, the model equation (1.1) is only valid for waves moving to the right. In a real channel, as soon as the wave motion reaches the right-hand end of the channel, comparison between theory and experiment needs to cease because the reflected waves are not described by this one-way equation (see [10]). Theory comparing solutions of (1.2) with those of an associated two-point boundary value problem suitable for use in computer simulations can be found in [7].

Under the assumptions (1.3), Bona and Bryant [4] showed that there exists a unique classical solution u of (1.2) which, for each $T > 0$, lies in $C(0,T;H^1(\mathbb{R}))$ and is such that $\partial_t^i \partial_x^j u$ is a bounded, continuous function for $(x,t) \in \mathbb{R}^+ \times [0,T]$ and $i = 0,1$, $j = 0,1,2$. The theory developed in [4] began by noting that with the assumptions (1.3), the solution of the problem (1.2) satisfies the integral equation

$$\begin{aligned} u(x,t) = g(x) + \big(h(t) - h(0)\big)e^{-x} \\ + \int_0^t \int_0^\infty k(x,y)\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)dyds, \end{aligned} \tag{2.1}$$

where

$$k(x,y) = \frac{1}{2}\Big(e^{-(x+y)} + \operatorname{sign}(x-y)e^{-|x-y|}\Big). \tag{2.2}$$

This integral equation is derived by noting that if the differential equation is written in the form

$$\big(I - \partial_x^2\big)u_t \;=\; \big(I - \partial_x^2\big)v \;=\; -\partial_x\big(u + \frac{1}{2}u^2\big) = f, \tag{2.3}$$

where t is viewed as fixed, $v(0)$ is given and $v(x)$ is required to remain bounded as $x \to \infty$, then necessarily

$$v(x) = (I - \partial_x^2)^{-1} f = v(0)e^{-x} - \int_0^\infty \frac{1}{2}\Big(e^{-(x+y)} - e^{-|x-y|}\Big) f(y)\, dy. \tag{2.4}$$

In this case, $f(y) = -\partial_y(u + \frac{1}{2}u^2)(y,t)$ and $v(0) = u_t(0,t)$, so an integration by parts in the y–variable followed by an integration in time over $[0,t]$ yields (2.1).

Local well posedness is then a straightforward application of the contraction mapping principle applied to the mapping defined by the right-hand side of (2.1). Global well posedness obtains by deriving *a priori* bounds on the H^1–norm of the solution and using these to show that iterating the local theory leads to solutions defined on arbitrarily large temporal intervals.

In this article, we deal with well-posedness, globally in time, for the problem (1.2) supplemented with initial and boundary data having the weaker regularity conditions

$$g(x) \in L_2(\mathbb{R}^+), \qquad h(t) \in L^\infty_{loc}(\mathbb{R}^+) \tag{2.5}$$

and g and h are such that the consistency condition

$$g(0) \;=\; h(0) \tag{2.6}$$

makes sense and holds good. The condition (2.6) is formally derived from the observation that $u(0,0)$ has both the values $h(0)$ and $g(0)$, depending upon how the point $(0,0)$ is approached. Thus, both g and h are taken to be continuous at the origin and the condition (2.6) is presumed to hold. It will be convenient to have the notation

$$L_T^0 = \Big\{(g,h) \in L_2(\mathbb{R}^+) \times L_\infty([0,T]) \ : \quad g,\, h \text{ are continuous at 0 and } g(0) = h(0)\Big\} \tag{2.7}$$

for the class of admissible data. This class is viewed as a linear subspace of $L_2(\mathbb{R}^+)\times L_\infty([0,T])$. Of course, it is not closed in the inherited norm

$$\|(g,h)\|_{L_T^0} \ = |g|_2 + |h|_{L_\infty([0,T])}.$$

Under the weak regularity assumptions (2.5), it is no longer expected that there is a classical solution to the BBM–equation (1.1). Instead, we search for solutions in the space $\mathcal{L}_T = L_\infty(0,T;L_2(\mathbb{R}^+))$. It will turn out that the solutions obtained have a trace at $x = 0$ and at $t = 0$, which is of course not implied simply by membership in $\mathcal{L}_T$.

A function $u(x,t) \in \mathcal{L}_T$ solves the equation (1.1) on the half-line $\mathbb{R}^+$ in the *integral sense* if u satisfies (2.1). Elementary considerations reveal that if (g,h) satisfies (2.5) and (2.6), then solutions of the integral equation (2.1) solve the initial-boundary-value problem (1.2) in the sense of distributions. Indeed, it is clear that a solution of (2.1) satisfies the initial condition since $h(t) \to h(0)$ and the double integral term vanishes as $t \downarrow 0$. Because $k(x,y) \to 0$ as $x \downarrow 0$, for all $y \geq 0$, the dominated convergence theorem implies that the double integral again tends to zero in the limit $x \downarrow 0$. Since $g(x) \to h(0)$ as $x \downarrow 0$, it thus follows that the boundary condition is also satisfied. The fact that a solution of (2.1) is a distributional solution of (1.1) follows from the observation that

$$u_t \ = h'(t)e^{-x} + \int_0^\infty k(x,y)\Big(u(y,s) + \frac{1}{2}u^2(y,s)\Big)\, dyds$$

by the fundamental theorem of calculus together with the the formula

$$(I - \partial_x^2)\int_0^\infty k(x,y)f(y)\,dy = f_x$$

(see (3.5)). Of course, h' and f_x are taken in the sense of distributions.

Conversely, distributional solutions of the initial-boundary-value problem (1.2) that lie in $\mathcal{L}_T$ are solutions of the integral equation (2.1). This latter point is seen by following the steps outlined earlier for the derivation of the integral equation. Thus the two problems are equivalent as far as $\mathcal{L}_T$–solutions are concerned and we will not continue to distinguish between them.

The following theorem is the principal result of this article.

Theorem 2.1. *Suppose admissible initial and boundary data (g,h) are provided that satisfy (2.5), have (g,h) continuous at the origin and are such that the consistency condition*

$$g(0) = h(0)$$

holds. Then, there exists a unique solution $u = u(x,t) \in L^\infty_{loc}(\mathbb{R}^+;L_2(\mathbb{R}^+))$ that solves the initial-boundary-value problem (1.2). Moreover, the correspondence between initial and boundary data (g,h) and the associated solution u of (1.2) is, for any $T > 0$, a Lipschitz continuous mapping from $L_2(\mathbb{R}^+) \times L_\infty([0,T])$ into $\mathcal{L}_T = L_\infty([0,T];L_2(\mathbb{R}^+))$.

Proof. This theorem will be proved in three steps. First, it is shown that the quarter-plane problem (1.2) is locally well posed in $\mathcal{L}_{T^*}$ for small $T^* > 0$. Second, for arbitrarily given $T > 0$, we prove that the solution exists on the temporal interval $[0, T]$ provided the initial and boundary data are small enough. Once this latter result is in hand, general initial and boundary data are split into two parts. The first part is small whilst the second part, which need not be small, is smooth. Solve (1.2) with the small part of the data first. If the small part is small enough, the resulting solution will exist on the time-interval $[0, T]$. With this function in hand, pose a perturbed initial-boundary-value problem with the smooth part of the original data as auxiliary conditions. We prove that the solution of this perturbed equation survives at least for $0 \leq t \leq T$. The solution of the quarter-plane problem with the original data is then recovered by summing the two partial solutions. The well-posedness on the time interval $[0, T]$ then follows from the local well-posedness result. Since the time T is arbitrarily given, it is concluded that the initial-boundary-value problem (1.2) is globally well posed. The details of this scheme will be worked out in the remainder of the paper.

3. **Local well-posedness.**

As in [4], we search for a function u which is the solution of the integral equation (2.1) where g, h satisfy the regularity and compatibilty conditions described in (2.5) and (2.6).

Introduce the new dependent variable

$$v(x,t) = u(x,t) - h(t)e^{-x}. \tag{3.1}$$

Then, the function u is a solution of (2.1) if and only if v is a solution of the integral equation

$$\begin{aligned} v(x,t) = \; & g(x) - h(0)e^{-x} + \int_0^t \int_0^\infty k(x,y)\Big(v(y,s) + \frac{1}{2}v^2(y,s) \\ & + h(s)e^{-y}v(y,s) + h(s)e^{-y} + \frac{1}{2}h^2(s)e^{-2y}\Big)dyds, \end{aligned}$$

where k is as in (2.2). The explicit expression (2.2) for k and exact integration of exponentials reduces the last equation to

$$\begin{aligned} v(x,t) = \; & g(x) - h(0)e^{-x} + \frac{1}{2}xe^{-x}\int_0^t h(s)\,ds + \frac{1}{3}e^{-x}(1-e^{-x})\int_0^t h^2(s)ds \\ & + \int_0^t \int_0^\infty k(x,y)\Big(v(y,s) + h(s)e^{-y}v(y,s) + \frac{1}{2}v^2(y,s)\Big)dyds. \end{aligned} \tag{3.2}$$

Notice that the operator

$$\mathcal{K}(f)(x) = \int_0^\infty k(x,y)f(y)\,dy \tag{3.3}$$

is a bounded linear operator with

$$|\mathcal{K}f|_\infty \leq |f|_\infty, \quad |\mathcal{K}f^2|_2 \leq |f^2|_1 = |f|_2^2 \quad \text{and} \quad |\mathcal{K}f|_2 \leq |f|_2. \tag{3.4}$$

In fact, $\mathcal{K}$ is also a smoothing operator, mapping $L_2(\mathbb{R}^+)$ to $H^1(\mathbb{R}^+)$. More precisely, if $f \in L_2(\mathbb{R}^+)$, then $\mathcal{K}f \in H^1(\mathbb{R}^+)$ and

$$\frac{d}{dx}\mathcal{K}f(x) = f(x) - \int_0^\infty m(x,y)f(y)\,dy = f(x) - \mathcal{M}f(x), \tag{3.5}$$

pointwise almost everywhere and in the sense of distributions, where

$$m(x,y) \;=\; \frac{1}{2}\Big(e^{-(x+y)}+e^{-|x-y|}\Big). \tag{3.6}$$

It follows immediately from this formula and (3.4) that

$$\|\mathcal{K}f\|_{H^1(\mathbb{R}^+)} \le 3|f|_2. \tag{3.7}$$

For given initial and boundary data (g,h) and $v \in C([0,T];L_2(\mathbb{R}^+))$, define the operator $\mathcal{A}=\mathcal{A}_{(g,h)}$ by

$$\begin{aligned}\mathcal{A}_{(g,h)}(v)(x,t) \;=\;& g(x)-h(0)e^{-x}\\ &+\frac{1}{2}xe^{-x}\int_0^t h(s)\,ds+\frac{1}{3}e^{-x}(1-e^{-x})\int_0^t h^2(s)ds\\ &+\int_0^t\int_0^\infty k(x,y)\Big(v(y,s)+h(s)e^{-y}v(y,s)+\frac{1}{2}v^2(y,s)\Big)dyds.\end{aligned} \tag{3.8}$$

Since the boundary data h is assumed to lie in $L^\infty_{loc}(\mathbb{R}^+)$, it follows that $\int_0^t h(s)ds$, and, $\int_0^t h^2(s)ds$ lie in $C(\mathbb{R}^+)$. This, together with the fact that $g\in L_2(\mathbb{R}^+)$, implies $\mathcal{A}$ maps $C(\mathbb{R}^+;L_2(\mathbb{R}^+))$ to itself.

From (3.4), it is also inferred that, for arbitrary $T>0$ and v in the space $\mathcal{C}_T = C([0,T];L_2(\mathbb{R}^+))$,

$$\begin{aligned}\sup_{0\le t\le T}|\mathcal{A}_{(g,h)}(v)|_2 \le\;& |g|_2+|h|_{L_\infty([0,T])}\\ &+T\left(\frac{1}{4}|h|_{L_\infty([0,T])}+\frac{1}{6}|h|^2_{L_\infty([0,T])}\right)\\ &+T\left(\Big(1+|h|_{L_\infty([0,T])}\Big)\sup_{0\le t\le T}|v(\cdot,t)|_2+\frac{1}{2}\sup_{0\le t\le T}|v(\cdot,t)|_2^2\right),\end{aligned} \tag{3.9}$$

or, what is the same,

$$\begin{aligned}\|\mathcal{A}(v)\|_{\mathcal{C}_T} \le\;& |g|_2+|h|_{L_\infty([0,T])}+T\left(\frac{1}{4}|h|_{L_\infty([0,T])}+\frac{1}{6}|h|^2_{L_\infty([0,T])}\right)\\ &+T\Big(\big(1+|h|_{L_\infty([0,T])}\big)\|v\|_{\mathcal{C}_T}+\frac{1}{2}\|v\|^2_{\mathcal{C}_T}\Big).\end{aligned} \tag{3.10}$$

Similarly, if $v,w\in\mathcal{C}_T$, then

$$\|\mathcal{A}(w)-\mathcal{A}(v)\|_{\mathcal{C}_T} \le T\Big(1+|h|_{L_\infty([0,T])}+\frac{1}{2}\|u+v\|_{\mathcal{C}_T}\Big)\|u-v\|_{\mathcal{C}_T}. \tag{3.11}$$

Hence, if the positive numbers R and T are chosen so that

$$R=2\left(|g|_2+|h|_{L_\infty([0,T])}+T\Big(\frac{1}{4}|h|_{L_\infty([0,T])}+\frac{1}{6}|h|^2_{L_\infty([0,T])}\Big)\right),$$

and

$$(1+R)T=\frac{1}{2},$$

it follows immediately from the last two inequalities that $\mathcal{A}$ is a contraction mapping of the closed ball $B_R(0)$ of radius R centered at the origin in $\mathcal{C}_T$. It is concluded that there is a unique solution v of (3.2) which lies in $\mathcal{C}_T$.

Moreover, the solution map $(g,h)\mapsto v$ that carries initial data into the corresponding solution of (3.2) is locally Lipschitz continuous. This fact derives from the triangle inequality. Indeed, let v correspond to (g,h) and let $\widetilde{v}$ correspond to $(\widetilde{g},\widetilde{h})$.

Then, on some positive time interval $[0,T^*]$, $v = \mathcal{A}_{(g,h)}(v)$ and $\widetilde{v} = \mathcal{A}_{(\widetilde{g},\widetilde{h})}(\widetilde{v})$. It thus transpires that

$$\begin{aligned}\|v-\widetilde{v}\|_{\mathcal{C}_{T^*}} &= \|\mathcal{A}_{(g,h)}(v) - \mathcal{A}_{(\widetilde{g},\widetilde{h})}(\widetilde{v})\|_{\mathcal{C}_{T^*}} \\ &\leq \|\mathcal{A}_{(g,h)}(v) - \mathcal{A}_{(g,h)}(\widetilde{v})\|_{\mathcal{C}_{T^*}} + \|\mathcal{A}_{(g,h)}(\widetilde{v}) - \mathcal{A}_{(\widetilde{g},\widetilde{h})}(\widetilde{v})\|_{\mathcal{C}_{T^*}} \\ &\leq \theta\|v-\widetilde{v}\|_{\mathcal{C}_{T^*}} + |g-\widetilde{g}|_2 + |h-\widetilde{h}|_{L_\infty([0,T^*])} \\ &\quad + T^*\left(\frac{1}{4} + \frac{1}{6}|h+\widetilde{h}|_{L_\infty([0,T^*])}\right)|h-\widetilde{h}|_{L_\infty([0,T^*])}\end{aligned}$$

where $\theta < 1$ is the contraction constant for the operator $\mathcal{A}_{(g,h)}$. In consequence,

$$\|v-\widetilde{v}\|_{\mathcal{C}_{T^*}} \leq \frac{1}{1-\theta} C\,\|(g,h) - (\widetilde{g},\widetilde{h})\|_{L_2(\mathbb{R}^+)\times L_\infty([0,T^*])},$$

where C depends upon the radius R of the ball where $\mathcal{A}_{(g,h)}$ is contractive.

The following lemma has been established.

Lemma 3.1. *Suppose compatible initial and boundary data (g,h) are given satisfying (2.5)-(2.6). Then, there is a $T^* > 0$ and a unique solution $v = v(x,t) \in \mathcal{C}_{T^*} = C([0,T^*];L_2(\mathbb{R}^+))$ to the integral equation (3.2). Moreover, the correspondence between initial and boundary data (g,h) and the associated solution v of (3.2) is a Lipschitz continuous mapping of any bounded subset B of the class L_T^0 of consistent initial data into $\mathcal{C}_{T^*}$ provided $T^* = T^*(B)$ is chosen sufficiently small.*

Corollary 3.2. *Suppose initial and boundary data (g,h) lies in $L_2(\mathbb{R}^+)\times L_{loc}^\infty(\mathbb{R}^+)$ and that the consistency condition*

$$\lim_{t\downarrow 0} g(t) = g(0) = h(0) = \lim_{x\downarrow 0} h(x)$$

holds. Then, there is a $T^* > 0$ and a unique solution $u = u(x,t) = v(x,t) + h(t)e^{-x} \in \mathcal{L}_{T^*}$ that solves the initial-boundary-value problem (1.2). Moreover, the correspondence between initial and boundary data (g,h) and the associated solution u of (1.2) is Lipschitz continuous on any bounded subset B of the class $L_{T^*}^0$ (see (2.7)) into $\mathcal{L}_{T^*}$ provided $T^* = T^*(B)$ is chosen sufficiently small.

4. Long time existence.

As the problem is locally well-posed, establishing global well-posedness only requires showing that the solution obtained from the local theory can be extended to any time interval $[0,T]$, where $T>0$ is arbitrary. This is accomplished in two steps. The first contemplates only small auxiliary data.

4.1. Long time existence for small data.

Consider again the dependent variable $v(x,t) = u(x,t) - h(t)e^{-x}$ introduced in (3.1). It satisfies the initial-boundary-value problem

$$\begin{aligned} v_t + v_x + vv_x + h(t)(e^{-x}v)_x - v_{xxt} &= h^2(t)e^{-2x} + h(t)e^{-x} \\ v(x,0) = g(x) - h(0)e^{-x} &= g(x) - g(0)e^{-x}, \quad v(0,t) \equiv 0. \end{aligned} \tag{4.1}$$

The integral equation corresponding to (4.1) is expressed in (3.2) and is known on account of Lemma 3.1 to have a solution $v \in \mathcal{C}_{T^*}$ for $T^* > 0$ sufficiently small. To establish that the solution v corresponding to sufficiently small auxiliary data (g,h) has a time-interval of existence that contains $[0,T]$ for an arbitrarily specified $T > 0$, it suffices to derive *a priori* bounds in $\mathcal{C}_T$ on a putative solution v of (4.1) and iterate the local result in Lemma 3.1. Once v is known to exist on the time interval $[0,T]$, then $u(x,t) = v(x,t) + h(t)e^{-x}$ is inferred to exist on the same time interval and solves the original problem (1.2).

The derivation of appropriate *a priori* bounds is the subject of the next lemma.

Lemma 4.1. *For any given $T > 0$, there exists an $\epsilon = \epsilon(T) > 0$ such that if $\|(g,h)\|_{L_T^0} \le \epsilon$, then the solution v of (3.2) exists on the time interval $[0,T]$ and is an element of $\mathcal{C}_T$.*

Proof. Let $\epsilon > 0$ connote a bound on the norm of the initial and boundary data (g,h) in L_T^0, which is to say, $|g|_2 + |h|_{L_\infty([0,T])} \le \epsilon$. In due course, ϵ will be taken small, but for the moment it is unrestricted.

Upon differentiating the integral equation (3.2) with respect to t, there emerges the formula

$$\begin{aligned} v_t(x,t) = &\frac{1}{2}xe^{-x}h(t) + \frac{1}{3}e^{-x}(1-e^{-x})h^2(t) \\ &+ \int_0^\infty k(x,y)\Big(v(y,t) + \frac{1}{2}v^2(y,t) + h(t)e^{-y}v(y,t)\Big)dy. \end{aligned} \tag{4.2}$$

One sees immediately that v_t lies in $\mathcal{L}_{T^*}$. Multiply the integral equation (4.2) by v and integrate the result over the half-line $\mathbb{R}^+$ with respect to the spatial variable x to obtain the relation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^\infty v^2(x,t)\,dx \\ &\quad= \frac{1}{2}h(t)\int_0^\infty xe^{-x}v(x,t)\,dx + \frac{1}{3}h^2(t)\int_0^\infty e^{-x}(1-e^{-x})v(x,t)\,dx \\ &\quad+ \int_0^\infty\int_0^\infty k(x,y)\,v(x,t)\Big(v(y,t) + \frac{1}{2}v^2(y,t)\Big)\,dydx \\ &\quad+ \int_0^\infty\int_0^\infty k(x,y)\Big(v(x,t)h(t)e^{-y}v(y,t)\Big)\,dydx. \end{aligned} \tag{4.3}$$

This formal calculation is easily justified (see, for example, [25]).

The following collection of integral inequalities comes to our aid in the analysis of (4.3) (see (3.4), (3.5)). First observe that

$$I_0 = \frac{1}{2}h(t)\int_0^\infty xe^{-x}v(x,t)\,dx + \frac{1}{3}h^2(t)\int_0^\infty e^{-x}(1-e^{-x})v(x,t)\,dx$$

is bounded in terms of ϵ, *viz*

$$|I_0| \le \left(\frac{1}{4}\epsilon + \frac{1}{6}\epsilon^2\right)|v(\cdot,t)|_2 \tag{4.4}$$

and that

$$\begin{aligned} I_1 &= \int_0^\infty\int_0^\infty e^{-(x+y)}v(y,t)v(x,t)dydx \\ &= \big\langle e^{-x}, v(x,t)\big\rangle^2 \le \frac{1}{2}|v(\cdot,t)|_2^2. \end{aligned} \tag{4.5}$$

Define I_2 to be the integral

$$I_2 = \int_0^\infty\int_0^\infty \operatorname{sign}(x-y)e^{-|x-y|}v(y,t)v(x,t)dydx$$

and remark that

$$\left|\int_0^\infty \int_0^\infty \operatorname{sign}(x-y)e^{-|x-y|}v(y,t)v(x,t)dydx\right| \le |v(\cdot,t)|_2^2.$$

Hence, I_2 is finite and since $I_2 = -I_2$ by Fubini's Theorem, so $I_2 = 0$. It follows that

$$\left|\int_0^\infty \int_0^\infty k(x,y)v(y,t)v(x,t)dydx\right| \le \frac{1}{2}\Big(|I_1| + |I_2|\Big) \le \frac{1}{4}|v(\cdot,t)|_2^2. \tag{4.6}$$

Next, consider the two integrals

$$I_3 = \int_0^\infty \int_0^\infty e^{-(x+y)}v^2(y,t)v(x,t)dydx$$

and

$$I_4 = \int_0^\infty \int_0^\infty \operatorname{sign}(x-y)e^{-|x-y|}v^2(y,t)v(x,t)dydx.$$

It is readily ascertained that

$$|I_3| = \Big|\big\langle e^{-y}, v^2(y,t)\big\rangle\big\langle e^{-x}, v(x,t)\big\rangle\Big| \le \frac{1}{\sqrt{2}}|v(\cdot,t)|_2^3$$

and, by first estimating the integral with respect to x, that

$$|I_4| \le |v(\cdot,t)|_2 \int_0^\infty \big(1 - \frac{1}{2}e^{-2y}\big)v^2(y,t)\,dy \le |v(\cdot,t)|_2^3.$$

Consequently, the inequality

$$\Big|\int_0^\infty \int_0^\infty k(x,y)v^2(y,t)v(x,t)dydx\Big| \le \frac{1}{2}\big(|I_3| + |I_4|\big) \le |v(\cdot,t)|_2^3 \tag{4.7}$$

emerges. Continuing in this vein, it is found that

$$\Big|\int_0^\infty \int_0^\infty k(x,y)v(x,t)v(y,t)h(t)e^{-y}\,dydx\Big| \le \frac{3}{4}\epsilon|v(\cdot,t)|_2^2. \tag{4.8}$$

If (4.4), (4.5), (4.7) and (4.8) are combined and used to estimate the size of the right-hand side of (4.3), there obtains the differential inequality

$$\frac{d}{dt}|v(\cdot,t)|_2^2 \le \frac{1}{2}|v(\cdot,t)|_2^2 + |v(\cdot,t)|_2^3 + \frac{3}{2}\epsilon|v(\cdot,t)|_2^2 + \left(\frac{1}{4}\epsilon + \frac{1}{6}\epsilon^2\right)|v(\cdot,t)|_2. \tag{4.9}$$

If it is presumed that $\epsilon < 1$, then the latter inequality implies that

$$\frac{d}{dt}|v(\cdot,t)|_2 \le (1+\epsilon)|v(\cdot,t)|_2 + |v(\cdot,t)|_2^2 + \epsilon. \tag{4.10}$$

Let $\sigma(t)$ satisfy the differential equation

$$\sigma'(t) = (1+\epsilon)\sigma(t) + \sigma(t)^2 + \epsilon = \big(\sigma(t)+1\big)\big(\sigma(t)+\epsilon\big) \tag{4.11}$$

with $\sigma(0) = \epsilon$. Then, so long as $\sigma(t)$ remains finite, it will be positive and provide a finite upper bound for $|v(\cdot,t)|_2$.

Elementary considerations reveal that

$$\sigma(t) = \frac{\left(\frac{2\epsilon}{\epsilon+1}\right)e^{(1-\epsilon)t} - \epsilon}{1 - \left(\frac{2\epsilon}{\epsilon+1}\right)e^{(1-\epsilon)t}}. \tag{4.12}$$

The right-hand side of the last equation is certainly positive and finite as long as

$$t < \log\Big(\frac{1}{2\epsilon}\Big).$$

It follows that if $T > 0$ is given, then the solution v of the integral equation (3.2) exists at least on the time interval $[0, T]$ whenever $\epsilon = \epsilon(T) < \frac{1}{2}e^{-T}$. □

Corollary 4.2. *For any $T > 0$, there is an $\epsilon = \epsilon(T) > 0$ such that if $(g, h) \in L^0_T$ with $\|(g, h)\|_{L^0_T} \le \epsilon$, then the initial-boundary-value problem (1.2) has a unique distributional solution u lying in $\mathcal{L}_T$.*

4.2. **An auxiliary BBM-equation with a variable coefficient.** Consider the variable-coefficient initial-boundary-value problem

$$\begin{cases} w_t + w_x + ww_x + (vw)_x - w_{xxt} = 0, \\ w(x, 0) = g(x), \\ w(0, t) = h(t), \end{cases} \tag{4.13}$$

posed for $x, t \ge 0$, where $v = v(x, t)$ need not be constant. This equation will intervene in the analysis of the original initial-boundary-value problem (1.2) with rough auxiliary data.

Lemma 4.3. *Suppose that $g \in H^1(\mathbb{R}^+)$ and $h \in L^\infty_{loc}(\mathbb{R}^+)$ with h continuous at 0 and $g(0) = h(0) = \lim_{t\downarrow 0} h(t)$. If $v \in \mathcal{L}_T$ for some positive number T, then the initial-boundary-value problem (4.13) has a unique distributional solution $w \in \mathcal{Z}_T = L_\infty([0, T]; H^1(\mathbb{R}^+))$.*

Proof. In analogy with (2.1), the integral equation for the solution of (4.13) is

$$\begin{aligned} w(x, t) =& g(x) + \big(h(t) - h(0)\big)e^{-x} \\ &+ \int_0^t \int_0^\infty k(x, y)\Big(w(y, \tau) + \frac{1}{2}w^2(y, \tau) + (vw)(y, \tau)\Big)dyd\tau, \end{aligned} \tag{4.14}$$

where k is as in (2.2).

First, it is asserted that this integral equation has a solution in the space $\mathcal{Z}_{T^*} = L_\infty([0, T^*]; H^1(\mathbb{R}^+))$, at least for suitably small, positive values of T^*. Define the operator $\mathcal{B} = \mathcal{B}_{(g,h)}$ by

$$\begin{aligned} \mathcal{B}_{(g,h)}(w) =& g(x) + \big(h(t) - h(0)\big)e^{-x} \\ &+ \int_0^t \int_0^\infty k(x, y)\Big(w(y, \tau) + \frac{1}{2}w^2(y, \tau) + (vw)(y, \tau)\Big)dyd\tau. \end{aligned} \tag{4.15}$$

For any $T^* \in [0, T]$, we have

$$\begin{aligned} \sup_{0\le t\le T^*} \big|\mathcal{B}(w)(\cdot, t)\big|_2 \le& |g|_2 + 2|h|_{L_\infty([0,T^*])} \\ &+ T^*\Big(\sup_{0\le t\le T^*} |w(\cdot, t)|_2 + \frac{1}{2}\sup_{0\le t\le T^*} |w(\cdot, t)|_2^2 + \|v\|_{\mathcal{L}_T} \sup_{0\le t\le T^*} |w(\cdot, t)|_2\Big), \end{aligned} \tag{4.16}$$

which is to say,

$$\begin{aligned} &\|\mathcal{B}(w)\|_{\mathcal{L}_{T^*}} \\ \le& |g|_2 + 2|h|_{L_\infty([0,T^*])} + T^*\Big(\|w(\cdot, t)\|_{\mathcal{L}_{T^*}} + \frac{1}{2}\|w(\cdot, t)\|^2_{\mathcal{L}_{T^*}} + \|v\|_{\mathcal{L}_T}\|w(\cdot, t)\|_{\mathcal{L}_{T^*}}\Big). \end{aligned}$$

Formula (3.5) implies that

$$\frac{d}{dx}\mathcal{K}\Big(w+\frac{1}{2}w^2+vw\Big)(x,t) = \Big(w+\frac{1}{2}w^2+vw\Big)(x,t)-\mathcal{M}\Big(w+\frac{1}{2}w^2+vw\Big)(x,t). \quad (4.17)$$

Consequently, the inequality

$$\begin{aligned}\Big|\frac{d}{dx}\mathcal{K}\Big(w+\frac{1}{2}w^2+vw\Big)(\cdot,t)\Big|_2 &\le 2\Big(\|w(\cdot,t)\|_{H^1(\mathbb{R}^+)}\\ &+\frac{1}{2}\|w(\cdot,t)\|^2_{H^1(\mathbb{R}^+)}+\|v\|_{\mathcal{L}_T}\|w(\cdot,t)\|_{H^1(\mathbb{R}^+)}\Big)\end{aligned} \quad (4.18)$$

is seen to hold. It follows that

$$\begin{aligned}\sup_{0\le t\le T^*}\Big|\frac{d}{dx}\mathcal{B}(w)(\cdot,t)\Big|_2 &\le \|g\|_{H^1(\mathbb{R}^+)}+2|h|_{L_\infty([0,T])}\\ &+2T^*\Big(\sup_{0\le t\le T^*}\|w(\cdot,t)\|_{H^1(\mathbb{R}^+)}+\frac{1}{2}\sup_{0\le t\le T^*}\|w(\cdot,t)\|^2_{H^1(\mathbb{R}^+)}\\ &+\|v\|_{\mathcal{L}_T}\sup_{0\le t\le T^*}\|w(\cdot,t)\|_{H^1}\Big).\end{aligned} \quad (4.19)$$

Combining (4.16) and (4.19) yields

$$\begin{aligned}\sup_{0\le t\le T^*}\big\|\mathcal{B}(w)\big\|_{H^1} \;&\le\; 2\|g\|_{H^1}+4|h|_{L_\infty([0,T])}+3T^*\Bigg[\sup_{0\le t\le T^*}\|w(\cdot,t)\|_{H^1}\\ &+\left(\frac{1}{2}\sup_{0\le t\le T^*}\|w(\cdot,t)\|_{H^1}+\|v\|_{\mathcal{L}_T}\right)\sup_{0\le t\le T^*}\|w(\cdot,t)\|_{H^1}\Bigg].\end{aligned} \quad (4.20)$$

From these inequalities, one concludes that if $w_1, w_2 \in B_R(0)$, the ball of radius R centered at the zero function in $\mathcal{Z}_{T^*}$, then

$$\|\mathcal{B}_{(g,h)}(w_1)-\mathcal{B}_{(g,h)}(w_2)\|_{\mathcal{Z}_{T^*}} \le 3T^*\big(1+R+\|v\|_{\mathcal{L}_T}\big)\|w_1-w_2\|_{\mathcal{Z}_{T^*}}. \quad (4.21)$$

Also, if w lies in $B_R(0)$, then

$$\|\mathcal{B}_{(g,h)}(w)\|_{\mathcal{Z}_{T^*}} \le 2\|g\|_{H^1}+4|h|_{L_\infty([0,T])}+3T^*\big(1+R+\|v\|_{\mathcal{L}_T}\big)\|w\|_{\mathcal{Z}_{T^*}}. \quad (4.22)$$

If the radius R is chosen to be $4\big(\|g\|_{H^1}+2\|h\|_{L_\infty([0,T])}\big)$ and then $T^* < T$ is fixed so that $0 < 3T^*(1+R+\|v\|_{\mathcal{L}_T}) \le \frac{1}{2}$, it follows immediately from the last two inequalities that $\mathcal{B}$ is a contraction mapping of the closed ball $B_R(0) \subset \mathcal{Z}_{T^*}$ into itself. It is concluded that there is a unique solution $w \in \mathcal{Z}_{T^*}$ to the integral equation (4.15).

To obtain the existence theory for (4.14) over the full time interval $[0,T]$, it is only necessary to derive an *a priori* estimate for $\|w(\cdot,t)\|_{H^1(\mathbb{R}^+)}$. To achieve this, consider the dependent variable $z(x,t) = w(x,t) - h(t)e^{-x}$ which satisfies the equation

$$z_t + z_x + zz_x + (vz)_x - z_{xxt} = h(t)e^{-x} - \big(h(t)e^{-x}z\big)_x + h^2e^{-2x} - \big(vhe^{-x}\big)_x, \quad (4.23)$$

with initial and boundary data

$$z(x,0) = g(x) - h(0)e^{-x} \quad \text{and} \quad z(0,t) \equiv 0. \quad (4.24)$$

Define an energy-type functional

$$E(t) = \int_0^\infty \big(z^2(x,t)+z_x^2(x,t)\big)\,dx \;=\; \|z(\cdot,t)\|^2_{H^1(\mathbb{R}^+)}. \quad (4.25)$$

The energy identity

$$\frac{1}{2}\frac{d}{dt}E(t) = \int_0^\infty vzz_x dx + \int_0^\infty he^{-x} z dx + \frac{1}{2}\int_0^\infty he^{-x}z^2 dx \\ + \int_0^\infty h^2 e^{-2x} z dx + \int_0^\infty vhe^{-x} z_x dx \tag{4.26}$$

is readily derived by multiplying (4.23) by z, integrating the result over the half-line $\mathbb{R}^+$ and integrating by parts suitably. Straightforward estimates allow the conclusion

$$\frac{1}{2}\frac{d}{dt}\|z\|_{H^1}^2 \leq \Big(\|v\|_{\mathcal{L}_T} + 2|h|_{L_\infty([0,T])}\Big)\|z\|_{H^1}^2 \\ + \Big(|h|^2_{L_\infty([0,T])} + |h|_{L_\infty([0,T])} + |h|_{L_\infty([0,T])}\|v\|_{\mathcal{L}_T}\Big)\|z\|_{H^1}, \tag{4.27}$$

from which the differential equality

$$\frac{d}{dt}\|z(\cdot,t)\|_{H^1} \leq C_1\|z\|_{H^1} + C_2, \tag{4.28}$$

follows directly. Here, the constants C_1 and C_2 are

$$C_1 = \|v\|_{\mathcal{L}_T} + 2|h|_{L_\infty([0,T])}\,, \\ C_2 = |h|^2_{L_\infty([0,T])} + |h|_{L_\infty([0,T])} + |h|_{L_\infty([0,T])}\|v\|_{\mathcal{L}_T}\,. \tag{4.29}$$

A Gronwall-type argument thus implies that

$$\|z(\cdot,t)\|_{H^1} \leq e^{C_1 t}\|z(\cdot,0)\|_{H^1} + \frac{C_2}{C_1}\Big(e^{C_1 t} - 1\Big)$$

for $0 \leq t \leq T$. This *a priori* bound on $\|z(\cdot,t)\|_{H^1(\mathbb{R}^+)}$, and hence on $\|w(\cdot,t)\|_{H^1(\mathbb{R}^+)}$, completes the proof of the lemma. □

4.3. **Long-time existence for data of arbitrary size.** In the present subsection, the argument in favor of Theorem 2.1 is completed. As mentioned earlier, long-time existence is established by breaking the solution into two parts, one small and one smooth. More precisely, let $T > 0$ be given and let $(g,h) \in L_T^0$ be arbitrarily-sized initial and boundary data.

Proposition 4.4. *Suppose $g \in L_2(\mathbb{R}^+)$ and that it is continuous at $x = 0$. Then for any $\epsilon > 0$, there exists a decomposition*

$$g = g_v + g_w \tag{4.30}$$

with the properties:

$$g_w \in H^1(\mathbb{R}^+), \quad |g_v|_2 \leq \epsilon \quad \textit{and} \quad |g_v(0)| < \epsilon.$$

Of course, if such a decomposition exists, it is automatic that g_v is continuous at 0 so that $g_v(0)$ makes sense. A proof that such a decomposition always obtains is sketched in Appendix B. To complete the decomposition of the auxiliary data, define

$$h_v(t) \equiv g_v(0) \quad \text{and} \quad h_w(t) = h(t) - h_v(t) \tag{4.31}$$

for $t \geq 0$.

With this decomposition in mind, reconsider problem (1.2) with $g = g_v, h = h_v$. This auxiliary data certainly satisfies (2.5) and (2.6). For given $T > 0$, choose $\epsilon = \epsilon(T)$ sufficiently small that the initial-boundary-value problem

$$\begin{cases} v_t + v_x + vv_x - v_{xxt} = 0, \\ v(x,0) = g_v(x), \\ v(0,t) = h_v(t) \end{cases} \tag{4.32}$$

posed for $x, t \geq 0$ has a unique solution $v \in \mathcal{L}_T$. That this occurs for small enough ϵ is guaranteed by Corollary 4.2.

Next, consider the variable-coefficient initial-boundary-value problem

$$\begin{aligned} &w_t + w_x + ww_x + (vw)_x - w_{xxt} = 0, \\ &w(x,0) = g_w(x), \\ &w(0,t) = h_w(t), \end{aligned} \tag{4.33}$$

where $v \in \mathcal{L}_T$ is the solution of (4.32). From their definitions, it is clear that $(g_w, h_w) \in L^0_T$. Lemma 4.3 implies immediately that (4.33) has a solution w that lies in $\mathcal{Z}_T = L^\infty([0,T]; H^1(\mathbb{R}^+))$.

If we define $u = v + w$, then $u \in \mathcal{L}_T$ and u solves the original initial-boundary-value problem (1.2). As $T > 0$ was arbitrary, this completes the proof of Theorem 2.1 asserting global well posedness for the initial-boundary-value problem (1.2) for arbitrary auxiliary data $(g, h) \in L^0_T$. □

5. **Appendix A: Function classes.** The following notation for function classes is used throughout the discussion. All functions are real–valued and if a spatial domain is not specified for a function class, it is presumed to be the positive real axis $\mathbb{R}^+$. For any Banach space X, its norm will be denoted $\|\cdot\|_X$ with the exceptions noted below.

1. For $1 \leq p < \infty$ and I a real interval, the Lebesgue space $L_p = L_p(I)$ of p^{th}–power integrable functions has its standard norm written $|\cdot|_{L_p(I)}$. The usual modification will be in force when $p = \infty$. In case the interval $I = \mathbb{R}^+$, we write simply $|\cdot|_p$ for the $L_p(\mathbb{R}^+)$–norm. The local version $L^\infty_{loc} = L^\infty_{loc}(\mathbb{R}^+)$, also appears, as does the closed subspace $C(\mathbb{R}^+) \subset L^\infty_{loc}(\mathbb{R}+)$. However, their standard Fréchet-space topologies do not intervene in the analysis.
2. The Sobolev space of $L_2(\mathbb{R}^+)$–functions whose distributional derivatives up to order $m \geq 0$ also lie in $L_2(\mathbb{R}^+)$ is denoted $H^m(\mathbb{R}^+) = H^m$. These spaces carry their standard Hilbert-space structures. The inner product of $f, g \in L_2(\mathbb{R}^+)$ is denoted by $< f, g >$.
3. If X is a Banach space, the space $L_\infty([0,T]; X)$ of measurable and essentially bounded functions from $[0,T]$ to X finds use. The norm on this space is

$$\|u\|_{L_\infty(0,T;X)} = \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(t)\|_X.$$

The notation $C([0,T]; X)$ is used for the closed subspace of $L_\infty([0,T]; X)$ consisting of continuous functions from the interval $[0,T]$ to X. Three spaces of this sort are singled out in our development, namely, $\mathcal{L}_T = L_\infty([0,T]; L_2(\mathbb{R}^+))$, $\mathcal{Z}_T = L_\infty([0,T]; H^1(\mathbb{R}^+))$ and the closed subspace $\mathcal{C}_T = C([0,T]; L_2(\mathbb{R}^+))$ of $\mathcal{L}_T$.

4. The special subset L_T^0 of the product space $L_2(\mathbb{R}^+) \times L_\infty([0,T])$ of pairs such that (g,h) is continuous at $(0,0)$ and satisfies the consistency condition
$$g(0) = h(0)$$
(see (2.6)) is the class from which the auxiliary data is drawn.

6. **Appendix B: Proof of Proposition 4.4.** Let $J \in C^\infty(\mathbb{R})$ have the following properties:
$$J(-x) = J(x) \geq 0, \quad J(x) = 0 \text{ for } |x| > 1 \quad \text{and} \quad \int_{-\infty}^{\infty} J(x)\,dx = 1.$$
For any $\lambda > 0$, define
$$J_\lambda(x) = \frac{1}{\lambda} J\left(\frac{x}{\lambda}\right).$$
If $g \in L_2(\mathbb{R}^+)$, g continuous at 0 and $\lambda > 0$, define
$$g^\lambda(x) = J_\lambda * \widetilde{g}(x) = \int_{-\infty}^{\infty} J_\lambda(x-y)\widetilde{g}(y)\,dy,$$
where $\widetilde{g}$ is the even extension of g. For any fixed $\lambda > 0$, $g^\lambda \in H^1(\mathbb{R})$ is an even function. Moreover,
$$\lim_{\lambda\downarrow 0} g^\lambda(0) = g(0) \quad \text{and} \quad \lim_{\lambda\downarrow 0} \|g^\lambda - g\|_{L_2(\mathbb{R}^+)} = \frac{1}{2}\lim_{\lambda\downarrow 0} \|g^\lambda - \widetilde{g}\|_{L_2(\mathbb{R})} = 0.$$

If g^λ is restricted to $\mathbb{R}^+$, then by choosing λ sufficiently small, we obtain a function g_w as advertised in Proposition 4.4. □

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REFERENCES

[1] L. Abdelohaub, J. L. Bona, M. Felland and J.-C. Saut, Nonlocal models for nonlinear, dispersive waves, *Physica D,* **40** (1989), 360–392.

[2] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. Royal Soc. London, Series A*, **272** (1972), 47–78.

[3] B. Boczar-Karakiewicz, J. L. Bona, W. Romanczyk and E. B. Thornton, Seasonal and interseasonal variability of sand bars at Duck, NC, USA. Observations and model predictions, submitted.

[4] J. L. Bona and P. J. Bryant, A mathematical model for long waves generated by wavemakers in non-linear dispersive systems, *Proc. Cambridge Philos. Soc.,* **73** (1973), 391–405.

[5] J. L. Bona, On solitary waves and their role in the evolution of long waves, in *Applications of Nonlinear Analysis in the Physical Sciences* (ed. H. Amann, N. Bazley and K. Kirchgässner), Pitman, London, 183–205.

[6] J. L. Bona and H. Chen, Well-posedness for regularized nonlinear dispersive wave equations, *Discrete Continuous Dyn. Systems, Series A,* **23** (2009), 1253–1275.

[7] J. L. Bona, H. Chen, S. Sun and B.-Y. Zhang, Comparison of quarter-plane and two-point boundary value problems: The BBM-equation, *Discrete Continuous Dyn. Systems,* **13** (2005), 921–940.

[8] J. L. Bona and V. A. Dougalis, An initial and boundary value problem for a model equation for propagation of long waves, *J. Math. Anal. Appl.,* **75** (1980), 503–522.

[9] J. L. Bona and L. Luo, Initial-boundary-value problems for model equations for the propagation of long waves, in *Evolution Equations* (ed. G. Gerreyra, G. Goldstein and F. Neubrander), Lecture Notes in Pure and Appl. Math., 168, Marcel Dekker, New York, 1995, 65–94.

[10] J. L. Bona, W. G. Pritchard and L. R. Scott, An evaluation of a model equation for water waves, *Philos. Trans. Royal. Soc. London, Series A,* **302** (1981), 457–510.

[11] J. L. Bona, S. Sun and B.-Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane, *Trans. American Math. Soc.,* **354** (2002), 427–490.

[12] J. L. Bona, S. Sun and B.-Y. Zhang, A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain, *Comm. Partial Differential Eqns.*, **28** (2003), 1391–1436.

[13] J. L. Bona, S. Sun and B.-Y. Zhang, Nonhomogeneous boundary-value problems for the one-dimensional nonlinear Schrödinger equation, to appear.

[14] J. L. Bona and N. Tzvetkov, Sharp well-posedness results for the BBM-equation, *Discrete Continuous Dyn. Systems, Series A,* **23** (2009), 1241–1252.

[15] J. L. Bona and V. Varlamov, Wave generation by a moving boundary, *Contemp. Math.,* **371** (2005), 41–71.

[16] J. L. Bona and R. Winther, The Korteweg-de Vries equation posed in a quarter plane, *SIAM J. Math. Anal.,* **14** (1983), 1056–1106.

[17] J. L. Bona and R. Winther, The Korteweg-de Vries equation in a quarter plane, continuous dependence results, *Differential and Integral Eq.,* **2** (1989), 228–250.

[18] M. Chen, Equations for bi-directional waves over an uneven bottom, *Mathematics and Computers in Simulation,* **62** (2003), 3–9.

[19] T. Colin and J.-M. Ghidaglia, An initial-boundary-value problem for the Korteweg-de Vries equation posed on a finite interval, *Adv. Differential Eq.*, **6** (2001), 1463–1492.

[20] T. Colin and M. Gisclon, An initial-boundary-value problem that approximates the quarter-plane problem for the Korteweg-de Vries equation, *Nonlinear Analysis: Theory, Method & Application*, **46** (2001), 869–892.

[21] J. E. Colliander and C. E. Kenig, The generalized Korteweg-de Vries equation on the half line, *Commun. Partial Differential Eqns.,* **27** (2002), 2187–2266.

[22] B. Guo and J. Shen, Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval, *Numer. Math.,* **86** (2000), 635–654.

[23] J. Hammack, A note on tsunamis: Their generation and propagation in an ocean of uniform depth, *J. Fluid Mech.,* **60** (1973), 769–799.

[24] J. Holmer, The initial-boundary value problem for the Korteweg-de Vries equation, *Comm. Partial Differential Eqns.,* **31** (2006), 1151–1190.

[25] H. Kalisch and J. L. Bona, Models for internal waves in deep water, *Discrete Cont. Dynamical Systems,* **6** (2000), 1–20.

[26] D. H. Peregrine, Calculations of the development of an undular bore, *J. Fluid Mech.,* **25** (1966), 321–330.

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