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PROPAGATION OF LONG-CRESTED WATER WAVES. II. BORE PROPAGATION

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Abstract. This essay is concerned with long-crested waves such as those arising in bore propagation. Such motions obtain on rivers when a surge of water invades an otherwise constantly flowing stretch and in the run-up of waves in the near-shore zone of large bodies of water. The dominating feature of the motion is that, in a standard $xyz-$coordinate system in which z increases in the direction opposite to which gravity acts and x increases in the principal direction of propagation, the depth of the fluid approaches a constant value $h_0 > 0$ as $x \to +\infty$ and another value $h_1 > h_0$ as $x \to -\infty$. In an earlier work, the authors developed theory for an idealized model for such waves based on a Boussinesq system of equations. The local well-posedness theory developed in that article applies to the sort of initial data arising in modeling bore propagation. However, well-posedness on the longer, Boussinesq time scale was not dealt with in the case of bore propagation, though such results were established for motions where $h_1 = h_0$.

We argue that without a well-posedness theory at least on the Boussinesq time scale, such models for bore-propagation may not be of any practical use. The issue of well-posedness is complicated by the fact that the total energy of the idealized initial data is infinite.

The theory makes its way via the derivation of suitable approximations with which to compare the full solution. An interesting feature of the theory is the determination of dynamical boundary behavior that is not prescribed, but which the solution necessarily satisfies.

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1. **Introduction.** The present study is concerned with surface water waves. Of particular interest will be long-crested waves whose propagation is primarily along one direction, say the x–coordinate in a standard xyz–Cartesian coordinate system in which the vertical coordinate z increases in the direction opposite to that in which gravity acts. A three-dimensional theory is needed, as variations in the y–directions are allowed. In the present development, both x and y are taken to run over the entire real axis, thus avoiding impermeable, lateral boundaries and allowing us to focus upon the free motion of the fluid under the influence of gravity. It is presumed, however, that the variations in the y–directions subside as y goes to $\pm\infty$, so that at least formally, a two-dimensional description is appropriate there. The problem focussed upon here corresponds to the propagation of bores, or other surges such as those arising in the near-shore zone of a large body of water.

The propagation of tidally generated bores on rivers and bores arising in the later stages of run-up of waves on a beach has attracted attention for centuries. Early theoretical study of this phenomena appears in Airy's article [1]. Well considered accounts of the development of models for bore propagation may be found in the papers of Rajopadhye and her collaborators [12], [23], [24], [25]. Recent field work in the area is reported in the research of Bonneton and his collaborators (see [14] and the references contained therein).

The idealized, physical context of the present study is a layer of incompressible, irrotational, perfect fluid resting upon a horizontal, featureless bottom represented by the plane

$$\Big\{(x, y, z) \colon\ z = -h_0\Big\}.$$

Consistent with the observed properties of bores, the height h of the water column above the point $(x, y, -h_0)$ on the bottom at time t will have the form

$$h = h(x, y, t) = \eta(x, y, t) + h_0, \tag{1}$$

where $h_0 > 0$, $\eta(x, y, t) \to 0$ as $x \to +\infty$ and $\eta(x, y, t) \to \eta_1 = h_1 - h_0 > 0$ as $x \to -\infty$. The value h_0 corresponds to the undisturbed depth of the fluid prior to the invasion of the bore while $\eta(x, y, t)$ is the deviation of the free surface from its undisturbed position $(x, y, 0)$ at time t. The aymptotic behavior of η is meant to mimic an incoming tidal surge of height η_1. Here, and throughout, we posit motion whose free surface remains a graph over the bottom, so that h is a well-defined, positive function. Thus, in the standard parlance, only undular bores are in view, as in the classical theoretical work of Benjamin and Lighthill [5] and in some of the beautiful experiments of Favre [18]. Of course, the presumptions (1) and the asymptotic conditions as $x \to \pm\infty$ pertaining to h are easily enforced for an initial disturbance, but it is part of the theory that such a property continues in time.

Undular bore propagation such as that seen on some rivers fits within the Boussinesq regime and our theory revolves around a Boussinesq system of equations whose validity subsists on the relatively small-amplitude, long-wavelength properties of the flow.

The plan of the paper is the following. Section 2 is devoted to preliminaries, including further commentary about the modeling and a precise mathematical formulation of the problem. One-dimensional, long-time theory is presented in Section 3. This is new in the context of bore propagation and finds essential use in the later sections. Section 4 is devoted to a general result of long-time well-posedness for the full model system. This theory is developed by constructing an approximate solution on the Boussinesq time scale and then providing theory for the difference between the exact and approximate solutions using energy estimates. The approximate solution makes use of the one-dimensional solutions that were constructed in Section 3.

2. **Mathematical models.**

We commence with a brief indication of the mostly standard notation in force. After this is settled, the mathematical model is introduced and an initial-boundary-value problem appropriate to bore propagation set forth.

2.1. **Notation.**

Derivatives with respect to spatial or temporal variables are designated by subscripts x, y, z or t, *e.g.* u_x, u_t, $\cdots$, and also, when convenient, by $\partial_x u, \partial_y u$ or $\partial_t u$. The differential operators Δ and ∇ are always taken with respect to the spatial variables x and y. Thus, $\Delta = \partial_x^2 + \partial_y^2$. We occasionally use the standard multi-index notation ∂^γ, $\gamma \in \mathbb{Z}_+^n$, for n-variable partial derivatives, $n = 2, 3, \cdots$.

Except for the abbreviations noted below, the norm of an element f in a Banach space Z is denoted by $\|f\|_Z$. For $1 \le p \le \infty$, $L_p(\mathbb{R}^n)$ is the space of real-valued, p^{th}–power Lebesgue integrable functions defined on $\mathbb{R}^n$, $n = 1, 2$, with the usual modification if $p = \infty$. The L_p–norm of a function or of a vector-valued function f is indicated by $|f|_p$.

The L_2–based Sobolev spaces appear frequently and the norm of f in $H^k = H^k(\mathbb{R}^n) = W_2^k(\mathbb{R}^n)$ is abbreviated to $\|f\|_k$. It will sometimes be convenient to hang a subscript x or y on the H^k–Sobolev spaces, *viz.* $H_x^k(\mathbb{R})$ or $H_y^k(\mathbb{R})$ to indicate in which variable the norm is being computed. The space $H^\infty = \cap_{k\ge1} H^k$ will appear a couple of times, but its Fréchet space structure will not be needed. The space $\mathcal{C}_b = \mathcal{C}_b(\mathbb{R}^n)$ is the collection of bounded, continuous functions on $\mathbb{R}^n$ with the $L_\infty(\mathbb{R}^n)$–norm. The subspace $\mathcal{C}_b^k(\mathbb{R}^n)$ of k–times continuously differentiable functions whose derivatives up to order k are bounded is likewise given its standard, Banach-space norm.

Spaces that single out the temporal variable will also appear. If $T > 0$ and if Z is a Banach space, the Banach space $\mathcal{C}(0, T; Z)$ is comprised of the continuous mappings from $[0, T]$ to Z with its usual norm. If $k \ge 0$ is an integer, $\mathcal{C}^k(0, T; Z)$ are those functions u such that the Z–valued distributional derivative $\partial_t^j u$ lies in $\mathcal{C}(0, T; Z)$, for all $0 \le j \le k$, with the norm

$$\|u\|_{\mathcal{C}^k(0,T;Z)} = \sum_{j=0}^{k} ||\partial_t^j u||_{\mathcal{C}(0,T;Z)} = \sum_{j=0}^{k} \sup_{0\le t\le T} ||\partial_t^j u(t)||_Z\,.$$

2.2. **Mathematical formulation.**

As mentioned above, a layer of perfect fluid of depth h_0 is presumed to be resting on the plane $\{(x, y, z)\colon\ z = -h_0\}$. It is assumed that the wave motion resulting from a disturbance of the equilibrium has a resulting free surface that is a graph over the flat bottom. In this circumstance, the free surface may be described by the function $\eta = \eta(x, y, t)$ as indicated already in (1). With the additional assumptions that the fluid is incompressible (a good assumption for water in ordinary circumstances) and the flow irrotational (an assumption that requires the scale to be relatively large and holds only over limited time scales), one formulation of the water-wave problem is

$$\begin{cases} \beta\Delta\phi + \phi_{zz} = 0 & \text{in } \{-1 \le z \le \alpha\eta\}, \\ \phi_z = 0 & \text{on } \{z = -1\}, \\ \eta_t + \alpha\nabla\phi\cdot\nabla\eta = \frac{1}{\beta}\phi_z & \text{on } \{z = \alpha\eta\}, \\ \phi_t + \frac{1}{2}\Big(\alpha|\nabla\phi|^2 + \frac{\alpha}{\beta}(\phi_z)^2\Big) + \eta = 0 & \text{on } \{z = \alpha\eta\}, \end{cases} \tag{2}$$

where, as mentioned, Δ and ∇ are the Laplacian and the gradient operators with respect to the variables x and y. The variables in these equations have been non-dimensionalized using the scheme

$$\tilde{x} = \ell x, \quad \tilde{y} = \ell y, \quad \tilde{z} = h_0 z, \quad \tilde{\eta} = A\eta, \quad \tilde{t} = \frac{\ell}{c_0}t\,, \quad \tilde{\phi} = \frac{\ell g A}{c_0}\phi\,, \tag{3}$$

where those surmounted with a tilde are the original, dimensional quantities, $A = \max_{x,y}|\tilde{\eta}(x,y,0)|$ is the maximum amplitude in the initial wave motion, ℓ is the smallest wavelength for which the flow has significant energy, $c_0 = \sqrt{gh_0}$ is the kinematic wave velocity, with g the gravity constant, while the unknown function $\phi = \phi(x,y,z,t)$ is the velocity potential, whose existence follows from incompressibility and irrotationality. The velocity field U is therefore given by $U = (\nabla\phi, \phi_z) = (\phi_x, \phi_y, \phi_z)$ where ∇ denotes the gradient operator in only the (x,y)–variables.

The Boussinesq regime of the water-wave problem is characterized by the parameters

$$\alpha = \frac{A}{h_0} \quad \text{and} \quad \beta = \left(\frac{h_0}{\ell}\right)^2$$

where A, ℓ and h_0 are as above. Assume that both α and β are relatively small compared to one, and that the Stokes number $S = \alpha/\beta$ is of order one, throughout the time interval during which the motion is considered. That such a presumption can be inferred from conditions on the initial data is a consequence of the work of, for example, Alvarez-Samaniego and Lannes [3]. In the circumstances just delineated, a formal expansion of the velocity potential in the vertical coordinate, followed by ignoring all terms of quadratic order or higher in the quantities α and β, leads to the set of *abcd*-systems (coupled systems of three nonlinear evolution equations, see [7, 8]),

$$\begin{cases} V_t + \nabla\eta + \frac{\alpha}{2}\nabla|V|^2 + \beta\Big(a\,\Delta\nabla\eta - b\,\Delta V_t\Big) = 0, \\ \eta_t + \nabla\cdot V + \alpha\nabla\cdot(\eta V) + \beta\Big(c\,\Delta\nabla\cdot V - d\,\Delta\eta_t\Big) = 0. \end{cases} \tag{4}$$

The coefficients a, b, c and d are

$$a = \frac{1-\theta^2}{2}\mu, \quad b = \frac{1-\theta^2}{2}(1-\mu), \quad c = \Big(\frac{\theta^2}{2} - \frac{1}{6}\Big)\lambda, \quad d = \Big(\frac{\theta^2}{2} - \frac{1}{6}\Big)(1-\lambda),$$

where λ and μ are real parameters that, formally, may be be chosen without restriction, and θ lies in the interval $[0,1]$. The dependent variable $z = \eta(x,y,t)$ is the deviation of the free surface from its rest position $(x,y,0)$ at the time t, as already discussed. (Thus the free surface is $\{(x,y,\eta(x,y,t))\colon (x,y)\in\mathbb{R}^2\}$ at time t.) The variable $V = V_\theta(x,y,t) = (u_\theta, v_\theta)$ is the horizontal velocity field at the height θ above the bottom. Notice that because z has been scaled by h_0, the undisturbed depth in these variables is 1.

Members of the subclass of locally well-posed systems provide approximations of the solutions of the Euler system (2) (see [10] for rigourous theory in this direction). Indeed, the systems in this subclass provide direct approximations of the deviation of the free surface and of the horizontal velocity field $V = V_\theta$ at the height θ above the bottom (at the vertical coordinate $z = \theta - 1$), where θ has a fixed value (again, with $0 \le \theta \le 1$, since the scaled height is measured in depths). A short additional calculation using the formula

$$\begin{aligned} V(x,y,z,t) &= \left(1 - \frac{(1-\theta)^2 - z^2}{2}\beta^2\Delta\right) V(x,y,\theta-1,t) \\ &= \left(1 - \frac{(1-\theta)^2 - z^2}{2}\beta^2\Delta\right) V_\theta(x,y,t) \end{aligned}$$

yields an approximation to the horizontal velocity field at heights other than θ above the bottom. At the Boussinesq level of approximation, the vertical velocity is quadratic in the small parameter β, and so ignored. There is a substantial theory pertaining to the initial-value problem for these systems on the Boussinesq time scale $O(\frac{1}{\varepsilon})$ when the initial data is assumed to decay to zero (see [2], [20], [21], [26] for the long-crested situation where variations in the y–directions are ignored and [16] when the waves are three-dimensional). Work on bore-type problems for two-dimensional Boussinesq systems appeared in [9] and in the recent independent, but related work of Burtea [15].

If we take $\theta = \sqrt{2/3}$ and $\lambda = \mu = 0$, the system (4) reduces to

$$\begin{cases} \eta_t + \nabla \cdot V + \alpha \nabla \cdot (\eta V) - \frac{\beta}{6}\,\Delta \eta_t = 0, \\ V_t + \nabla \eta + \frac{\alpha}{2}\nabla |V|^2 - \frac{\beta}{6}\,\Delta V_t = 0, \end{cases} \tag{5}$$

where $V = V_{\sqrt{2/3}}$. This is the so-called BBM-BBM Boussinesq system (see e.g. [4], [17]). The zeroes on the right-hand side are in reality the terms in the expansion that are neglected in coming to the Boussinesq approximation. These terms are of second order, which is to say, of order α^2, $\alpha\beta$ and β^2. It is worth recalling that this level of approximation has been shown in laboratory experiments to yield reasonably accurate predictions of real wave motion, even for relatively large values of the Stokes number (*c.f.* [6], [11], [19], [27]).

An order-one rescaling of the variables (x, y, t) allows us to rewrite the system (5) as

$$\begin{cases} \eta_t + \nabla \cdot V + \varepsilon \nabla \cdot (\eta V) - \varepsilon\,\Delta \eta_t = 0, \\ V_t + \nabla \eta + \frac{\varepsilon}{2}\nabla |V|^2 - \varepsilon\,\Delta V_t = 0, \end{cases} \tag{6}$$

in terms of the single small parameter $\varepsilon = \alpha$, say. Writing $V = \big(u(x,y,t), v(x,y,t)\big)$, the system (6) satisfied by (η, u, v) in $\mathbb{R}^2 \times \mathbb{R}^+$ is, in detail,

$$\begin{cases} \eta_t + u_x + v_y + \varepsilon\Big[(\eta u)_x + (\eta v)_y - \eta_{xxt} - \eta_{yyt}\Big] = 0, \\ \\ u_t + \eta_x + \varepsilon\Big[u u_x + v v_x - u_{xxt} - u_{yyt}\Big] = 0, \\ \\ v_t + \eta_y + \varepsilon\Big[u u_y + v v_y - v_{xxt} - v_{yyt}\Big] = 0, \end{cases} \tag{7}$$

posed with order-one initial conditions

$$\eta(x,y,0) = \eta_0(x,y), \quad u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y),$$

defined for $(x,y) \in \mathbb{R}^2$.

The behavior as $x, y \to \pm\infty$ that captures the type of wave motion in view here is that the free surface is asymptotically constant in the x–directions and that variations with respect to the y–variable vanish in the limit of large $|y|$, the so-called long-crested regime. In particular, because of the normalizations in place and the choice of dependent variables, the problem of bore propagation is modeled by the specifications

$$\begin{cases} \eta(x,y,t) \longrightarrow \begin{cases} \eta_1 & \text{as } x \to -\infty, \\ 0 & \text{as } x \to +\infty, \end{cases} \\ \\ (u(x,y,t), v(x,y,t)) \longrightarrow \begin{cases} (u_1, 0) & \text{as } x \to -\infty, \\ (u_2, 0) & \text{as } x \to +\infty, \end{cases} \\ \\ \partial_y \longrightarrow 0, \quad v(x,y,t) \longrightarrow 0 \ \text{ as } y \to \pm\infty, \\ \\ \left.\begin{array}{l} \eta(x,y,t) \longrightarrow \eta^\pm(x,t) \\ u(x,y,t) \longrightarrow u^\pm(x,t) \end{array}\right\} \quad \text{as } y \to \pm\infty. \end{cases} \tag{8}$$

Here, $\eta_1 > 0$ is the height of the incoming surge of liquid propagating upstream that is invading the steadily flowing river, so that the actual depth is $1 + \eta_1$ as $x \to -\infty$. In the present scaling, the value of η_1 is the physical value divided by h_0; it is assumed to be of order one. The condition $\partial_y \longrightarrow 0$ as $y \to \pm\infty$ is taken to mean that variations of (η, u, v) in the y–directions die out for large values of $|y|$. Thus, the dependent variables $(\eta(x,y,t), u(x,y,t), v(x,y,t))$ settle down to $(\eta^\pm(x,t), u^\pm(x,t), 0)$ in the limit as $y \to \pm\infty$. The functions $u^\pm = u^\pm(x,t)$ are therefore the horizontal velocities in the x–directions at $y = \pm\infty$. The values u_1 and u_2 are the inflow and outflow velocities far downstream and upstream and are taken to be constant, independent of y and t, throughout. Such steady boundary conditions will hold only for a limited period of time in real situations.

Remark that in the light of these specifications, the third equation in (7) is formally satisfied identically in the limit $y \to \pm\infty$, leaving the functions $(\eta^\pm, u^\pm)$ to satisfy the reduced Boussinesq system

$$\begin{cases} (\eta^\pm)_t + (u^\pm)_x + \varepsilon\Big[(\eta^\pm u^\pm)_x - (\eta^\pm)_{xxt}\Big] = 0, \\ \\ (u^\pm)_t + (\eta^\pm)_x + \varepsilon\Big[u^\pm u^\pm_x - (u^\pm)_{xxt}\Big] = 0, \end{cases} \tag{9}$$

whose initial values should presumably be

$$\big(\eta_0^\pm(x), u_0^\pm(x)\big) = \lim_{y\to\pm\infty} \Big(\eta(x,y,0), u(x,y,0)\Big).$$

Naturally, it will follow that

$$\begin{cases} \eta_0^\mp(x) \quad \longrightarrow \begin{cases} \eta_1 & \text{as } x \to -\infty, \\ 0 & \text{as } x \to +\infty, \end{cases} \\ \\ u_0^\mp(x) \quad \longrightarrow \begin{cases} u_1 & \text{as } x \to -\infty, \\ u_2 & \text{as } x \to +\infty. \end{cases} \end{cases}$$

We remark that in our earlier paper [9], we took $u_1 = u_2 = 0$, corresponding to completely quiescent water. In the present study, this is relaxed. It is staightforward to check that all the major results in [9] go over for this slightly more general situation. Indeed, all that is required is to substract from u_0 the function χ to be introduced presently and work with this new dependent variable.

With this proviso, we take it as established that for any $\varepsilon > 0$ and even for (η_0, u_0, v_0) which are merely bounded and continuous, *e.g.* lying in $\mathcal{C}_b(\mathbb{R}^2)^3$, which satisfy

$$\begin{cases} \eta_0(x,y) \longrightarrow \begin{cases} \eta_1 & \text{as } x \to -\infty, \\ 0 & \text{as } x \to +\infty, \end{cases} \\ \\ (u_0(x,y), v_0(x,y)) \longrightarrow \begin{cases} (u_1, 0) & \text{as } x \to -\infty, \\ (u_2, 0) & \text{as } x \to +\infty, \end{cases} \\ \\ (\eta_0(x,y), u_0(x,y), v_0(x,y)) \longrightarrow (\eta_0^{\pm}(x), u_0^{\pm}(x), 0) \ \text{ as } y \to \pm\infty, \end{cases} \tag{10}$$

there exists a unique bounded continuous triple (η, u, v) which is a distributional solution of (7) on a time interval of order one. Moreover it transpires that as $y \to \pm\infty$, $(\eta, u, v) \to (\eta^{\pm}, u^{\pm}, 0)$, where $(\eta^{\pm}, u^{\pm})$ is the unique solution of (9) with initial data $(\eta_0^{\pm}, u_0^{\pm})$. Regularity theory corresponding to further restrictions on the initial data was also established as well as continuous dependence of solutions on the initial data.

Missing from our previous analysis was theory that extended to the Boussinesq time scale $\frac{1}{\varepsilon}$. Such a result was obtained in case $\eta_1 = 0$ under regularity assumptions that imply the initial data is localized in the x–directions (see again [9]). This longer-time theory does not apply when $\eta_1 > 0$. As mentioned earlier, our principal goal here is to extend the earlier, large-time theory so that it encompasses bore propagation.

3. **One-dimensional long-time theory.** In the present section, a theory for long-time existence of bore-like solutions is developed in the case of purely planar waves where no variation along the crest is present. Thus the horizontal velocity in the y–directions is identically zero and all derivatives with respect to y vanish identically. Hence, all the terms in the third equation in (7) vanish identically and the first two equations devolve to the pair

$$\begin{cases} \eta_t + u_x + \varepsilon(u\eta)_x - \varepsilon\eta_{xxt} = 0, \\ \quad u_t + \eta_x + \varepsilon u u_x - \varepsilon u_{xxt} = 0, \end{cases} \tag{11}$$

of coupled, nonlinear dispersive wave equations in one space- and one time-variable (x, t). Here, $x \in \mathbb{R}$ while $t \geq 0$. The system is supplemented with initial data

$$\eta(x, 0) = \eta_0(x) \ \text{ and } \ u(x, 0) = u_0(x), \tag{12}$$

for $x \in \mathbb{R}$, which has a bore-like structure as in (8). More precisely, it is presumed that

$$\eta_0 - \xi \in L_2(\mathbb{R}), \qquad \eta_0' \in H^k(\mathbb{R}), \qquad u_0 - \chi \in L_2(\mathbb{R}), \qquad u_0' \in H^k(\mathbb{R}), \tag{13}$$

with $k \geq 2$. Here $\xi(x)$ is a smooth version of the step-function that takes the value 0 to the right of the jump and the value η_1 to the left and $\chi(x)$ is a smooth function that rapidly takes the values u_1 as $x \to -\infty$ and u_2 as $x \to +\infty$. For example, we could define φ by

$$\begin{cases} \varphi(x) = \dfrac{1 + \tanh(-x)}{2} \quad \text{and take} \\ \\ \xi(x) = \eta_1\, \varphi(x), \quad \chi(x) = u_1\, \varphi(x) + u_2\, \varphi(-x), \end{cases} \tag{14}$$

for $x \in \mathbb{R}$, which is convenient since ξ' and χ' lie in $H^\infty(\mathbb{R})$. In particular, $\eta_0 - \xi$ and $u_0 - \chi$ both lie in $H^{k+1}(\mathbb{R})$.

Problems posed in this form go back at least to the early work of Peregrine [22] and the subsequent, more mathematically precise study in [12] for unidirectional models with dissipation. Study of a KP-type model which made allowance for weak variations in the y–directions and which included dissipation was undertaken a little later by S. Rajopadhye [24]. The present work allows for much stronger variations in the y–directions, albeit still maintaining the long-crested hypothesis that the motion becomes two-dimensional as $y \to \pm\infty$. As mentioned, local well-posedness for exactly this problem is in hand (see [9]).

Going forward, it will be useful to keep track of constants that depend upon the initial data and upon the general aspects of the bore, namely the asymptotic height $\eta_1 > 0$ of the bore and the asymptotic velocities u_1 and u_2. To this end, we propose the following conventions. The quantity

$$\lambda = \max\Big\{\eta_1,\ |u_1|,\ |u_2|\Big\} \tag{15}$$

will be used to provide a restriction on the overall size of the large-$|x|$ boundary values pertaining to the given initial data. Notice that any Sobolev or Lebesgue norm of the j^{th}–derivative of ξ or χ, $j = 1, 2, \cdots$, is bounded by λ times a constant depending only on j. The same is true of the L_∞–norms of these functions, $j = 0, 1, \cdots$.

The functionals Λ_j provide more detailed control of the initial data, *viz*,

$$\Lambda_0 = \max\Big\{|u_0-\chi|_2,\ |\eta_0-\xi|_2\Big\}, \qquad \Lambda_j = \max\Big\{|\partial^j u_0|_2, |\partial^j \eta_0|_2\Big\}, \quad j = 1, 2, \cdots k+1.$$

It is convenient to have a pair of comparison functions to aid in the task of obtaining estimates of (η, u) on the Boussinesq time interval $[0, \frac{1}{\varepsilon}]$. To this end, consider the linear initial-value problem

$$\left\{ \begin{array}{l} \bar\eta_t + \bar u_x - \varepsilon \bar\eta_{xxt} = 0, \quad \bar\eta(\cdot, 0) = \eta_0, \\ \\ \bar u_t + \bar\eta_x - \varepsilon \bar u_{xxt} = 0, \quad \bar u(\cdot, 0) = u_0, \end{array} \right. \tag{16}$$

obtained from (11) by discarding the nonlinear terms. Suppose that $\varepsilon \le 1$ from now on. Some facts about $\bar\eta$ and $\bar u$ are needed in the effort to obtain helpful bounds on (η, u). First, decouple the system (16) by letting $Y = \bar\eta + \bar u$ so that

$$Y_t + Y_x - \varepsilon Y_{xxt} = 0, \quad Y(\cdot, 0) = \eta_0 + u_0. \tag{17}$$

Of course, both $\bar\eta$ and $\bar u$, and hence Y, all depend upon ε, but this dependence is suppressed for ease of reading. The linear group $S(t)$ associated to the initial-value problem for the purely dispersive equation (17) is, for any $t \ge 0$, an isometry on $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. In particular, since any spatial derivative $Y_{(j)} = \partial_x^j Y$ satisfies the same linear equation with initial data

$$Y_{(j)}(x, 0) = \partial_x^j\big(\eta_0 + u_0\big), \tag{18}$$

it follows that for $1 \le j \le k+1$, the globally defined solutions of the linear BBM-equation in (17) with initial data as in (18) preserve their L_2–norms, which is to say, $|Y_{(j)}(\cdot, t)|_2 = |Y_{(j)}(\cdot, 0)|_2$ for all $t \ge 0$, $j = 1, 2, \cdots, k+1$. Of course, the L_2–norm of Y itself is not finite for bore-like initial data.

The next step is to show that $|Y(\cdot, t)|_\infty$ is bounded by a quantity that is independent of $\varepsilon \le 1$, say, and that this bound is finite on any bounded interval and uniform on the Boussinesq time interval $[0, \frac{1}{\varepsilon}]$. To see this, let $\rho(x) = \xi(x) + \chi(x)$

where ξ and χ are as above in (14) and define

$$W(x,t) = Y(x,t) - \rho(x-t).$$

The initial-value problem satisfied by W is

$$W_t + W_x - \varepsilon W_{xxt} = -\varepsilon\rho'''. \tag{19}$$

Because of our presumptions (13) about the data, $W(\cdot,0) \in H^{k+1}(\mathbb{R})$. Moreover, examination of the formulas (14) for ξ and χ shows that the right-hand side of (19) lies in $H^\infty(\mathbb{R})$. Solving this by Duhamel's formula and using the fact that the linear group $S(t)$ generated by (17) is an isometry on $L_2(\mathbb{R})$, it is determined immediately that

$$|W(\cdot,t)|_2 \le |W(\cdot,0)|_2 + \varepsilon t|\rho'''|_2 = |W(\cdot,0)|_2 + c\lambda\varepsilon t, \tag{20}$$

where c is an absolute constant and λ is as in (15). Thus W is bounded independently of $\varepsilon > 0$ on the time interval $[0, \frac{1}{\varepsilon}]$. Since $Y_{(1)}$ is bounded in L_2, independently of ε and t, it follows that $W_{(1)}(x,t) = Y_{(1)}(x,t) - \rho'(x-t)$ is also bounded in L_2 by $\Lambda_1 + c\lambda$, independently of ε and t. Together with the elementary inequality $|f|_\infty^2 \le |f|_2|f'|_2$, (20) and the last observation show that there is a constant c independent of ε such that $|W(\cdot,t)|_\infty \le c(\Lambda_0 + \Lambda_1 + \lambda)$, uniformly for $t \in [0, \frac{1}{\varepsilon}]$. This in turn implies that $|Y(\cdot,t)|_\infty \le |W(\cdot,t)|_\infty + |\rho|_\infty \le c(\Lambda_0 + \Lambda_1 + \lambda)$, independently of $\varepsilon \le 1$ and $t \in [0, \frac{1}{\varepsilon}]$.

If instead, we take the difference $H = \bar\eta - \bar u$, then

$$H_t - H_x - \varepsilon H_{xxt} = 0 \quad \text{and} \quad H(\cdot,0) = \eta_0 - u_0.$$

Exactly the same line of argument serves to establish identical boundedness results for H, though this time one defines $W(x,t) = H(x,t) - \rho(x+t)$. The outcome of these ruminations is summarized in the following proposition.

Proposition 1. *With the preceding notation, $k \ge 2$ and with the assumptions (13) on the auxiliary data,*

$$|Y_{(j)}(\cdot,t)|_2 \le 2\Lambda_j \quad \textit{and} \quad |H_{(j)}(\cdot,t)|_2 \le 2\Lambda_j,$$

for $1 \le j \le k+1$ and for all $t \ge 0$. There is also a constant C_∞ of the form $c(\Lambda_0 + \Lambda_1 + \lambda)$, where c is an absolute constant, such that

$$|Y(\cdot,t)|_\infty \le C_\infty \quad \textit{and} \quad |H(\cdot,t)|_\infty \le C_\infty,$$

independently of $\varepsilon \in (0,1]$ and $t \in [0, \frac{1}{\varepsilon}]$.

Since $\bar\eta = \frac{1}{2}(Y+H)$ and $\bar u = \frac{1}{2}(Y-H)$, exactly the same bounds apply to these functions as do to Y and H, again independently of $\varepsilon \le 1$ and $t \in [0, \frac{1}{\varepsilon}]$.

Corollary 1. *For $k \ge 2$, the bounds*

$$|\bar\eta_{(j)}(\cdot,t)|_2 \le 2\Lambda_j \quad \textit{and} \quad |\bar u_{(j)}(\cdot,t)|_2 \le 2\Lambda_j$$

hold for $j = 1, \cdots k+1$, independently of $\varepsilon \in (0,1]$ and $t \ge 0$. For $\varepsilon \in (0,1]$ and $t \in [0, \frac{1}{\varepsilon}]$,

$$|\bar\eta(\cdot,t)|_\infty \le C_\infty \quad \textit{and} \quad |\bar u(\cdot,t)|_\infty \le C_\infty. \tag{21}$$

where C_∞ is as in Proposition 1.

It will be convenient to define the parameter R to be

$$R = \lambda + \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3. \tag{22}$$

This quantity provides a coarse characterization of the size of the data for the problem. Indeed, according to Corollary 1, R provides bounds on various norms of $\bar\eta$ and $\bar u$, at least on the Boussinesq time scale.

Form the difference between the solution (η, u) of (11), known to exist at least for a short time, and $(\bar\eta, \bar u)$, *viz.*

$$N = \eta - \bar\eta, \text{ and } U = u - \bar u.$$

Notice that $N(\cdot, 0) = U(\cdot, 0) \equiv 0$. The system satisfied by (N, U) is

$$\begin{cases} N_t + U_x + \varepsilon\Big[(NU)_x + (\bar\eta U)_x + (\bar u N)_x\Big] - \varepsilon N_{xxt} = -\varepsilon(\bar\eta\,\bar u)_x, \\ \\ U_t + N_x + \varepsilon\Big[UU_x + (\bar u U)_x\Big] - \varepsilon U_{xxt} = -\varepsilon\bar u\,\bar u_x. \end{cases} \tag{23}$$

It is known from the local theory for (η, u) that the latter system (23) is locally well posed for bore-like data as specified in (13). Thus there are solutions N and U in $\mathcal{C}(0, T; H^{k+1}(\mathbb{R}))$ at least for $T > 0$ small enough and these solutions depend continuously upon variations of the data.

Remark 1. The continuous dependence result includes that the time interval $[0, T]$ over which solutions can be inferred to exist only depends upon the size of the initial data in the relevant space. In consequence, one may approximate data with a given regularity by smoother data, make calculations with the resulting smoother solutions and then pass to the limit of the resulting bounds as long as they do not depend upon the extra smoothness. This standard ploy will be used without further comment.

If appropriate *a priori* bounds are available, it is only required to iterate the local well-posedness theory in [9], made via a contraction-mapping argument, to obtain long-time existence of solutions. To establish such bounds, begin by multiplying the first equation in (23) by N, the second one by U, sum what transpires and integrate the result over $\mathbb{R}$. After integrations by parts, to which the boundary terms do not contribute, there emerges the integro-differential equation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[N^2 + U^2 + \varepsilon\Big(N_x^2 + U_x^2\Big)\Big]dx \\ &\qquad = \varepsilon\int_{-\infty}^{+\infty}\Big[UNN_x + \bar u NN_x + \bar\eta UN_x + \bar u UU_x - N(\bar u\,\bar\eta)_x - U\bar u\,\bar u_x\Big]dx. \end{aligned} \tag{24}$$

If $X = X(t)$ is defined by

$$X^2 = \int_{-\infty}^{+\infty}\Big(N^2 + U^2 + \varepsilon(N_x^2 + U_x^2) + N_{xx}^2 + U_{xx}^2 + \varepsilon(N_{xxx}^2 + U_{xxx}^2)\Big)dx, \tag{25}$$

then (24) implies that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\Big[N^2 + U^2 + \varepsilon\Big(N_x^2 + U_x^2\Big)\Big]dx \le \varepsilon\Big[X^3 + 2RX^2 + 3R^2X\Big]. \tag{26}$$

Essentially the same result obtains at the H^1–level, but is not needed here. We go straight to the derivation of H^2–bounds by calculating

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{+\infty}&\Big[\Big(1+\varepsilon(N+\bar\eta)\Big)U_{xx}^2+N_{xx}^2+\varepsilon\Big(U_{xxx}^2+N_{xxx}^2\Big)\Big]dx\\ &=\varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)_tU_{xx}^2\,dx+2\varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}U_{xxt}\,dx\\ &\quad+2\int_{-\infty}^{+\infty}\Big[U_{xx}\Big(U_{xxt}-\varepsilon U_{xxxxt}\Big)+N_{xx}\Big(N_{xxt}-\varepsilon N_{xxxxt}\Big)\Big]dx\\ &=I_1+I_2+I_3.\end{aligned}\tag{27}$$

The last three integral quantities are examined separately.

To obtain a satisfactory bound on I_1, note that

$$N_t=-(1-\varepsilon\partial_x^2)^{-1}\partial_x\Big[U+\varepsilon(UN+\bar uN+\bar\eta U)+\varepsilon\bar u\,\bar\eta\Big],$$

whence

$$|N_t|_\infty\le X+c\varepsilon\Big[X^2+RX+R^2\Big],\tag{28}$$

where here and below, c denotes various absolute constants, so not depending on the data, on ε nor t (*e.g.* in (28), taking $c=3$ suffices). A similar argument shows that

$$|\bar\eta_t|_\infty\le 2R.$$

It follows that

$$|(N+\bar\eta)_t|_\infty\le 2R+X+c\varepsilon\Big[X^2+RX+R^2\Big].$$

In consequence, it is seen that

$$\begin{aligned}I_1&\le c\varepsilon\Big[(R+\varepsilon R^2)X^2+(1+\varepsilon R)X^3+\varepsilon X^4\Big]\\ &\le c\varepsilon\Big[(R+R^2)X^2+(1+R)X^3+\varepsilon X^4\Big].\end{aligned}\tag{29}$$

Attention is turned to I_3. Integrating by parts leads to

$$\begin{aligned}\frac12 I_3&=\int_{-\infty}^{+\infty}\Big[U_{xx}\left(U_{xxt}-\varepsilon U_{xxxxt}\right)+N_{xx}\left(N_{xxt}-\varepsilon N_{xxxxt}\right)\Big]dx\\ &=-\int_{-\infty}^{+\infty}\Bigg[U_{xx}\partial_x^3\left(N+\varepsilon\frac{U^2}{2}+\varepsilon\bar uU+\varepsilon\frac{\bar u^2}{2}\right)\\ &\qquad\qquad+N_{xx}\partial_x^3\Big(U+\varepsilon UN+\varepsilon\bar uN+\varepsilon\bar\eta U+\varepsilon\bar u\,\bar\eta\Big)\Bigg]dx\\ &=-\varepsilon\int_{-\infty}^{+\infty}\left[\frac52\left(U_{xx}^2+N_{xx}^2\right)U_x+2N_xN_{xx}U_{xx}\right]dx\\ &\qquad\qquad+\varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}N_{xxx}\,dx+\varepsilon\int_{-\infty}^{+\infty}\Phi(U,N)\,dx,\end{aligned}\tag{30}$$

where $\Phi(U,N)$ is a polynomial of degree 2 in U and N and in their spatial derivatives up to order 2, with coefficients depending on $\bar u$, $\bar\eta$ and their derivatives up to order

3. Note that the $O(1)$-terms cancel leaving only terms of formal order $O(\varepsilon)$. The formula (30) implies the estimate

$$I_3 \leq c\varepsilon\Big[R^2X + RX^2 + X^3\Big] + 2\varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}N_{xxx}\,dx. \tag{31}$$

The more involved term I_2 is now considered. Use the equation satisfied by U to determine that

$$\begin{aligned}
\frac{1}{2\varepsilon}I_2 &= \int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}U_{xxt}\,dx \\
&= -\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}\partial_x^2\Big[N_x + \varepsilon\Big(UU_x + (\bar u U)_x - U_{xxt} + \bar u\,\bar u_x\Big)\Big]dx \\
&= -\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}N_{xxx}dx - \varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}\Big(UU_x + (\bar u U)_x\Big)_{xx}dx \\
&\quad +\varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}U_{xxxxt}\,dx - \varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}(\bar u\,\bar u_x)_{xx}\,dx \\
&= -\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}N_{xxx}\,dx + J_1 + J_2 + J_3.
\end{aligned} \tag{32}$$

The integrals J_1, J_2 and J_3 are examined one at a time. Estimating J_3 directly yields

$$J_3 = -\varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}(\bar u\,\bar u_x)_{xx}\,dx \leq c\varepsilon\Big(R^3X + R^2X^2\Big).$$

For J_1, calculate as follows:

$$\begin{aligned}
J_1 &= -\varepsilon\int_{-\infty}^{+\infty}(N+\bar\eta)U_{xx}\Big(UU_x + (\bar u U)_x\Big)_{xx}dx \\
&= \varepsilon\int_{-\infty}^{+\infty}\Big[\frac{1}{2}((N+\bar\eta)U)_x - 3(N+\bar\eta)U_x\Big]U_{xx}^2\,dx \\
&\quad -\varepsilon\int_{-\infty}^{+\infty}\Big[(N+\bar\eta)UU_{xx}\bar u_{xxx} + 3(N+\bar\eta)U_xU_{xx}\bar u_{xx} \\
&\qquad +3(N+\bar\eta)U_{xx}^2\bar u_x - \tfrac{1}{2}((N+\bar\eta)\bar u)_xU_{xx}^2\Big]dx \\
&= \varepsilon\int_{-\infty}^{+\infty}\Big[\frac{1}{2}((N+\bar\eta)U)_x - 3(N+\bar\eta)U_x\Big]U_{xx}^2\,dx \\
&\quad +\varepsilon\int_{-\infty}^{+\infty}\Big[\frac{1}{2}((N+\bar\eta)\bar u)_x - 3N\bar u_x\Big]U_{xx}^2\,dx \\
&\quad -\varepsilon\int_{-\infty}^{+\infty}\Big[(N+\bar\eta)UU_{xx}\bar u_{xxx} - \frac{3}{2}U_x^2\Big((N+\bar\eta)\bar u_{xx}\Big)_x\Big]dx.
\end{aligned}$$

The inequality

$$J_1 \leq c\varepsilon\Big[R^2X^2 + RX^3 + X^4\Big]$$

now follows.

To get control of J_2, write

$$
\begin{aligned}
J_2 \quad &= \varepsilon \int_{-\infty}^{+\infty} (N+\bar{\eta}) U_{xx} U_{xxxxt}\, dx \\
&= -\varepsilon \int_{-\infty}^{+\infty} \Big((N+\bar{\eta}) U_{xxx} + (N+\bar{\eta})_x U_{xx} \Big) U_{xxxt}\, dx \\
&= -\frac{\varepsilon}{2}\frac{d}{dt} \int_{-\infty}^{+\infty} (N+\bar{\eta}) U_{xxx}^2\, dx + \frac{\varepsilon}{2} \int_{-\infty}^{+\infty} (N+\bar{\eta})_t U_{xxx}^2\, dx \\
&\qquad\qquad -\varepsilon \int_{-\infty}^{+\infty} (N+\bar{\eta})_x U_{xx} U_{xxxt}\, dx.
\end{aligned}
\tag{33}
$$

By using the estimate (28), the second term on the right-hand side of (33) may be bounded above thusly:

$$
\begin{aligned}
\frac{\varepsilon}{2} \int_{-\infty}^{+\infty} (N+\bar{\eta})_t U_{xxx}^2\, dx \quad &\le c\varepsilon \Big[X + \varepsilon \left(X^2 + RX + R^2 \right) \Big] \int_{-\infty}^{+\infty} U_{xxx}^2\, dx \\
&\le c \Big[X^3 + \varepsilon \Big(X^4 + RX^3 + R^2 X^2 \Big) \Big].
\end{aligned}
$$

For the third term on the right-hand side of (33), use is again made of the equation satisfied by U to see that

$$
\varepsilon U_{xxxt} = -\varepsilon \partial_x^4 (1-\varepsilon\partial_x^2)^{-1} \Big(N + \varepsilon \frac{U^2}{2} + \varepsilon \bar{u} U + \varepsilon \frac{\bar{u}^2}{2} \Big),
$$

whence

$$
\varepsilon |U_{xxxt}|_2 \le X + \varepsilon \Big(X^2 + RX + R^2 \Big).
$$

It is concluded that

$$
\varepsilon \int_{-\infty}^{+\infty} (N+\bar{\eta})_x U_{xx} U_{xxxt}\, dx \le c \Big(X^4 + RX^3 + \varepsilon \left[R^2 X^2 + R^3 X \right] \Big).
$$

Combining these inequalities and recalling that $\varepsilon \le 1$, the estimate

$$
J_2 \le -\frac{\varepsilon}{2}\frac{d}{dt} \int_{-\infty}^{+\infty} (N+\bar{\eta}) U_{xxx}^2\, dx + \Lambda_2 \Big(X^2 + X^4 \Big)
$$

follows.

If the estimates for the integrals J_1, J_2 and J_3 are used in (32), the bound

$$
\begin{aligned}
I_2 \le -\varepsilon^2 \frac{d}{dt} \int_{-\infty}^{+\infty} &(N+\bar{\eta}) U_{xxx}^2\, dx - 2\varepsilon \int_{-\infty}^{\infty} (N+\bar{\eta}) U_{xx} N_{xxx}\, dx \\
&+ c\varepsilon \Big(X^4 + (1+R) X^3 \Big) + c\varepsilon^2 \Big(X^4 + RX^3 + R^2 X^2 + R^3 X \Big)
\end{aligned}
$$

on I_2 emerges. Combining this inequality with those in (29) and (31) on I_1 and I_3, putting this into equation (27) and adding the result to the earlier differential inequality (26) leads to

$$
\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{+\infty} \Big[N^2 + U^2 + \varepsilon (N_x^2 + U_x^2) + N_{xx}^2 + \Big(1 + \varepsilon (N+\bar{\eta}) \Big) U_{xx}^2 \\
+ \varepsilon \Big(N_{xxx}^2 + \Big(1 + \varepsilon (N+\bar{\eta}) \Big) U_{xxx}^2 \Big) \Big] dx \ \le \ \varepsilon P_\varepsilon (X),
\end{aligned}
\tag{34}
$$

where P_ε is a polynomial of degree 4 in X with coefficients depending only on R and absolute constants. Notice the important cancellation of the integrals on the right-hand sides of (31) and (32). Referring back to the detailed inequalities derived earlier, it is seen that P_ε is composed of monomial terms of the form cR^pX^q where c is an absolute constant and the integers p and q are such that $p+q \geq 3$, $p \leq 3$ and $q \leq 4$.

Theorem 3.1. *Let ξ and χ be as in (14) and suppose that $0 < \varepsilon \leq 1$. Suppose also that bore-like initial data (η_0, u_0) is specified so that*

$$\eta_0 - \xi \in L_2(\mathbb{R}), \quad \eta_0' \in H^2(\mathbb{R}), \quad u_0 - \chi \in L_2(\mathbb{R}), \quad u_0' \in H^2(\mathbb{R}). \tag{35}$$

There is an $R_0 > 0$ such that if R as defined in (22) lies in $(0, R_0]$, then the solution (η, u) to the system (11)-(12) emanating from (η_0, u_0) exists for at least the time interval $[0, \frac{1}{\varepsilon}]$ and satisfies

$$(\eta - \xi, u - \chi) \in \mathcal{C}\left(0, \frac{1}{\varepsilon}; L_2(\mathbb{R})^2\right), \quad (\eta_x, u_x) \in \mathcal{C}\left(0, \frac{1}{\varepsilon}; H^1(\mathbb{R})^2\right).$$

Proof. This result of long-time existence for data drawn from a given bounded subset

$$|\eta_0 - \xi|_2 + |u_0 - \chi|_2 + ||\eta_0'||_{k-1} + ||u_0'||_{k-1} + \varepsilon\Big(||\eta_0'||_k + ||u_0'||_k\Big) \leq R \tag{36}$$

of data satisfying (35) is a consequence of the differential inequalities (26) and (34).

In a little more detail, define first $Y = Y(t)$ by

$$Y = \left\{X^2 + \varepsilon \int_{-\infty}^{+\infty} (N + \bar{\eta})\big(U_{xx}^2 + \varepsilon U_{xxx}^2\big)\, dx\right\}^{\frac{1}{2}},$$

where $X = X(t)$ is as in (25). Integrating the differential inequality in (34) over the time interval $[0, t]$, there appears

$$Y^2(t) \leq Y^2(0) + \varepsilon \int_0^t P_\varepsilon(X(s))\, ds,$$

where P_ε is the quartic polynomial discussed earlier. As long as $\varepsilon|(N+\bar{\eta})(\cdot, t)|_\infty < \frac{1}{2}$, it follows that

$$\frac{1}{2}X^2(t) \leq Y^2(t) \leq Y^2(0) + \varepsilon \int_0^t P_\varepsilon(X(s))\, ds \leq \frac{3}{2}X^2(0) + \varepsilon \int_0^t P_\varepsilon(X(s))\, ds. \tag{37}$$

Fix an $\varepsilon_0 \in (0, 1]$. Let $R_0 > 0$ be such that if $\varepsilon \leq \varepsilon_0$ and $R \leq 2R_0$, then

$$X(t) \leq R \quad \text{implies that} \quad \varepsilon|(N + \bar{\eta})(\cdot, t)|_\infty \leq \frac{1}{2}.$$

That this is possible follows from (21) and the fact that $X(t)$ bounds above the L_∞–norm of $N(\cdot, t)$. Note that control of $|\bar{\eta}|_\infty$ is guaranteed at least on the Boussinesq time interval $[0, \frac{1}{\varepsilon}]$.

The next step is to show that if $R_0 \leq \frac{1}{2}$ is chosen appropriately, but independently of ε, then for $R \leq R_0$, it must be the case that $X(t) \leq 2R$ as long as $t \in [0, \frac{1}{\varepsilon}]$. By continuity, if $X(0) \leq R$, then $X(t) \leq 2R$ at least for t in some positive time interval, say $[0, t_0]$. The goal is to show $t_0 \geq \frac{1}{\varepsilon}$. For $t \in [0, t_0]$, we have $P_\varepsilon(X(t)) \leq c_0 R^3$ where c_0 is again an absolute constant. As a consequence of inequality (37), a calculation shows that

$$X^2(t) \leq 3X^2(0) + 2c_0\varepsilon t R^3 \leq 3R_0^2 + 2c_0\varepsilon t R_0^3$$

for $t \in [0, t_0]$. We require this quantity to be bounded above by $(2R_0)^2 = 4R_0^2$, uniformly for values of $t \leq t_0$, so the inequality

$$3R_0^2 + 2c_0\varepsilon t_0 R_0^3 \leq 4R_0^2$$

is imposed. The latter inequality holds if

$$R_0 \leq \frac{1}{2c_0 \varepsilon t_0}.$$

Choosing $R_0 = \min(\frac{1}{2c_0}, \frac{1}{2})$ allows us to take $t_0 = 1/\varepsilon$. The conclusions of the theorem now follow. $\square$

Corollary 2. *Fix $k \geq 2$. There exists an $R_0 > 0$ such that for all $\varepsilon \in (0,1]$ and $R \in (0, R_0]$, if bore-like initial data (η_0, u_0) satisfying (36) is specified, then the solution $(\eta, u) = (\eta_\varepsilon, u_\varepsilon)$ of (11)-(12) with initial value (η_0, u_0) exists at least on the time interval $[0, \frac{1}{\varepsilon}]$,*

$$(\eta - \xi, u - \chi) \in \mathcal{C}\left(0, \frac{1}{\varepsilon}; L_2(\mathbb{R})^2\right), \quad (\eta_x, u_x) \in \mathcal{C}\left(0, \frac{1}{\varepsilon}; H^{k-1}(\mathbb{R})^2\right),$$

and (η, u) is such that

$$|\eta - \xi|_2 + |u - \chi|_2 + ||\eta_x||_{k-1} + ||u_x||_{k-1} + \varepsilon\Big(||\eta_x||_k + ||u_x||_k\Big) \leq C_1 R,$$

where $C_1 = C_1(R_0)$ is independent of ε. Moreover, the operator that associates the solution (η, u) to initial data (η_0, u_0) satisfying (36) is uniformly Lipschitz continuous for $0 < \varepsilon \leq 1$, which is to say that for all $t \in [0, \frac{1}{\varepsilon}]$,

$$||\eta - \tilde{\eta}||_k + ||u - \tilde{u}||_k \leq C_2\Big(||\eta_0 - \tilde{\eta}_0||_k + ||u_0 - \tilde{u}_0||_k\Big), \tag{38}$$

where C_2 is independent of ε and $(\tilde{\eta}, \tilde{u})$ is the solution of the reduced system (9) with initial value $(\tilde{\eta_0}, \tilde{u_0})$.

Proof. Long-time well-posedness for higher-order Sobolev classes follows from higher-order energy estimates completely analogous to the H^2–estimates appearing above. Argue by induction on k, the case $k = 2$ being in hand. Supposing bounds are available up to order k, consider the energy-type functional

$$\int_{-\infty}^{+\infty} \Big[\Big(1 + \varepsilon(N + \bar{\eta})\Big)U_{(k)}^2 + N_{(k)}^2 + \varepsilon\Big(N_{(k+1)}^2 + (1 + \varepsilon(N + \bar{\eta}))U_{(k+1)}^2\Big)\Big]dx, \tag{39}$$

and differentiate with respect to t. As before, $U_{(k)} = \partial_x^k U$ and so on. Essentially the same calculations as appear at the H^2–level provide a differential inequality of the form appearing in (34). The argument then proceeds as already outlined. Note that further restrictions on ε and R are not needed to maintain the positivity of the quantity $1 + \varepsilon(N + \bar{\eta})$ during these calculations.

For the Lipschitz continuity, the bounds in $H^k(\mathbb{R})$, uniform for $t \in [0, \frac{1}{\varepsilon}]$, together with energy estimates performed on $(\eta - \tilde{\eta}, u - \tilde{u})$ yield (38). Indeed, once bounds have been obtained, we can replace $(\bar{\eta}, \bar{u})$ by $(\tilde{\eta}, \tilde{u})$ and perform exactly the same computations as those leading to the inequality displayed in (34). In this case, $N = \eta - \tilde{\eta}$ and $U = u - \tilde{u}$. This differential inequality leads to the advertised Lipschitz continuity where the Lipschitz constant depends upon the H^{k+1}–norms of the initial data. $\square$

Remark 2. Notice that the long-time result explained above suffers from a loss of a derivative. The data is assumed to be one derivative smoother than the long-time solution that is obtained. We presume this is a defect in the method, and not the real state of affairs. Practically, this makes no difference. As discussed in [10], for solutions of equations like the ones considered here to be good approximations of the full water-wave problem on the Boussinesq time scale, considerably more smoothness than the minimal H^2-level needs to be provided. The loss of a derivative could be circumvented by an ε–dependent regularization of the initial data as in [13], but this point is not pursued here.

4. **Long-time theory in two dimensions.** The aim of the present section is a theorem of existence for the two-dimensional system (7)-(8) on the large time interval $[0, \frac{1}{\varepsilon}]$. This will be shown for any $\varepsilon \in (0, 1]$, but will require size restrictions on the auxiliary data. These restrictions will be characterized by a parameter R. The final result will have the following form, here stated informally. Precise statements come later after details are provided, just as in the one-dimensional case.

Theorem 4.1. *There is a positive R_0 such that for any $R \in (0, R_0]$ and bore-like auxiliary data (η_0, u_0, v_0) which is vorticity free and $O(R)$ (in a sense to be explained), the unique solution (η, u, v) of (7)-(8) emanating from it exists at least on the time interval $[0, \frac{1}{\varepsilon}]$.*

4.1. **Construction of approximate solutions.** The present subsection has as its goal the construction of approximate solutions $(\eta_\varepsilon, u_\varepsilon, v_\varepsilon)$ defined on $\mathbb{R}^2 \times [0, \frac{1}{\varepsilon}]$ corresponding to given auxiliary data (η_0, u_0, v_0). Once constructed, these solutions will be compared to the exact solution, whose existence over at least a short time interval is guaranteed.

First, the hypotheses on the auxiliary data are set out. Let $k \geq 4$ be an integer.

A1. As $y \to \pm\infty$,

$$\begin{cases} \eta_0(x, y) & \longrightarrow \eta_0^\pm(x), \\ u_0(x, y) & \longrightarrow u_0^\pm(x), \end{cases}$$

in the sense that

$$||\eta_0(x, y) - \eta_0^+(x)||_{H^k(\mathbb{R}\times[-1,\infty))} \leq C_1, \quad ||\eta_0(x, y) - \eta_0^-(x)||_{H^k(\mathbb{R}\times(-\infty,1])} \leq C_1,$$

and similarly for $u_0 - u_0^\pm$. This means that η_0 and its various partial derivatives take on the large value asymptotics in both the $x-$ and $y-$variables fast enough that the difference lies in the relevant L_2-based Sobolev space.

A2. $v_0 \in H^k(\mathbb{R}^2)$ and $||v_0||_{H^k(\mathbb{R}^2)} \leq C_2$.

A3. The functions $\eta_0^\pm - \xi$, $u_0^\pm - \chi \in L_2(\mathbb{R})$, and $\partial_x \eta_0^\pm, \partial_x u_0^\pm \in H^k(\mathbb{R})$, where ξ and χ are the functions introduced in (14) used to indicate the large–x asymptotics of the one-dimensional boundary data. Moreover, there is a constant C_3 such that

$$||\eta_0^\pm - \xi||_{L_2(\mathbb{R})}, \; ||u_0^\pm - \chi||_{L_2(\mathbb{R})}, \; ||\partial_x \eta_0^\pm||_{H^{k-1}(\mathbb{R})}, \; ||\partial_x u_0^\pm||_{H^{k-1}(\mathbb{R})} \leq C_3.$$

A4. It is also assumed that there is a constant C_0 such that

$$||u_0^+ - u_0^-||_{H^k(\mathbb{R})} + ||\eta_0^+ - \eta_0^-||_{H^k(\mathbb{R})} \leq C_0\, \varepsilon. \tag{40}$$

The constants C_0, C_1, C_2 and C_3 are all independent of ε.

Notice that condition A1 implies that $\partial_y \eta_0$ and $\partial_y u_0$ lie in $H^{k-1}(\mathbb{R}^2)$ and they are bounded there. It also implies that $(\eta_0, u_0) \to (\eta_1, u_1)$ as $x \to -\infty$, $(\eta_0, u_0) \to$

$(0, u_2)$ as $x \to +\infty$, where η_1, u_1 and u_2 are the large–$|x|$ asymptotic values for η_0 and u_0 introduced in (8). It also follows that $(\eta_0, u_0) \longrightarrow (\eta_0^\pm, u_0^\pm)$ as $y \to \pm\infty$. These limits are all pointwise and uniform in the other variable. That is, $|\eta_0(x, y) - \eta_1| \to 0$ as $x \to -\infty$, uniformly in y, $|\eta_0(x, y) - \eta_0^+(x)| \to 0$ as $y \to +\infty$, uniformly in x, and so on.

The class $\mathcal{D} = \mathcal{D}(C_0, C_1, C_2, C_3, \varepsilon) = \mathcal{D}(R_0, \varepsilon)$ of initial data will be those satisfying conditions A1, A2, A3 and A4. Here R_0 is an upper bound for the constants C_0, C_1, C_2, and C_3 that define $\mathcal{D}$. The local theory in [9] guarantees existence of a solution (η, u, v) corresponding to initial data in $\mathcal{D}$, which is bounded in the various norms that characterize $\mathcal{D}$, uniformly in some non-trivial time interval $[0, t_0]$. Let $Q = Q(\eta, u, v, \varepsilon)$ be some norm of the solution (η, u, v). We say that Q is $O(R)$ if there is a constant C, depending only on the bound R_0 that defines the class $\mathcal{D}$, such that $Q \le C\,R$ for $t \in [0, \frac{1}{\varepsilon}]$.

Let φ^+ be a $\mathcal{C}^\infty$, monotone non-decreasing function defined on all of $\mathbb{R}$ such that

$$\varphi^+(y) = \begin{cases} 1 & \text{for } y \ge 1, \\ 0 & \text{for } y \le -1, \end{cases}$$

and so that $\varphi^+(y) \in [0, 1]$ for all y. If

$$\varphi^-(y) = 1 - \varphi^+(y),$$

then

$$\varphi^-(y) = \begin{cases} 0 & \text{for } y \ge 1, \\ 1 & \text{for } y \le -1, \end{cases}$$

φ^- is $\mathcal{C}^\infty$, monotone non-increasing and also has $\varphi^-(y) \in [0, 1]$. Since $\varphi^+ + \varphi^- \equiv 1$, it follows that $(\varphi^+)'(y) = -(\varphi^-)'(y)$ for all $y \in \mathbb{R}$.

The first step in the construction to follow is to solve the two one-dimensional problems (11)-(12) for initial data (η_0^+, u_0^+) and (η_0^-, u_0^-). On applying Corollary 2 of Theorem 3.1, we find a value R_0 such that for data bounded by $R \le R_0$ as in (36), solutions to the system exist on $\mathbb{R} \times [0, \frac{1}{\varepsilon}]$. Moreover, the various norms of these solutions are $O(R)$. Note also that because of Hypothesis A3, $\eta_0^+ - \eta_0^-$ and $u_0^+ - u_0^-$ lie in $H^k(\mathbb{R})$. In consequence, the Lipschitz result (38) in Corollary 2 implies that

$$||\eta^+ - \eta^-||_{H^k(\mathbb{R})} \quad \text{and} \quad ||u^+ - u^-||_{H^k(\mathbb{R})}$$

are $O(R)$ quantities, uniformly for $\varepsilon \le 1$.

Consider the functions

$$\tilde{\eta}(x, y, t) = \varphi^+(y)\,\eta^+(x, t) + \varphi^-(y)\,\eta^-(x, t),$$
$$\tilde{u}(x, y, t) = \varphi^+(y)\,u^+(x, t) + \varphi^-(y)\,u^-(x, t),$$

defined on

$$\Omega_\varepsilon = \left\{(x, y, t) :\, (x, y) \in \mathbb{R}^2,\, t \in \left[0, \frac{1}{\varepsilon}\right]\right\}.$$

These functions are $\mathcal{C}^\infty$ in y, and k-times differentiable in x, at least in the local-L_2 sense.

We claim that the functions $\underline{\eta}_0(x, y) = \eta_0(x, y) - \tilde{\eta}(x, y, 0)$ and $\underline{u}_0(x, y) = u_0(x, y) - \tilde{u}(x, y, 0)$ lie in $H^k(\mathbb{R}^2)$ and are $O(R)$ there. This follows since

$$\begin{aligned} &\eta_0(x, y) - \varphi^+(y)\,\eta_0^+(x) - \varphi^-(y)\,\eta_0^-(x) \\ &\qquad = \varphi^+(y)\Big(\eta_0(x, y) - \eta_0^+(x)\Big) + \varphi^-(y)\Big(\eta_0(x, y) - \eta_0^-(x)\Big), \end{aligned}$$

and similarly for $\underline{u}_0(x,y)$. The support of φ^+ is $[-1,\infty)$, and φ^+ and its derivatives are bounded there. Similarly, the support of φ^- is $(-\infty,1]$, and φ^- and its derivatives are equally well behaved. Combining this with Hypothesis A1 yields the desired result.

Subject to a size restriction R_1, say, the initial-value problem for the two-dimensional system (7)-(8) posed with $H^k(\mathbb{R}^2)^3$ initial data $(\underline{\eta_0},\underline{u_0},v_0)$ is known to have a unique solution $(\underline{\eta},\underline{u},\underline{v})$ on Ω_ε, which is $O(R)$. This follows from the early work [17], our previous paper [9] or the recent paper [15] of Burtea. This finite-energy solution lies in $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^k(\mathbb{R}^2))$. The time derivatives $\partial_t^j(\underline{\eta},\underline{u},\underline{v})$ are also $O(R)$ in $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^{k-j}(\mathbb{R}^2))$ as well, provided $j\le k-1$.

The approximate solutions to be used presently are taken to be

$$
\left\{
\begin{array}{rcl}
\overline{\eta}(x,y,t) &=& \underline{\eta}(x,y,t)+\varphi^+(y)\,\eta^+(x,t)+\varphi^-(y)\,\eta^-(x,t),\\
&=& \underline{\eta}(x,y,t)+\tilde{\eta}(x,y,t),\\
\\
\overline{u}(x,y,t) &=& \underline{u}(x,y,t)+\varphi^+(y)\,u^+(x,t)+\varphi^-(y)\,u^-(x,t),\\
&=& \underline{u}(x,y,t)+\tilde{u}(x,y,t),\\
\\
\overline{v}(x,y,t) &=& \underline{v}(x,y,t).
\end{array}
\right.
\tag{41}
$$

What has been done is to take out the dynamic boundary conditions, solve the resulting homogeneous problem using existing theory and then put the dynamic boundary conditions back. Going forward in the next subsection, an analysis is provided of the differences between the exact solution, known to exist at least over a short time interval, and the approximate solution displayed in (41), which is defined and $O(R)$ on all of Ω_ε.

4.2. *A priori* **bounds.** Let $(\overline{\eta},\overline{u},\overline{v})$ be the approximate solution just constructed. Of course, $(\overline{\eta},\overline{u},\overline{v})=(\overline{\eta}_\varepsilon,\overline{u}_\varepsilon,\overline{v}_\varepsilon)$ depends upon ε, but this dependence is suppressed for ease of reading. Let (η,u,v) be the local in time solution of the initial-boundary-value problem (7)-(8) under study. Such a solution exists and has regularity properties outlined below (see [9]). Define the residual functions (N,U,V) by

$$N=\eta-\overline{\eta},\quad U=u-\overline{u},\quad V=v-\overline{v}.$$

Inspection reveals that $N(\cdot,\cdot,0)=U(\cdot,\cdot,0)=V(\cdot,\cdot,0)\equiv 0$. Energy estimates are established on the triple (N,U,V) with the aim of deriving *a priori* bounds which allow the local solution to be continued to all of Ω_ε.

Toward this goal, a calculation shows that the triple (N,U,V) satisfies the forced, variable coefficient system

$$
\left\{
\begin{array}{l}
N_t+U_x+V_y\\
\qquad +\varepsilon\Big[(NU)_x+(NV)_y+(\bar{\eta}U)_x+(\bar{u}N)_x+(\bar{v}N)_y+(\bar{\eta}V)_y\Big]-\varepsilon\Delta N_t\\
\qquad\qquad =-(\bar{\eta}_t+\bar{u}_x+\bar{v}_y)-\varepsilon\Big[(\bar{\eta}\,\bar{u})_x+(\bar{\eta}\,\bar{v})_y-\Delta\bar{\eta}_t\Big],\\
\\
U_t+N_x+\varepsilon\Big[UU_x+(\bar{u}U)_x+VV_x+(\bar{v}V)_x\Big]-\varepsilon\Delta U_t\\
\qquad\qquad =-(\bar{u}_t+\bar{\eta}_x)-\varepsilon(\bar{u}\,\bar{u}_x+\bar{v}\,\bar{v}_x-\Delta\bar{u}_t),\\
\\
V_t+N_y+\varepsilon\Big[UU_y+(\bar{u}U)_y+VV_y+(\bar{v}V)_y\Big]-\varepsilon\Delta V_t\\
\qquad\qquad =-(\bar{v}_t+\bar{\eta}_y)-\varepsilon(\bar{u}\,\bar{u}_y+\bar{v}\,\bar{v}_y-\Delta\bar{v}_t),
\end{array}
\right.
\tag{42}
$$

with zero initial- and boundary-conditions. It will be helpful to examine the coefficients appearing in (42).

Lemma 4.2. *The functions* $(\overline{\eta},\overline{u},\overline{v})$, $\nabla(\overline{\eta},\overline{u},\overline{v})$ *and* $\partial_t(\overline{\eta},\overline{u},\overline{v})$ *are* $O(R)$ *in* $L_\infty(\Omega_\varepsilon)$ *for* $\varepsilon\in(0,1]$.

Proof. Since $\overline{\eta}(x,y,t)=\underline{\eta}(x,y,t)+\tilde{\eta}(x,y,t)$ and $\underline{\eta}$ is $O(R)$ in $\mathcal{C}(0,\frac{1}{\varepsilon};H^k(\mathbb{R}^2))\cap\mathcal{C}^1(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}^2))$ and $k\geq 4$, $\underline{\eta}$ clearly satisfies the conclusions. Similarly for $\underline{u}$ and $\underline{v}$. The solutions $(\eta^\pm,u^\pm)$ to the reduced problems (9) are also known to be $O(R)$ in $L_\infty(\mathbb{R})$ along with their spatial derivatives up to order two. As $|\varphi^+|$, $|\varphi^-|\leq 1$ and $|(\varphi^+)'|_\infty=|(\varphi^-)'|_\infty=C$, where C is a constant only dependent on the choice of φ^+, the result follows. $\square$

Attention is turned to the forcing functions on the right-side of (42), which also depend upon the approximate solution and so are defined everywhere in Ω_ε, for $\varepsilon\in(0,1]$. Consider first the function

$$\overline{\eta}_t+\overline{u}_x+\overline{v}_y+\varepsilon\Big((\overline{\eta}\,\overline{u})_x+(\overline{\eta}\,\overline{v})_y-\Delta\overline{\eta}_t\Big)$$

on the right-hand side of the first equation in (42). Using the definitions of the various quantities, this function has the detailed form

$$\left\{\begin{aligned}&\underline{\eta}_t+\underline{u}_x+\underline{v}_y+\varepsilon\Big((\underline{\eta}\,\underline{u})_x+(\underline{\eta}\,\underline{v})_y-\Delta\underline{\eta}_t\Big)\\ &\quad+\tilde{\eta}_t+\tilde{u}_x+\varepsilon\Big((\tilde{\eta}\,\tilde{u})_x-\Delta\tilde{\eta}_t\Big)\\ &\quad+\varepsilon\Big((\tilde{\eta}\,\underline{u})_x+(\underline{\eta}\,\tilde{u})_x+(\tilde{\eta}\,\underline{v})_y\Big).\end{aligned}\right.\tag{43}$$

The first line in (43) is zero since $(\underline{\eta},\underline{u},\underline{v})$ solves the system (7). The second line, written out in detail, is

$$\left\{\begin{aligned}&\varphi^+\,\eta^+_t+\varphi^-\,\eta^-_t+\varphi^+\,(\eta^+)_x+\varphi^-\,(\eta^-)_x-\varepsilon\Big(\varphi^+\,(u^+)_{xxt}+\varphi^-\,(u^-)_{xxt}\Big)\\ &\qquad+\varepsilon\Big[(\varphi^+)^2(\eta^+\,u^+)_x+\varphi^+\,\varphi^-(\eta^-\,u^++\eta^+\,u^-)_x+(\varphi^-)^2(\eta^-\,u^-)_x\\ &\qquad\quad-\Big((\varphi^+)''\eta^+_t+(\varphi^-)''\eta^-_t\Big)\Big]\\ &\quad=\varepsilon\Big[\varphi^+\,\varphi^-(\eta^-\,u^++\eta^+\,u^-)_x-\varphi^+\,\varphi^-(\eta^+\,u^+)_x-\varphi^+\,\varphi^-(\eta^-\,u^-)_x\\ &\qquad-(\varphi^+)''(\eta^+_t-\eta^-_t)\Big],\end{aligned}\right.$$

since (η^+,u^+) and (η^-,u^-) solve the reduced system (9). The functions $\varphi^+\,\varphi^-$ and $(\varphi^\pm)''$ are smooth and have compact support in y. And the various functions $(\eta^\pm\,u^\pm)_x$, $(\eta^\pm)_t$ and $(u^\pm)_t$ are all $O(R)$ in $H^{k-1}_x(\mathbb{R})$, uniformly for $\varepsilon\in(0,1]$. It follows that the function in the square bracket in the last display is $O(R)$, whence the second line of (43) is $\varepsilon\,O(R)$ in $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}^2))$. The third line of the aforementionned quantity (43) is calculated to be

$$\left\{\begin{aligned}&\varepsilon\Big[\varphi^+\Big((\eta^+\underline{u}+\underline{\eta}u^+)_x+(\eta^+\underline{v})_y\Big)+\varphi^-\Big((\eta^-\underline{u}+\underline{\eta}u^-)_x+(\eta^-\underline{v})_y\Big)\\ &\quad+\Big((\varphi^+)'\eta^++(\varphi^-)'\eta^-\Big)\underline{v}+\Big(\varphi^+\eta^++\varphi^-\eta^-\Big)\underline{v}_y\Big].\end{aligned}\right.$$

All the functions $\varphi^\pm$ and $(\varphi^\pm)'$ are smooth and bounded in y. And the functions $(\eta^\pm\underline{u})_x$, $(\underline{\eta}u^\pm)_x$ and $(\eta^\pm\underline{v}_y)$ are all $O(R)$ in $H^{k-1}(\mathbb{R}^2)$, uniformly for $\varepsilon\in(0,1]$. The remaining term has the form

$$\Big((\varphi^+)'\eta^+ + (\varphi^-)'\eta^-\Big)\underline{v} = (\varphi^+)'(\eta^+ - \eta^-)\underline{v},$$

a function of x and t times a function of y times $\underline{v}$. Thus

$$\begin{aligned} \Big|\Big|\Big((\varphi^+)'\eta^+ + (\varphi^-)'\eta^-\Big)\underline{v}\Big|\Big|_{H^{k-1}(\mathbb{R}^2)} & \\ \le C||(\varphi^+)'||_{H_y^{k-1}(\mathbb{R})}\,||\eta^+ - \eta^-||_{H_x^{k-1}(\mathbb{R})}||\underline{v}||_{H^k(\mathbb{R}^2)} & \\ \le C||\eta_0^+ - \eta_0^-||_{H_x^{k-1}(\mathbb{R})} & \\ \le C\,C_0\,\varepsilon, & \end{aligned} \tag{44}$$

where the Lispchitz estimate (38) and the hypothesis A3 have been used. The final conclusion is that this term is also $\varepsilon\,O(R)$. It follows that the whole quantity (43) is $\varepsilon\,O(R)$ in $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}^2))$.

A nearly identical analysis applied to the forcing terms

$$\bar\eta_t + \bar u_x + \bar v_y + \varepsilon\Big((\bar\eta\,\bar u)_x + (\bar\eta\,\bar v)_y - \Delta\bar\eta_t\Big)$$

and

$$\bar u_t + \bar\eta_x + \varepsilon\Big(\bar u\,\bar u_x + \bar v\,\bar v_x - \Delta\bar u_t\Big)$$

yields the conclusion that these quantities are also $\varepsilon\,O(R)$ in $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}^2))$.

The third forcing term is

$$-(\bar v_t + \bar\eta_y) - \varepsilon(\bar u\,\bar u_y + \bar v\,\bar v_y - \Delta\bar v_t),$$

which is written out as

$$\begin{aligned} &-(\underline{v}_t + \underline{\eta}_y) - \varepsilon(\underline{u}\,\underline{u}_y + \underline{v}\,\underline{v}_y - \Delta\underline{v}_t) \\ &\quad+\tilde\eta_y + \varepsilon(\underline{u}\,\tilde u)_y. \end{aligned} \tag{45}$$

The first line in (45) is zero since $(\underline{\eta},\underline{u},\underline{v})$ solves the system (11). The second term in the second line is $\varepsilon\,O(R)$ since $\underline{u}$ is $O(R)$ in $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^k(\mathbb{R}^2))$ and $\tilde u$ has the relevant number of bounded derivatives. The remaining term has the form

$$(\varphi^+)'\eta^+ + (\varphi^-)'\eta^- = (\varphi^+)'(\eta^+ - \eta^-),$$

which was already dealt with in (44). The final result is that this term is also $\varepsilon O(R)$.

The conclusions coming from the last several considerations are summarized here.

Lemma 4.3. *The forcing functions on the right-hand side of (42), defined via the approximate solution $(\overline{\eta},\overline{u},\overline{v})$, are all $\varepsilon\,O(R)$ in $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}^2))$.*

With these lemmas in hand, attention is returned to the functions (N,U,V) which begin at the origin in function space at $t=0$, exist in $\mathcal{C}_b(0,t_0;H^k(\mathbb{R}^2)^3$ for at least some positive time interval $[0,t_0]$, and solve (42) on that interval. Local existence for (42) follows readily just as in [9], [10], [15] or [16] because of the regularity of the coefficients and the forcing functions established in Lemmas 4.2 and 4.3. To produce a solution on the time interval $[0,\frac{1}{\varepsilon}]$ *a priori* bounds are now derived.

A straightforward energy-type calculation reveals that

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\int\Big[N^2+\varepsilon|\nabla N|^2\Big]+\int N(U_x+V_y) \\ \qquad +\varepsilon\int N\Big[(NU)_x+(NV)_y+(\bar\eta U)_x+(\bar u N)_x+(\bar v N)_y+(\bar\eta V)_y\Big] \\ \quad =-\int N\Big[\bar\eta_t+\bar u_x+\bar v_y-\varepsilon\Big(\Delta\bar\eta_t+(\bar\eta\,\bar u)_x+(\bar\eta\,\bar v)_y\Big)\Big], \\ \frac{1}{2}\frac{d}{dt}\int\Big[U^2+\varepsilon|\nabla U|^2\Big]+\int UN_x+\varepsilon\int U\Big[UU_x+(\bar u U)_x+VV_x+(\bar v V)_x\Big] \\ \quad =-\int U\Big[\bar u_t+\bar\eta_x-\varepsilon\Big(\Delta\bar u_t+\bar u\,\bar u_x\Big)\Big], \\ \frac{1}{2}\frac{d}{dt}\int\Big[V^2+\varepsilon|\nabla V|^2\Big]+\int VN_y+\varepsilon\int V\Big[UU_y+(\bar u U)_y+VV_y+(\bar v V)_y\Big] \\ \quad =-\int V\Big[\bar v_t+\bar\eta_y-\varepsilon\Big(\Delta\bar v_t+\bar u\,\bar u_y\Big)\Big], \end{cases} \tag{46}$$

where unadorned integrals are henceforth taken over $\{(x,y)\in\mathbb{R}^2\}$. Define $X(t)$ by

$$\begin{aligned} X^2(t) \;=\; & \|N(\cdot,t)\|_3^2+\|U(\cdot,t)\|_3^2+\|V(\cdot,t)\|_3^2 \\ & +\varepsilon\Big(|\Delta^2N(\cdot,t)|_2^2+|\Delta^2U(\cdot,t)|_2^2+|\Delta^2V(\cdot,t)|_2^2\Big). \end{aligned}$$

Note that when the three equations (46) are summed, the quadratic terms on the left-hand side cancel. According to Lemma 4.3, the right-hand side is bounded by $c\,\varepsilon\,X(t)$ where c depends only on R_0. Estimating the various cubic terms on the left-hand sides appearing in (46), it is determined that

$$\frac{d}{dt}\int\Big[N^2+U^2+V^2+\varepsilon\Big(|\nabla N|^2+|\nabla U|^2+|\nabla V|^2\Big)\Big]\le\varepsilon P_0\big(X\big) \tag{47}$$

where P_0 is a cubic polynomial in X and R of the form

$$P_0=c\left(X^3+RX^2+(R^2+R)X\right)$$

with a coefficient c that is independent of $\varepsilon\le 1$, at least on the time interval $[0,\frac{1}{\varepsilon}]$. This makes use of the properties of the coefficients described in Lemma 4.2 and the fact that the L_∞–norms of the first partial derivatives of the dependent variables are bounded by the H^3–norms of the variable itself (e.g. $|\nabla N|_\infty\le c\|N\|_3$, etc.).

The differential inequality (47) by itself is not helpful toward obtaining bounds of any kind on the variables (N,U,V) since the right-hand side features derivatives not under the control of the left-hand side. To close the inequality, an H^3–bound is now undertaken. The calculations to follow are straightforward, but somewhat tedious:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int & \Big[\Big(1+\varepsilon N+\varepsilon\bar\eta\Big)\Big(|\Delta\nabla U|^2+|\Delta\nabla V|^2\Big)+|\Delta\nabla N|^2 \\ & +\varepsilon\Big(|\Delta^2U|^2+|\Delta^2V|^2+|\Delta^2N|^2\Big)\Big] \\ = \int & \Big[\Delta\nabla U\cdot\Delta\nabla(U_t-\varepsilon\Delta U_t)+\Delta\nabla V\cdot\Delta\nabla(V_t-\varepsilon\Delta V_t) \\ & +\Delta\nabla N\cdot\Delta\nabla(N_t-\varepsilon\Delta N_t)\Big] \end{aligned} \tag{48}$$

$$
\begin{aligned}
&+\varepsilon \int (N+\bar\eta)(\Delta\nabla U\cdot\Delta\nabla U_t+\Delta\nabla V\cdot\Delta\nabla V_t)\\
&+\frac{\varepsilon}{2}\int (N_t+\bar\eta_t)\Big(|\Delta\nabla U|^2+|\Delta\nabla V|^2\Big)\\
=\;& I_1+I_2+I_3.
\end{aligned}
$$

The I_3–terms are easily estimated using the boundedness in L_∞ of $\bar\eta_t$ and N_t. The boundedness of the norm of N_t in L_∞ is obtained via the first equation in (42). Writing it in the form

$$
\begin{aligned}
N_t = -(I-\varepsilon\Delta)^{-1}\Big[&U_x+V_y+\bar\eta_t+\bar u_x+\bar v_y+\varepsilon\Big((\bar\eta\,\bar u)_x+(\bar\eta\,\bar v)_y-\Delta\bar\eta_t\Big)\\
&+\varepsilon\Big((NU)_x+(NV)_y+(\bar\eta U)_x+(\bar u N)_x+(\bar v N)_y+(\bar\eta V)_y\Big)\Big]
\end{aligned}
$$

and using the boundedness of $(I-\varepsilon\Delta)^{-1}$ on the L_2–based Sobolev spaces $H^j(\mathbb{R}^2)$, $j\in\mathbb{Z}$, uniform in $\varepsilon>0$, it transpires that

$$
I_3 \le c\varepsilon\Big(X+R+\varepsilon\left(X^2+RX+R^2\right)\Big)X^2.
$$

Estimating I_1 and I_2 is a little more subtle. Making use of the evolution equations (42) satisfied by (N,U,V), I_1 may be expressed in the form

$$
\begin{aligned}
I_1 = &-\int \Delta\nabla U\cdot\Delta\nabla\Big[N_x+(\bar u_t+\bar\eta_x)+\varepsilon(\bar u\,\bar u_x+\bar v\,\bar v_x-\Delta\bar u_t)\\
&\qquad+\varepsilon\Big(UU_x+(\bar u U)_x+VV_x+(\bar v V)_x\Big)\Big]\\
&-\int \Delta\nabla V\cdot\Delta\nabla\Big[N_y+(\bar v_t+\bar\eta_y)+\varepsilon(\bar u\,\bar u_y+\bar v\,\bar v_y-\Delta\bar v_t)\\
&\qquad+\varepsilon\Big(UU_y+(\bar u U)_y+VV_y+(\bar v V)_y\Big)\Big]\\
&-\int \Delta\nabla N\cdot\Delta\nabla\Big[U_x+V_y+(\bar\eta_t+\bar u_x+\bar v_y)+\varepsilon\Big((\bar\eta\,\bar u)_x+(\bar\eta\,\bar v)_y-\Delta\bar\eta_t\Big)\\
&\qquad+\varepsilon\Big((NU)_x+(NV)_y+(\bar\eta U)_x+(\bar u N)_x+(\bar v N)_y+(\bar\eta V)_y\Big)\Big].
\end{aligned}
\tag{49}
$$

The lowest-order terms cancel since two integrations by parts reveal that

$$
\int \Delta\nabla U\cdot\Delta\nabla N_x+\int\Delta\nabla V\cdot\Delta\nabla N_y+\int\Delta\nabla N\cdot\Delta\nabla\Big(U_x+V_y\Big)=0.
$$

The terms in (49) dependent upon the approximate solution $(\bar\eta,\bar u,\bar v)$ are easily estimated using the results of Lemma 4.3, *viz.*

$$
\begin{aligned}
&\int \Delta\nabla U\cdot\Delta\nabla\Big(\bar u_t+\bar\eta_x+\varepsilon(\bar u\,\bar u_x+\bar v\,\bar v_x-\Delta\bar u_t)\Big)\\
&\quad+\int \Delta\nabla V\cdot\Delta\nabla\Big(\bar v_t+\bar\eta_y+\varepsilon(\bar u\,\bar u_y+\bar v\,\bar v_y-\Delta\bar v_t)\Big)\\
&\quad+\int \Delta\nabla N\cdot\Delta\nabla\Big[\bar\eta_t+\bar u_x+\bar v_y+\varepsilon\Big((\bar\eta\,\bar u)_x+(\bar\eta\,\bar v)_y-\Delta\bar\eta_t\Big)\Big]\le c\varepsilon X.
\end{aligned}
$$

With a few exceptions, the other terms comprising I_1 are all bounded by εX^3. In more detail, let σ, τ and γ connote any of the residual variables N, U and V. The integrals appearing in I_1 with at most three derivatives on each residual variable have the form

$$
\int \partial^3\sigma\,\partial^3\tau\,\partial\gamma \qquad\text{or}\qquad \int\partial^3\sigma\,\partial^2\tau\,\partial^2\gamma.
$$

These may be bounded above thusly:

$$\begin{aligned}\Big|\int \partial^3\sigma\,\partial^3\tau\,\partial\gamma\Big| &\le |\partial\gamma|_\infty\,||\sigma||_3\,||\tau||_3\\ &\le c||\gamma||_2||\sigma||_3||\tau||_3 \le cX^3\end{aligned} \tag{50}$$

and

$$\begin{aligned}\Big|\int \partial^3\sigma\,\partial^2\tau\,\partial^2\gamma\Big| &\le |\partial^3\sigma|_2\,|\partial^2\tau|_4\,|\partial^2\gamma|_4\\ &\le c||\sigma||_3\,||\partial^2\tau||_1\,||\partial^2\gamma||_1 \le cX^3.\end{aligned} \tag{51}$$

Here ∂^3 and ∂^2 stand for any combination of 3 or 2 partial derivatives in the x– and y–variables. The fact that $H^2(\mathbb{R}^2)\subset L_\infty(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)\subset L_4(\mathbb{R}^2)$ has been used in (50) and (51), respectively. Thus, terms in I_1 of this form are all bounded by $c\varepsilon X^3$. This observation allows us to gain helpful control of two more of the integrals appearing in I_1, specifically

$$\begin{aligned}&\varepsilon\int \Delta\nabla U\cdot\Delta\nabla\Big(UU_x+(\bar{u}U)_x\Big)+\varepsilon\int\Delta\nabla V\cdot\Delta\nabla\Big(VV_y+(\bar{v}V)_y\Big)\\ &=\varepsilon\int\Big((U+\bar{u})\Delta\nabla U\cdot\Delta\nabla U_x+(V+\bar{v})\Delta\nabla V\cdot\Delta\nabla V_y\Big)\\ &\qquad+\varepsilon\{\text{terms with at most 3-derivatives on each residual variable}\}\\ &=-\frac{\varepsilon}{2}\int\Big(\,(U_x+\bar{u}_x)|\Delta\nabla U|^2+(V_y+\bar{v}_y)|\Delta\nabla V|^2\\ &\qquad+\varepsilon\{\text{terms with at most 3-derivatives on each residual variable}\}.\end{aligned}$$

Thus, these integrals have been reduced completely to ε times terms with at most three derivative on each residual variable. They are therefore bounded by εX^3 or by εX^2 when they involve $\bar{u}_x$ or $\bar{v}_y$.

The hypothesis that the initial velocity field is irrotational is now invoked.

Lemma 4.4. *The "vorticity" $U_y - V_x$ of the residual variables is $O(R)$ in the space $\mathcal{C}_b(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}))$.*

Proof. Since the vorticity of the initial velocity field (u_0,v_0) is assumed to be zero, the vorticity of the time-dependent velocity field also vanishes. This is because

$$\begin{aligned}\partial_t\nabla\times\begin{pmatrix}u\\v\end{pmatrix} &= \nabla\times\begin{pmatrix}u_t\\v_t\end{pmatrix}\\ &=\nabla\times(I-\varepsilon\Delta)^{-1}\left(\nabla\eta+\frac{\varepsilon}{2}\nabla\left|\begin{pmatrix}u\\v\end{pmatrix}\right|^2\right)\\ &=(I-\varepsilon\Delta)^{-1}\nabla\times\nabla\Big(\eta+\frac{\varepsilon}{2}(u^2+v^2)\Big)=0.\end{aligned}$$

It remains to understand the vorticity

$$\underline{u}_y+(\varphi^+)'u^++(\varphi^-)'u^--\underline{v}_x=\underline{u}_y-\underline{v}_x+(\varphi^+)'(u^+-u^-)$$

of the approximate flow. The right-hand is composed of terms known to be $O(R)$ in $H^{k-1}(\mathbb{R}^2)$, so the result follows. □

Remark 3. The assumption of irrotationality is in fact redundant. This was used already in the derivation of the model (6) (see [7], [8]). However, it is worth noting that an examination of the derivation of *a priori* bounds shows that one only needs that for the exact velocity field, the vorticity is $u_y - v_x$ is $O(R)$ and this will follow if $u_{0y} - v_{0x}$ is $\varepsilon\, O(R)$ in $H^{k-1}(\mathbb{R}^2)$. This point is not pursued here.

Using this information about the vorticity allows us to gain a useful bound on two more of the integral terms appearing in I_1, namely

$$\begin{aligned}
&\varepsilon \int \Delta\nabla U \cdot \Delta\nabla\Big(VV_x + (\bar{v}V)_x\Big) + \varepsilon \int \Delta\nabla V \cdot \Delta\nabla\Big(UU_y + (\bar{u}U)_y\Big)\\
&\quad = \varepsilon \int \Delta\nabla U \cdot \Delta\nabla\Big((V+\bar{v})V_x + \bar{v}_x V\Big) + \varepsilon \int \Delta\nabla V \cdot \Delta\nabla\Big((U+\bar{u})U_y + \bar{u}_y U\Big)\\
&\quad = \varepsilon\Bigg\{ \int \Big[(V+\bar{v})\Delta\nabla U \cdot \Delta\nabla U_y + (U+\bar{u})\Delta\nabla V \cdot \Delta\nabla V_x\Big]\\
&\qquad + \int \Big[(V+\bar{v})\Delta\nabla U \cdot \Delta\nabla(V_x - U_y) + (U+\bar{u})\Delta\nabla V \cdot \Delta\nabla(U_y - V_x)\Big]\\
&\qquad + \text{terms with at most 3-derivatives on each residual variable}\Bigg\}\\
&\quad \le -\frac{\varepsilon}{2} \int \Big[(V+\bar{v})_y |\Delta\nabla U|^2 + (U+\bar{u})_x |\Delta\nabla V|^2\Big] + cRX(t)^2\\
&\qquad + \varepsilon\big\{\text{terms with at most 3-derivatives on each residual variable}\big\},
\end{aligned}$$

where the vorticity Lemma 4.4 has been used to obtain the $RX(t)^2$ term in the last step.

Attention is now given to the integrals involving the dependent variable N in I_1. A calculation reveals that

$$\begin{aligned}
&\varepsilon \int \Delta\nabla N \cdot \Delta\nabla\Big[(NU)_x + (NV)_y + (\bar{\eta}U)_x + (\bar{u}N)_x + (\bar{v}N)_y + (\bar{\eta}V)_y\Big]\\
&\qquad = \varepsilon \int \Delta\nabla N \cdot \Delta\nabla\Big[(N+\bar{\eta})(U_x+V_y) + (N+\bar{\eta})_x U + (N+\bar{\eta})_y V\\
&\qquad\qquad + (\bar{u}N)_x + (\bar{v}N)_y\Big] \qquad\qquad (52)\\
&\qquad = \varepsilon \int (N+\bar{\eta})\Delta\nabla N \cdot \Delta\nabla\Big(U_x + V_y\Big)\\
&\qquad\qquad + \varepsilon\{\text{terms with at most 3-derivatives on each residual variable}\}.
\end{aligned}$$

The first term on the right-hand side of (52) is troublesome, but it will cancel with a similar term in I_2. To see this, calculate I_2 by using (42) to determine U_t and V_t, to wit,

$$\begin{aligned}
I_2 &= \varepsilon \int (N+\bar{\eta})(\Delta\nabla U \cdot \Delta\nabla U_t + \Delta\nabla V \cdot \Delta\nabla V_t)\\
&= -\varepsilon \int (N+\bar{\eta})\Delta\nabla U \cdot \Delta\nabla\Big[N_x + \bar{u}_t + \bar{\eta}_x\\
&\qquad + \varepsilon\Big(UU_x + (\bar{u}U)_x + VV_x + (\bar{v}V)_x + \bar{u}\,\bar{u}_x - \Delta\bar{u}_t - \Delta U_t\Big)\Big]\\
&\quad - \varepsilon \int (N+\bar{\eta})\Delta\nabla V \cdot \Delta\nabla\Big[N_y + \bar{v}_t + \bar{\eta}_y\\
&\qquad + \varepsilon\Big(UU_y + (\bar{u}U)_y + VV_y + (\bar{v}V)_y - \bar{u}\,\bar{u}_y - \Delta\bar{v}_t - \Delta V_t\Big)\Big]\\
&= -\frac{\varepsilon^2}{2}\frac{d}{dt} \int (N+\bar{\eta})\Big(|\Delta\nabla^2 U|^2 + |\Delta\nabla^2 V|^2\Big) + \varepsilon \int (N+\bar{\eta})\Delta\nabla(U_x+V_y) \cdot \Delta\nabla N
\end{aligned}$$

$$+\frac{\varepsilon^2}{2}\int(N_t+\bar{\eta}_t)\Big(|\Delta\nabla^2 U|^2+|\Delta\nabla^2 V|^2\Big)$$
$$+\varepsilon\{\text{terms with at most 3-derivatives on each residual variable}\}$$
$$+\text{ terms of order } \varepsilon^2.$$

Notice that the offending term

$$\varepsilon\int(N+\bar{\eta})\Delta\nabla(U_x+V_y)\cdot\Delta\nabla N$$

cancels between I_1 and I_2. Indeed, this was the whole point of introducing the integral

$$\varepsilon\int(N+\bar{\eta})\Big(|\Delta\nabla U|^2+|\Delta\nabla V|^2\Big)$$

into the "energy" on the left-hand side of (48).

The terms of order ε^2 do not present a problem. They come in two sorts; for example

$$\varepsilon^2\int N\Delta\nabla U\cdot\Delta\nabla(VV_x).$$

The most inconvenient term in the integral above is

$$\varepsilon^2\int N\Delta\nabla U\cdot\Big[V\Delta\nabla(V_x)\Big],$$

which may be bounded by

$$\varepsilon^{3/2}|N|_\infty\,||U||_3\,|V|_\infty\,\varepsilon^{1/2}||V||_4\le\varepsilon^{3/2}X^4.$$

Collecting all the estimates obtained for I_1, I_2, I_3, and using (48) yields the inequality

$$\frac{1}{2}\frac{d}{dt}\int\Big[\Big(1+\varepsilon N+\varepsilon\bar{\eta}\Big)(|\Delta\nabla U|^2+|\Delta\nabla V|^2\Big)+|\Delta\nabla N|^2$$
$$+\varepsilon\Big(1+\varepsilon N+\varepsilon\bar{\eta}\Big)\Big(|\Delta^2 U|^2+|\Delta^2 V|^2\Big)+\varepsilon|\Delta^2 N|^2\Big]$$
$$\le c\,\varepsilon\,X(t)\Big(1+X+X^2+X^3\Big)\Big(1+\varepsilon\int\Big(|\Delta\nabla^2 U|^2+|\Delta\nabla^2 V|^2\Big)\Big).$$

The same argument used earlier in the proof of Therorem 3.1 now applies, allowing us to complete the derivation of *a priori* bounds on the residual variables on the time interval $[0,\frac{1}{\varepsilon}]$ in the case $k=4$.

For larger values of k, one simply uses the 2-dimensional version of (39) and inducts on k, starting with $k=4$, which is in hand.

Theorem 4.5. *Fix $k\ge 4$ and let (η_0,u_0,v_0) be given bore-like initial data as described in (10) such that the hypotheses A1, A2, A3, A4 hold and the initial vorticity $\partial_y u_0-\partial_x v_0=0$. There is an $R_0>0$ such that if $0<R\le R_0$, then for $0<\varepsilon\le 1$, the unique bounded continuous solution (η,u,v) of the initial-boundary-value problem (7)-(8) for bore propagation is defined at least on the time interval $[0,\frac{1}{\varepsilon}]$ with*

$$\partial_x(\eta,u,v)\in\mathcal{C}\Big(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}^2)\Big)\ \ \textit{and}\ \ \partial_y(\eta,u,v)\in\mathcal{C}\Big(0,\frac{1}{\varepsilon};H^{k-1}(\mathbb{R}^2)\Big).$$

The differences $\eta-\bar{\eta},u-\bar{u},v-\bar{v}\in\mathcal{C}(0,\frac{1}{\varepsilon};L_2(\mathbb{R}^2))$. The solutions depend continuously in the function spaces just delineated on variations of the auxiliary data in the subset $\mathcal{D}(R_0,\varepsilon)$ defined below the hypotheses A1, $\cdots$, A4.

Remark 4. Continuity of the solution with respect to variations in the initial data is already available in the local existence theory in [9] made via a contraction-mapping argument.

Remark 5. It is worth pointing out that while relatively large negative values of η are not forbidden in the mathematical theory, they make no sense physically. In the original physical variables (see (3)), maintaining the free surface above the impermeable bottom requires $\tilde{\eta} > -h_0$, which, upon dividing by the maximum amplitude A, becomes

$$\eta = \frac{\tilde{\eta}}{A} > -\frac{h_0}{A} = -\frac{1}{\alpha} = -\frac{1}{\varepsilon}\,.$$

According to the theory, η remains of order 1, so the issue of the bottom running dry does not arise, at least on the Boussinesq time scale.

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