FIRST MIDTERM MATH 18.100B, ANALYSIS I

You may freely use Rudin's book Principles of Mathematical Analysis, your problem sets and your class notes. However, you may not use any other materials. In order to receive full credit on the problems you must prove any assertion that is not in Rudin or in the class notes. You may freely quote any theorems proved in Rudin or in class.

You have 80 minutes to complete the exam. The exam has TWO pages and will be graded over a 100 points. The total sum of points is 110, so you may earn 10 extra credit points.

Problem 1. Let $J = \{1, 2, 3, \dots\}$ denote the set of positive integers. Answer the following questions. You must justify your answers.

(1) Is the set of *all* subsets of J countable? Prove your answer. [5 pts]

(2) Is the set of all *finite* subsets of J countable? Prove your answer. [5 pts]

(3) Is the set of all subsets of J whose complement is bounded countable? Prove your answer. [5 pts]

Problem 2. [10 pts] Let X be a set. Suppose we let T be the collection of all subsets of X. Verify that T is a topology. Is there a metric on X such that T is the metric topology induced by this metric? Explain. (Hint: Think about a problem on problem set 3.)

Problem 3. [15 pts] Let Y be an open subset of a metric space X. As you showed on the homework, Y is also a metric space if we restrict the distance function to Y. Prove that every open subset of Y (open in the topology of Y) is also open in X. Is this assertion still true if we do not assume that Y is open? Justify your answer. (Throughout this problem assume that the topology on a metric space is the one induced by the metric.)

Problem 4. A topological space X is called Hausdorff if given any two distinct points $x \neq y$, there exists two disjoint open sets U_x, U_y (disjoint means $U_x \cap U_y = \emptyset$) such that $x \in U_x$ and $y \in U_y$.

(1) Show that metric spaces are Hausdorff. [5 pts]

(2) Give an example of a topological space that is not Hausdorff. [5 pts] (Hint: You've already seen an example in lecture.)

(3) Conclude that not every topological space is a metric space with the topology induced by the metric. [5 pts]

(4) Prove that a compact subset of a Hausdorff topological space is closed. [10 pts] (Hint: Imitate the proof of Theorem 2.34.)

(5) Give an example of a topological space X and a compact subset K such that K is not closed in X. [5 pts]

Problem 5. For each of the following subsets of the real numbers (endowed with the usual metric topology) answer the following four questions. Is the set open in $\mathbb{R}$? Is the set closed in $\mathbb{R}$? Is the set compact? What are all the limit points of the set in $\mathbb{R}$? You must justify your answers.

(1) $E = \cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$. [4 pts]

(2) $E = [0, 1] \cap \mathbb{Q}$, the set of rational numbers contained in the interval $[0, 1]$. [8 pts]

(3) $E = \{0\} \cup \{\frac{1}{n} : n = 1, 2, 3, \dots\}$ [8 pts]

PLEASE TURN OVER!

Problem 6. For each of the following subsets of $\mathbb{R}^2$ (endowed with the usual metric topology) answer the following three questions. Is the set open in $\mathbb{R}^2$? Is the set closed in $\mathbb{R}^2$? Is the set compact? You must justify your answers.

(1) $E = \{ (x, y) \in \mathbb{R}^2 \mid y = x^2 \}$. [8 pts]
(2) $E = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 3 \}$. [8 pts]
(3) $E = \{ (x, y) \in \mathbb{R}^2 \mid |x| + |y| < 2 \}$. [4 pts]