SECOND MIDTERM MATH 18.100B, ANALYSIS I

You may freely use Rudin's book Principles of Mathematical Analysis, your problem sets and your class notes. However, you may not use any other materials. You may not communicate about the exam with anyone but me. In order to receive full credit on the problems you must prove any assertion that is not proved in Rudin or in the class notes. You may freely quote any theorems proved in Rudin or in class. This is a take home exam. It is due on Monday, November 21 at the beginning of class.

Problem 1. (15 pts) Determine the radius of convergence R of the following power series. Discuss the boundary behavior (i.e at points $|z| = R$) of the series. You may use properties of the exponential and logarithm functions, but you must prove all other assertions you make.

(1)
$$\sum_{n=0}^{\infty} (\sqrt{n+1} - \sqrt{n})\ z^n$$

(2)
$$\sum_{n=4}^{\infty} \frac{z^n}{n\ \log(n)\ \log\log(n)}$$

(3)
$$\sum_{n=1}^{\infty} \frac{n!\ z^n}{n^n}$$

Problem 2. (15 pts) In this problem you will construct an example of a continuous function that is nowhere differentiable. For a real number x let $\{x\}$ denote the distance of x to the nearest integer. Consider the function
$$f : \mathbb{R}^1 \to \mathbb{R}^1$$
given by the formula
$$f(x) = \sum_{n=0}^{\infty} \frac{\{10^n x\}}{10^n}.$$

(1) Show that f is well-defined (i.e. the series converges) for every x in $\mathbb{R}^1$.
(2) Show that f is continuous in $\mathbb{R}^1$.
(3) Show that f is nowhere differentiable by showing that the limit
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
does not exist for any x in $\mathbb{R}^1$. (Hint: Consider the decimal expansion of x and take $h_m = \pm 10^{-m}$ depending on the m-th digit after the decimal point in the decimal expansion.)

Problem 3. (15 pts) Recall that a function $f : X \to Y$ is called open if the image of any open set in X is open in Y. Prove that if $f : \mathbb{R}^1 \to \mathbb{R}^1$ is an open, continuous function, then f is monotonic. (Hint: Show that on any closed interval the maximum and the minimum of f have to be assumed at the end points.)

Problem 4. (20 pts) Suppose f is a differentiable function on $[a, b]$ and that $f(a) = 0$. If $|f'(x)| \leq M|f(x)|$ for every $x \in [a, b]$ and some constant M, prove that $f(x) = 0$ for every $x \in [a, b]$. (Hint: This is problem 26 on page 119. Rudin gives a detailed hint.)

Please turn over!

Problem 5. (25 pts) Let $f : X \to X$ be a function from a metric space to itself. Let $0 < c < 1$ be a real number. Suppose for any two points $x_1 \neq x_2 \in X$,

$$d_X(f(x_1), f(x_2)) \leq \ c\, d_X(x_1, x_2).$$

(What you are going to prove below is a very important fixed point theorem.)

(1) Prove that f is uniformly continuous.

(2) Suppose that X is a complete metric space. Show that f must have a unique fixed point, i.e. there must exist a unique $x \in X$ such that $f(x) = x$. (Hint: Pick any point $x_0 \in X$. Inductively define $x_n = f(x_{n-1})$. Is this a Cauchy sequence? What can you say about the limit?) Deduce that if X is compact or $\mathbb{R}^k$, then f must have a unique fixed point.

(3) Show by giving an example that the previous assertion is false if we do not assume that X is complete.

(4) Does the conclusion of part (2) still hold if we let $c = 1$? Prove or give a counterexample.

Problem 6. (20 pts) Here are a couple of examples warning you against common misconceptions about functions $f : \mathbb{R}^2 \to \mathbb{R}^1$.

(1) Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that f is differentiable when restricted to the x and y axes. Show; however, that f is not continuous.

(2) Define $f(0, 0) = 0$ and

$$f(x, y) = x^2 + y^2 - 2x^2 y - \frac{4x^6 y^2}{(x^4 + y^2)^2}$$

if $(x, y) \neq (0, 0)$. Prove that f is continuous. Show that restricted to any line passing through the origin, f has a local minimum at the origin. Show; however, that f does not have a local minimum at the origin by considering points of the form $y = x^2$.