PRACTICE MATH 18.100B, ANALYSIS I

You may freely use Rudin's book Principles of Mathematical Analysis, your problem sets and your class notes. However, you may not use any other materials. In order to receive full credit on the problems you must prove any assertion that is not in Rudin or in the class notes. You may freely quote any theorems proved in Rudin or in class.

You have 80 minutes to complete the exam. The exam has TWO pages and will be graded over a 100 points. The total sum of points is 110, so you may earn 10 extra credit points.

Problem 1. Is the set of all sequences of 0's and 1's countable? Is the set of sequences of 0's and 1's all but finitely many of which are 1 countable? Is the sequences of 0's and 1's such that $s_{2i+1} = 1$ for every $i = 1, 2, 3, \dots$ countable? You must prove your answers.

Problem 2. Let d_1 denote the usual distance function on $\mathbb{R}^2$. Let $d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote the function

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

Show that d_2 is also a metric. Prove that $d_1 \neq d_2$. Each of these metrics induce a metric topology on $\mathbb{R}^2$. Show that a set is open for the topology induced by d_1 if and only if it is open for the topology induced by d_2. Define a third metric on $\mathbb{R}^2$ such that the topology induced by this metric is different from the topology induced by the metrics d_1 and d_2.

Problem 3. Let Y be an closed subset of a metric space X. As you showed on the homework, Y is also a metric space if we restrict the distance function to Y. Prove that every closed subset of Y (closed in the topology of Y) is also closed in X. Is this assertion still true if we do not assume that Y is closed? Justify your answer. (Throughout this problem assume that the topology on a metric space is the one induced by the metric.)

Problem 4. State whether the following statements are true or false. If the statement is true, prove it. If it is false, give a counterexample.

(1) A finite union of compact sets in a metric space is compact.

(2) The real numbers (under the usual topology) has a covering by countably many compact sets, i.e. there exists a sequence of compact sets $K_1, K_2, \dots$ so that $\mathbb{R} \subset \cup_{i=1}^{\infty} K_i$.

(3) Every infinite compact set in a metric space has uncountably many elements.

(4) If a subset of a compact metric space is not compact, then it is not closed.

(5) In a metric space a set consisting of a single point is closed.

(6) In a metric space a set consisting of a single point can never be open.

(7) In a metric space a set consisting of finitely many points is closed.

Problem 5. For each of the following subsets of the real numbers (endowed with the usual metric topology) answer the following four questions. Is the set open in $\mathbb{R}$? Is the set closed in $\mathbb{R}$? Is the set compact? What are all the limit points of the set in $\mathbb{R}$? You must justify your answers.

(1) The closed interval $[0, 1]$.

(2) The open interval $(0, 1)$.

(3) The Cantor set.

(4) The set of rational numbers that are finite sums $\sum_{i=1}^{N} s_i 2^{-i}$, where each s_i is 0 or 1.

(5) The set of points $\{ \; x \; | x = 3 \text{ or } |x| \leq 1 \; \}$.

Problem 6. For each of the following subsets of $\mathbb{R}^2$ (endowed with the usual metric topology) answer the following three questions. Is the set open in $\mathbb{R}^2$? Is the set closed in $\mathbb{R}^2$? Is the set compact? You must justify your answers.

(1) The set of points (x, y) in $[0, 1] \times [0, 1]$ where both of the coordinates x and y are rational numbers.

(2) $E = \{ \; (x, y) \in \mathbb{R}^2 \mid xy = 1 \; \}$.

(3) $E = \{ \; (x, y) \in \mathbb{R}^2 \mid x + y > 1 \; \}$.