MATH 417 HOMEWORK 12

This homework is due Wednesday December 3 in the beginning of class. You may collaborate on the homework. However, the final write-up must be yours and should reflect your own understanding of the problem. Please be sure to properly cite any help you get.

Problem 1 Find a linear fractional transformation that takes the points $1, i, -i$ to the points $1, 2, 3$, respectively.

Problem 2 Find a linear fractional transformation that takes the circle $|z| = 2$ to the circle $|z + 1| = 1$, the point -2 to the origin, and the origin to the point i.

Problem 3 Find a linear fractional transformation that takes the two circles/lines $x = 2$ and $|z| = 1$ to two concentric circles.

Problem 4 Find a 1-1 analytic map from the complement of the non-negative real numbers in the complex plane $\mathbb{C} - (\mathbb{R}_{\geq 0} \cup \{\infty\})$ onto the unit disc $|z| < 1$.

Problem 5 Suppose that f is an analytic function from the unit disc $|z| < 1$ into the unit disc (i.e., $|f(z)| < 1$) that has a zero of order n at the origin. Prove that $|f(z)| \leq |z|^n$. Furthermore, show that if $|f(a)| = |a|^n$ for some a with $|a| < 1$, then $f(z) = \epsilon z^n$ for some ϵ with $|\epsilon| = 1$.

Extra Credit Problem non-Euclidean Geometry: Let D denote the unit disc $|z| < 1$ and let C be the unit circle $|z| = 1$. Define a non-Euclidean point to be a complex number $z \in D$. Define a non-Euclidean line to be the intersection of any circle or any line in the complex plane that intersects C in two points and is orthogonal to C at those two points. Two non-Euclidean lines are called parallel if they do not intersect in D. You might find it amusing to show that with these definitions non-Euclidean points and lines satisfy all the axioms for points and lines in Euclidean geometry except the parallel postulate. Show that through any two non-Euclidean points there is a unique non-Euclidean line. Find a non-Euclidean line l and a point $z \notin l$ such that there are infinitely many non-Euclidean lines through z parallel to l. Show that there exists a linear fractional transformation that takes the unit disc D to itself and any non-Euclidean point z_1 to any other non-Euclidean point z_2. Define a non-Euclidean distance by $d(z_1, z_2) = \log((z_1, z_2, z_3, z_4))$ where z_3, z_4 are the two points on C where the non-Euclidean line through z_1 and z_2 meets C. (Here z_1, z_2, z_3, z_4 are ordered in the order they occur on the circle. The cross-ratio is positive.) Show that there exists a linear fractional transformation taking D to itself and a pair of non-Euclidean points (z_1, z_2) to another pair (w_1, w_2) if and only if $d(z_1, z_2) = d(w_1, w_2)$.