MATH 330 MIDTERM 2

This is the second midterm for Math 330. It will be handed out on Wednesday Oct 31 at the end of class. It is due on Wednesday Nov 7 before class. Late exams will not be accepted. You may use the class text book and your class notes. However, you may not use any other sources, such as other text books, the internet, math question centers. You may not discuss the questions with anyone or collaborate with anyone. All the work should be your own.

To receive full credit you must justify all your answers and calculations. Be sure to cite any theorem you are using.

Problem 1. (30 pts.) Let $\mathbb{Z}/n\mathbb{Z}$ denote the group of integers modulo n under addition.

i) Find the number of elements of order 15 in $\mathbb{Z}/60\mathbb{Z} \oplus \mathbb{Z}/20\mathbb{Z}$.

ii) Find the number of subgroups of order 14 in $\mathbb{Z}/28\mathbb{Z} \oplus \mathbb{Z}/49\mathbb{Z}$.

iii) Determine up to ismorphism the automorphism group of $\mathbb{Z}/70\mathbb{Z}$. Is this group cyclic? What is its order?

iv) Let D_n denote the dihedral group of order $2n$. Let p be an odd prime. Determine up to isomorphism the inner automorphism group $Inn(D_{2p})$ of D_{2p}.

v) Let f be a homomorphism from $\mathbb{Z}/60\mathbb{Z}$ onto a subgroup of order 12. Determine the kernel of f.

vi) Find all homomorphisms $f : \mathbb{Z}/30\mathbb{Z} \to \mathbb{Z}/45\mathbb{Z}$. Determine their kernels.

Problem 2. (10 pts.) Classify all abelian groups of order 360 up to isomorphism.

Problem 3. (30 pts.) Let G be a group.

i) Prove that the inner automorphism group of G is trivial if and only if G is abelian.

ii) Prove that if the index of $Z(G)$ in G is a prime number, then G is abelian.

iii) Give an example of a non-abelian group where $G/Z(G)$ is abelian.

iv) Give an example of two non-isomorphic, non-abelian groups whose inner automotphism groups are isomorphic.

Problem 4. (10 pts.) Consider the factor group $\mathbb{Z} \oplus \mathbb{Z}/ < (3, 6) >$. Is this group cyclic? Does it contain an element of order 3? If so, find one. Does it contain an element of order 4? If so, find one. Does it contain an element of infinite order? If so, find one.

Problem 5. (20 pts.) Let S_n denote the symmetric group on n letters.

i) Let π and σ be two permutations in S_n. Show that $\sigma^{-1} \circ \pi \circ \sigma$ is the permutation obtained by applying σ^{-1} to the numbers in the cycle decomposition of π. For example, if I would like to compute $(321)(2345)(123)$ where $\sigma = (123)$ and $\pi = (2345)$, then I can apply $\sigma^{-1} = (321)$ to the entries of π to get (1245).

ii) We say that two permutations written as a product of disjoint cycles have the same cycle structure if their number of r cycles are equal for every r. For example, $(12)(345)(678)$ has the same cycle structure as $(13)(268)(457)$, but does not have the same cycle structure as $(12)(34)(5678)$. Using part i) deduce that two permutations in S_n are conjugate if and only if they have the same cycle structure.

iii) Determine the number of elements in each of the conjugacy classes in S_5.

iv) Show that the only non-trivial, proper normal subgroup of S_5 is A_5. (Hint: A normal subgroup has to be the union of conjugacy classes.)

v) (Extra credit: 10 pts) As an extra credit problem try to prove that A_n does not have any non-trivial, proper normal subgroups when $n \geq 5$. This is not an easy problem, but it is very important. You should be able to solve it for small values of n such as $n = 5, 6$. You will get partial credit for doing the cases $n = 5, 6$.