MATH 330 FINAL

This exam has 10 problems, each worth 20 points. You have 2 hours to complete the exam. You may use your course notes and the class text book. However, you may not use any other resources. The work must be your own work. You may not collaborate with others. IN ORDER TO RECEIVE FULL CREDIT YOU MUST PROVE ALL YOUR ASSERTIONS.

Problem 1. Let G denote the cyclic group $\mathbb{Z}/240\mathbb{Z}$.

(1) Calculate the order of the automorphism group of G.
(2) Express the automorphism group of G as a direct product of cyclic groups.

Problem 2. Let G denote the group $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

(1) Calculate the number of elements of order 4 in G.
(2) Calculate the number of elements of order 2 in G.
(3) Find the number of cyclic subgroups of order 4 in G.

Problem 3. Classify all abelian groups of order 2000 up to isomorphism. How many abelian groups (up to isomorphism) of order 2000 have an element of order 25?

Problem 4. Find all ring homomorphisms $f : \mathbb{Z}/75\mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$. Determine their kernels.

Problem 5. Let R denote the ring $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

(1) Find the units in R.
(2) Find the zero divisors in R.

Problem 6. Let R_1 and R_2 be commutative rings with unity. Suppose that p_1 is a prime ideal in R_1 and m_1 is a maximal ideal in R_1. Suppose that p_2 is a prime ideal in R_2.

(1) Consider the ideal of $R_1 \oplus R_2$ defined by $\{(x, y) | x \in p_1, y \in R_2\}$. Show that this ideal is a prime ideal of $R_1 \oplus R_2$.
(2) Consider the ideal of $R_1 \oplus R_2$ defined by $\{(x, y) | x \in m_1, y \in R_2\}$. Show that this ideal is a maximal ideal of $R_1 \oplus R_2$.
(3) Consider the ideal of $R_1 \oplus R_2$ defined by $\{(x, y) | x \in p_1, y \in p_2\}$. Is this a prime ideal of $R_1 \oplus R_2$; prove or give a counterexample.

Problem 7. Consider the factor ring $R = \mathbb{Z}/5\mathbb{Z}[x]/ < x^2 + 2 >$.

(1) Prove that R is a field.
(2) How many elements does R have?
(3) Determine the multiplicative inverse of x in R.

Problem 8. Prove that the polynomial $2x^5 + 21x^3 + 49x^2 - 14x + 7$ is irreducible over $\mathbb{Q}$.

Problem 9. Prove that the polynomial $x^3 + 2x + 1$ is irreducible over $\mathbb{Q}$.

Problem 10. Determine the group of ring automorphisms of the field $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$.