MATH 417 COMPLEX ANALYSIS FINAL EXAMINATION

This exam has 10 problems each worth 20 points. You may use your class notes and the course text book. However, you cannot use any other materials. The work must be yours and yours only.

Problem 1. Find ALL the solutions of the equation $z^5 = 1 - i$.

Problem 2. Determine ALL the values of $(ie^{\pi})^i$. Indicate the principal value.

Problem 3. Let $u(x, y)$ and $v(x, y)$ be a conjugate pair of smooth harmonic functions defined in a domain D. Prove that $u^2 - v^2$ is a harmonic function in D.

Problem 4. Let $f(z)$ be an entire function. Suppose for any $z \neq 0$, $f(z)$ satisfies the equation

$$f(z) = f\left(\frac{1}{z}\right).$$

Prove that $f(z)$ must be constant.

Problem 5. Calculate the following contour integral

$$\int_C \frac{e^z}{z^2 - \frac{1}{4}} dz$$

where C is the unit circle $|z| = 1$ positively oriented.

Problem 6. Calculate the integral

$$\int_0^{\infty} \frac{\sqrt[3]{x}}{x^2 + 4} dx.$$

Be sure to include the main details of your calculation.

Problem 7. Calculate the integral

$$\int_0^{\infty} \frac{\cos(x) - 1}{x^2} dx$$

Be sure to include the main details of your calculation.

Problem 8. Find the number of zeros, counting multiplicities, of the polynomial $2z^8 + 3z^5 - 9z^3 + 2$ in the annulus $1 \leq |z| < 2$.

Problem 9. Let D_r denote the closed disc $|z| \leq r$. Let $f(z)$ be an analytic function on D_1 whose image is contained in D_r for some $r < 1$. Prove that $f(z)$ has a unique fixed point (i.e., a point z_0 such that $f(z_0) = z_0$). (Hint: Apply Rouché's theorem to $f(z)$ and $-z$.)

Problem 10. Find a linear fractional transformation $f(z)$ that maps the points $1, 2$ and i to the points $i, 1$ and 2, respectively (i.e., $f(1) = i$, $f(2) = 1$, $f(i) = 2$).