MATH 516 MIDTERM EXAM

This is the take-home midterm for Math 516. The exam has 16 problems and 3 extra credit problems. This is an open book exam. You have a week to do it. You may discuss the problems with people in the class and you may consult books. However, the final write-up must be yours. You may not collaborate while writing the solutions and you may not use anything other than the text book and your course notes.

Problem 1. List all the conjugacy classes in the group $\mathfrak{S}_6$. Determine the order of each conjugacy class. Determine the orders of all the conjugacy classes of the alternating group A_6. Prove by hand that A_6 is simple.

Problem 2. Classify all abelian groups of order 144 up to isomorphism. Determine the ones that have an element of order 8.

Problem 3. Let p be a prime number. Let G be a non-abelian group of order p^3.

- Determine the possible orders of the center $Z(G)$ of G?
- Classify all possible quotient groups $G/Z(G)$.
- Show that there can be non-isomorphic non-abelian groups of order p^3.

Problem 4. Let $p < q < r$ be three prime numbers such that $r \neq 1 \mod q$ and $q, r, qr \neq 1 \mod p$. Prove that a group of order pqr is cyclic. Conclude that any group of order $595 = 5 \cdot 7 \cdot 17$ or $1235 = 5 \cdot 13 \cdot 19$ is cyclic.

Problem 5. Prove that a group of order 4125 cannot be simple by showing that either the 5 or the 11 Sylow subgroup must be normal.

Problem 6. Let G be a non-trivial finite group. Suppose that given any two non-identity elements $a, b \in G$ there exists an automorphism ϕ of G such that $\phi(a) = b$.

- Prove that $G \cong \oplus_{i=1}^n \mathbb{Z}/p\mathbb{Z}$ for some positive integer n and some prime number p.
- Determine the order of $Aut_{Ab}(\oplus_{i=1}^n \mathbb{Z}/p\mathbb{Z})$?

Problem 7. Let $GL_n(\mathbb{Z}/p\mathbb{Z})$ denote $n \times n$ invertible matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ for a prime number p. $GL_n(\mathbb{Z}/p\mathbb{Z})$ is a group under matrix multiplication and is an example of a finite group of Lie type.

- Determine the order of $GL_n(\mathbb{Z}/p\mathbb{Z})$.
- Show that $GL_n(\mathbb{Z}/p\mathbb{Z}) \cong Aut_{Ab}(\oplus_{i=1}^n \mathbb{Z}/p\mathbb{Z})$.
- In particular, deduce that $GL_2(F_2) \cong \mathfrak{S}_3$.

Problem 8. Classify all groups of order 30.

Problem 9. Let R be a commutative ring with unit. Prove that if R is Noetherian, then the formal power series ring $R[[x]]$ is also Noetherian.

Problem 10. Do problem V.1.10 on page 250

Problem 11. Do problem V.1.17 on page 251

Problem 12. Do problem V.2.14 on page 259

Problem 13. Do problem V.2.18 on page 260

Problem 14. Do problem V.2.19 on page 260

Problem 15. Do problem V.2.20 on page 260

Problem 16. Do problem V.2.25 on page 260

Problem 17 (Extra Credit). Show that there are no non-cyclic simple groups of order between 60 and 168. Show that $\mathbb{P}SL_2(\mathbb{Z}/7\mathbb{Z}) = SL_2(\mathbb{Z}/7\mathbb{Z})/\{I, -I\}$ is a simple group of order 168. Show that this group is the automorphism group of the curve $y^7 = x^2(x-1)$. In fact, this curve is the unique curve of genus 3 that has an automorphism group of order 168. The curve is often expressed projectively in the more symmetric form $x^3y + y^3z + z^3x = 0$ and is known as the Klein quartic.

Problem 18 (Extra Credit). Determine the symmetry groups of the five Platonic solids.

Problem 19 (Extra Credit). A Coxeter group is a group with a presentation of the form

$$< r_1, r_2, \dots, r_n | (r_i r_j)^{m_{ij}} = 1 >,$$

where $m_{ii} = 1$ (i.e., $r_i^2 = 1$) and $m_{ij} \geq 2$ for $i \neq j$. Show that two generators commute if and only if $m_{ij} = 2$. A Coxeter group is denoted by a Coxeter graph obtained by the following rules:

(1) The vertices correspond to the generators of the group and are labeled by the subscripts.
(2) Vertices i and j are connected by an edge if and only if $m_{ij} \geq 3$.
(3) The edges are labeled by m_{ij} is $m_{ij} > 3$.

Show that the Coxeter group corresponding to the chain

$$\bullet - \bullet - \bullet - \cdots - \bullet$$

with n vertices is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$. Learn the classification of finite Coxeter groups.