HOMEWORK 5

You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that k is an algebraically closed field and R is a commutative ring with unit.

Problem 0.1. *Recall that "If $f : X \to Y$ is a surjective morphism of projective varieties such that*

1. *Y is irreducible,*
2. *Every fiber of f is irreducible,*
3. *Every fiber of f has the same dimension,*

then X is irreducible." Show that all three assumptions are necessary.

Problem 0.2. *Compute the multiplication table for the cohomology of $G(2,5)$.*

Problem 0.3. *Prove Pieri's formula*

$$\sigma_1 \cdot \sigma_{\lambda_1,\dots,\lambda_k} = \sum_{\lambda_i \leq \mu_i \leq \lambda_{i-1}, \sum \mu_i = 1 + \sum \lambda_i} \sigma_{\mu_1,\dots,\mu_k}$$

where $\sigma_{\lambda_1,\dots,\lambda_k}$ and $\sigma_{\mu_1,\dots,\mu_k}$ are Schubert cycles in $G(k,n)$.

Problem 0.4. *We say that a plane curve $F = 0$ has a cusp at p if the Taylor expansion of F at p has the form*

$$L^2 + h.o.t.$$

where L is a line containing p and h.o.t. denotes higher order terms. Show that for $d > 2$ plane curves of degree d that have a cusp form a projective subvariety of codimension two in $\mathbb{P}^{d(d+3)/2}$, the space of plane curves of degree d. (Hint: Linearize the problem by considering plane curves that have a cusp at p with tangent direction L.)

Problem 0.5. *Let $X \subset \mathbb{P}^n$ be a projective variety. The secant variety to X is the closure of the union of lines spanned by distinct points on X*

$$Sec(X) = \overline{\cup_{p,q \in X, p \neq q} \overline{pq}}.$$

Prove that $Sec(X)$ is a projective variety of dimension less than or equal to $\min(2\dim(X) + 1, n)$. We say that the secant variety is defective if $\dim(Sec(X)) < \min(2\dim(X) + 1, n)$. Prove that $Sec(X)$ is defective if and only if every point $x \in Sec(X)$ lies on infinitely many secant lines to X. Show that the secant variety of the Veronese image $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$ is defective. Hard Challenge: Show that a surface S in $\mathbb{P}^5$ which is not contained in any hyperplane has a defective secant variety if and only if S is the Veronese image $\nu_2(\mathbb{P}^2)$.

Problem 0.6. *More generally, let $X \subset \mathbb{P}^n$ be a projective variety. The r-secant variety $Sec_r(X)$ to X is the closure of the union of the $\mathbb{P}^{r-1}$'s spanned by r distinct points $p_1, \dots, p_r$ in X in general linear position. Prove that $Sec_r(X)$ is a projective variety of dimension less than or equal to $\min(r\dim(X)+r-1, n)$. We say that $Sec_r(X)$ is defective if the dimension of $Sec_r(X)$ is strictly less than $\min(r\dim(X) + r - 1, n)$. Show that $Sec_r(X)$ is defective if and only if every point on $Sec_r(X)$ is contained in infinitely many secant $\mathbb{P}^{r-1}$'s to X. Show that the fourth Veronese image $\nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ has a defective 5-secant variety $Sec_5(\nu_4(\mathbb{P}^2))$. Hard Challenge: Show that among the secant varieties to the Veronese images of $\mathbb{P}^2$, $Sec_2(\nu_2(\mathbb{P}^2))$ and $Sec_5(\nu_4(\mathbb{P}^2))$ are the only defective secant varieties.*