THE COHOMOLOGY OF THE MODULI SPACE OF SHEAVES ON SURFACES

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Let (X, H) be a smooth, irreducible, projective polarized surface and let $\mathbf{v}$ be a Chern character on X. Let $M_{X,H}(\mathbf{v})$ be the moduli space parameterizing S-equivalence classes of H-Gieseker semistable sheaves on X with Chern character $\mathbf{v}$. The spaces $M_{X,H}(\mathbf{v})$ play an important role in algebraic geometry, representation theory, combinatorics and mathematical physics.

When the rank r of $\mathbf{v}$ is 1, then the moduli space is closely related to the Hilbert scheme of points $X^{[n]}$ on X. Göttsche computed the Betti numbers of $X^{[n]}$ and discovered that the Betti numbers of $X^{[n]}$ stabilize as the number of points n tends to infinity [Got90]. Göttsche and Soergel obtained similar results for Hodge numbers [GS93]. In contrast, the Betti numbers of moduli spaces of sheaves of higher rank on a surface have been computed only in special cases (see [CW18, CH20] for references).

Let $\nu(\mathbf{v}) := \frac{c_1(\mathbf{v})}{r(\mathbf{v})}$ and $\Delta(\mathbf{v}) := \frac{\nu^2(\mathbf{v})}{2} - \frac{\mathrm{ch}_2(\mathbf{v})}{r(\mathbf{v})}$ be the *total slope* and the *discriminant*, respectively. In [CW18], we conjecture that the Betti numbers (and more generally the Hodge numbers) of $M_{X,H}(\mathbf{v})$ stabilize as $\Delta(\mathbf{v})$ tends to infinity and that the stable numbers are independent of $r(\mathbf{v})$, $c_1(\mathbf{v})$ and H and thus are computed by Göttsche. The main philosophy is that while the individual Betti numbers are hard to compute, the stable numbers behave well and are easy to compute.

Conjecture 1. [CW18, Conjecture 1.1] *Fix a rank $r > 0$ and a first Chern character c. Let $b_{i,\mathrm{Stab}}(X)$ denote the ith stable Betti number of $M_{X,H}(1, c, \Delta)$. Then given an integer k, there exists $\Delta_0(k)$ such that for $\Delta \geq \Delta_0(k)$ and $i \leq k$*

$$b_i(M_{X,H}(r, c, \Delta)) = b_{i,\mathrm{Stab}}(X).$$

Furthermore, if H is in a compact subset $\mathcal{C}$ of the ample cone of X, then $\Delta_0(k)$ can be chosen independently of $H \in \mathcal{C}$.

One can ask for effective bounds on $\Delta_0(k)$. Mandal provided such bounds in the case of $\mathbb{P}^2$ [M21].

Conjecture 1 fits well with the expectation of Donaldson, Gieseker and Jun Li that the geometry of $M_{X,H}(\mathbf{v})$ becomes better behaved as Δ tends

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to infinity. By work of Mukai, Huybrechts and Yoshioka, the conjecture is known for moduli spaces of sheaves with no strictly semistable sheaves on K3 or Abelian surfaces. We refer the reader to [CW18, CH20] for references. The conjecture is also compatible with the Atiyah-Jones conjecture (see [AJ78, CW18]).

Analogues of Conjecture 1 are expected to hold for moduli spaces of pure one-dimensional sheaves on surfaces, certain moduli spaces of Bridgeland stable objects on surfaces and the Matsuki-Wentworth moduli spaces of twisted stable sheaves [CW18].

Let $K_0(\mathrm{var}_{\mathbb{C}})$ denote the Grothendieck ring of varieties. Let $\mathbb{L}$ denote the class of $\mathbb{A}^1$ and let $R = K_0(\mathrm{var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. The ring R has a $\mathbb{Z}$-graded decreasing filtration $\mathcal{F}$ generated by

$$[X]\mathbb{L}^a \in \mathcal{F}^i \quad \text{if} \quad \dim(X) + a \leq -i.$$

Define the ring A^- as the completion of R with respect to this filtration. Note that elements of A^- have a natural notion of dimension generalizing the dimension of smooth, projective varieties. The virtual Poincaré and Hodge polynomials extend to A^-. The main theorem of [CW18] is the following.

Theorem 2. [CW18, Theorem 1.7] *Let X be a smooth, complex projective rational surface and let H be a polarization such that $H \cdot K_X < 0$. As Δ tends to ∞, the classes $[\mathcal{M}_{X,H}(r, c, \Delta)]$ of the moduli stacks of Gieseker semistable sheaves stabilize in A^- to*

$$\prod_{i=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{-i})^{\chi_{\mathrm{top}}(X)}}.$$

In particular, the virtual Betti and Hodge numbers of $\mathcal{M}_{X,H}(r, c, \Delta)$ stabilize, and the generating functions for the stable virtual numbers $b_{i,\mathrm{Stab}}$ and $h^{p,q}_{\mathrm{Stab}}$ are given by

$$\sum_{i=0}^{\infty} b_{i,\mathrm{Stab}} t^i = \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i})^{\chi_{\mathrm{top}}(X)}} \quad \textit{and}$$

$$\sum_{p,q=0}^{\infty} h^{p,q}_{\mathrm{Stab}} x^p y^q = \prod_{i=1}^{\infty} \frac{1}{(1 - (xy)^i)^{\chi_{\mathrm{top}}(X)}}.$$

When the moduli spaces $M_{X,H}(r, c, \Delta)$ have no strictly semistable sheaves, one obtains the following consequence.

Theorem 3. [CW18, Theorem 1.9] *Let X be a smooth, complex, projective rational surface and let H be a polarization such that $K_X \cdot H < 0$. Assume that the moduli spaces $M_{X,H}(r, c, \Delta)$ do not contain any strictly semistable sheaves. Then the Betti and Hodge numbers of $M_{X,H}(r, c, \Delta)$ stabilize to the stable Betti and Hodge numbers of the Hilbert scheme of points $X^{[\Delta]}$ as Δ tends to infinity.*

There is little evidence for Conjecture 1 for surfaces of general type. For small discriminants, the moduli spaces $M_{X,H}(\mathbf{v})$ can have arbitrarily many irreducible components [CH18]. During the talk, I was asked whether the Betti numbers monotonically increase to the stable value as is the case for Hilbert schemes of points and certain moduli spaces of sheaves on K3, abelian and rational surfaces. In [CHK21], we give examples of complete intersection surfaces of Picard rank 1, where the moduli space can have an arbitrarily large number of connected components. Hence, the answer is negative even for b_0, which stabilizes to 1 by a theorem of O'Grady [O'G96].

Conjecture 1 raises several questions of independent interest.

Question 4. *Is the cohomology of $M_{X,H}(r, c, \Delta)$ pure in increasing degrees as Δ tends to infinity? Does Poincaré duality hold in increasing degrees as Δ tends to infinity?*

Question 5. *If $\mathcal{X} \to B$ is a one-parameter family of smooth polarized surfaces and $\mathbf{v}$ is a Chern character that exists on every surface of the family, how does the topology of $M_{X_b,H_b}(\mathbf{v})$ vary? If Δ is sufficiently large, are small Betti numbers of the moduli spaces $M_{X_b,H_b}(\mathbf{v})$ independent of b?*

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