THE NORMAL BUNDLE OF A GENERAL CANONICAL CURVE OF GENUS AT LEAST 7 IS SEMISTABLE

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Abstract. Let C be a general canonical curve of genus g defined over an algebraically closed field of arbitrary characteristic. We prove that if $g \notin \{4, 6\}$, then the normal bundle of C is semistable. In particular, if $g \equiv 1$ or $3 \pmod 6$, then the normal bundle is stable.

1. Introduction

Let k be an algebraically closed field of arbitrary characteristic. Let C be a nonsingular, irreducible, non-hyperelliptic curve of genus $g \geq 3$ defined over k. Then the canonical linear system $|K_C|$ embeds C in $\mathbb{P}^{g-1}$. The image is called a canonical curve of genus g. Canonical curves of genus g lie in an irreducible component of the Hilbert scheme of curves of genus g in $\mathbb{P}^{g-1}$. Studying the properties of canonical curves is an essential tool in curve theory.

Given a vector bundle V on C of rank r and degree d, recall that the slope of V is defined by $\mu(V) := \frac{d}{r}$. The bundle V is called semistable if, for every proper subbundle W, we have $\mu(W) \leq \mu(V)$. The bundle is called stable if the inequality is always strict.

Since stable bundles are the atomic building blocks of all vector bundles on a curve, it is important to ask if naturally-defined vector bundles on canonical curves, such as the restricted tangent bundle $T_{\mathbb{P}^{g-1}}|_C$ or the normal bundle N_C, are stable. The first of the these is straightforward: the restricted tangent bundle of a general canonical curve of genus $g \geq 3$ is always stable. In fact, the restricted tangent bundle of a general Brill–Noether curve of any degree d and genus $g \geq 2$ in $\mathbb{P}^r$ is stable unless $(d, g) = (2r, 2)$ [FL22]. On the other hand, the normal bundle can fail to be stable in low genus (cf. Remark 1).

Aprodu, Farkas and Ortega [AFO16] conjectured that once the genus is sufficiently large, the normal bundle of a general canonical curve is stable. Previously, this was only known for $g = 7$ [AFO16] and for $g = 8$ [B17]. The proofs of these two results use explicit models of low genus canonical curves due to Mukai, and thus do not generalize to large genus. In this paper, we prove:

Theorem 1.1. *Let C be a general canonical curve of genus $g \notin \{4, 6\}$ defined over an algebraically closed field of arbitrary characteristic. Then the normal bundle of C is semistable.*

The rank of N_C is $g - 2$ and the degree of N_C is $2(g^2 - 1)$. Hence,

$$\mu(N_C) = 2g + 4 + \frac{6}{g-2}.$$

In particular, if $g - 2$ and 6 are relatively prime, the semistability of N_C implies the stability of N_C. We thus obtain the following corollary.

Corollary 1.2. *If $g \equiv 1$ or $3 \pmod 6$, then the normal bundle of the general canonical curve of genus g is stable.*

2010 *Mathematics Subject Classification.* Primary: 14H60. Secondary: 14B99.

Key words and phrases. Canonical curves, normal bundle, stability.

During the preparation of this article, I.C. was supported by NSF FRG grant DMS-1664296, and E.L. and I.V. were supported by NSF MSPRF grants DMS-1802908 and DMS-1902743, respectively.

Remark 1. When $g = 3$, the canonical curve is a plane quartic curve. Hence, $N_C \cong \mathcal{O}_C(4)$ and is stable. When $g = 5$, the general canonical curve is a complete intersection of three quadrics. Hence, $N_C \cong \mathcal{O}_C(2)^{\oplus 3}$. In particular, N_C is semistable but not stable. When $g = 4$ or 6, N_C is unstable, as we now explain. When $g = 4$, the canonical curve is a complete intersection of type $(2,3)$. The normal bundle of C in the quadric is a destabilizing line subbundle of N_C of degree 18. When $g = 6$, the general canonical curve is a quadric section of a quintic del Pezzo surface. The normal bundle of C in this del Pezzo surface gives a degree 20 destabilizing line subbundle of N_C.

We will prove Theorem 1.1 by specializing a canonical curve to the union of an elliptic normal curve E of degree g and a g-secant rational curve R of degree $g-1$ meeting E quasi-transversely in g points. In §3, we describe this degeneration and the Harder-Narasimhan (HN) filtration of $N_{E\cup R}|_R$. In §4, we will prove that $N_{E\cup R}|_E$ is semistable. This suffices to prove Theorem 1.1 when g is odd by [CLV22, Lemma 4.1], because $N_{E\cup R}|_R$ is balanced in this case. When g is even, $N_{E\cup R}|_R$ is not balanced. However, we have an explicit geometric understanding of the HN-filtration. In this case, we give two proofs of Theorem 1.1, one using the strong Franchetta Conjecture (see §4), and an elementary proof using the explicit HN-filtration and induction on g (in §5 and §6).

Acknowledgments. We would like to thank Atanas Atanasov, Lawrence Ein, Gavril Farkas, Joe Harris, Eric Riedl, Ravi Vakil, and David Yang for invaluable conversations.

2. Preliminaries

In this section, we collect basic facts about normal bundles of reducible curves and their stability. We refer the reader to [CLV22] for more details.

2.1. **Stability of vector bundles on nodal curves.**

In our argument, we will specialize canonical curves to certain nodal curves. We recall a natural extension of the notion of stability to nodal curves (see [CLV22, §2]). Let C be a connected nodal curve and write

$$\nu\colon \tilde{C} \to C$$

for the normalization of C. For any node p of C, let $\tilde{p}_1$ and $\tilde{p}_2$ be the two points of $\tilde{C}$ over p.

Given a vector bundle V on C, the fibers of the pullback ν^*V to $\tilde{C}$ over $\tilde{p}_1$ and $\tilde{p}_2$ are naturally identified. Given a subbundle $F \subseteq \nu^*V$, we can thus compare $F|_{\tilde{p}_1}$ and $F|_{\tilde{p}_2}$ inside $\nu^*V|_{\tilde{p}_1} \simeq \nu^*V|_{\tilde{p}_2}$.

Definition 2.1. Let V be a vector bundle on a connected nodal curve C. For a subbundle $F \subset \nu^*V$, define the adjusted slope μ_C^{adj} by

$$\mu_C^{\mathrm{adj}}(F) := \mu(F) - \frac{1}{\operatorname{rk} F} \sum_{p \in C_{\mathrm{sing}}} \operatorname{codim}_F \left(F|_{\tilde{p}_1} \cap F|_{\tilde{p}_2}\right),$$

where $\operatorname{codim}_F \left(F|_{\tilde{p}_1} \cap F|_{\tilde{p}_2}\right)$ refers to the codimension of the intersection in either $F|_{\tilde{p}_1}$ or $F|_{\tilde{p}_2}$ (which are equal since $\dim F|_{\tilde{p}_1} = \dim F|_{\tilde{p}_2}$). Note that if F is pulled back from C, then $\mu_C^{\mathrm{adj}}(F) = \mu(F)$. We say that V is (semi)stable if for all subbundles $F \subset \nu^*V$,

$$\mu^{\mathrm{adj}}(F) \underset{(-)}{<} \mu(\nu^*V) = \mu(V).$$

The advantage of this definition is that it specializes well.

Proposition 2.2. [CLV22, Proposition 2.3] *Let $\mathscr{C} \to \Delta$ be a family of connected nodal curves over the spectrum of a discrete valuation ring, and $\mathscr{V}$ be a vector bundle on $\mathscr{C}$. If the special fiber $\mathscr{V}_0 = \mathscr{V}|_0$ is (semi)stable, then the general fiber $\mathscr{V}^* = \mathscr{V}|_{\Delta^*}$ is also (semi)stable.*

Lemma 2.3. [CLV22, Lemma 4.1] *Suppose that $C = X \cup Y$ is a reducible curve and V is a vector bundle on C such that $V|_X$ and $V|_Y$ are semistable. Then V is semistable. Furthermore, if one of $V|_X$ or $V|_Y$ is stable, then V is stable.*

2.2. **Elementary modifications of normal bundles.** In this section, we briefly recall the definition of an elementary modification of a vector bundle on a curve. For a more detailed exposition, see [ALY19, §2-6].

Given a vector bundle V on a curve C, a subbundle $F \subset V$, and an effective Cartier divisor $D \subset C^{\mathrm{sm}}$, the negative elementary modification $V[D \xrightarrow{-} F]$ of V along D towards F is defined by the exact sequence

$$0 \to V[D \xrightarrow{-} F] \to V \to V|_D/F|_D \to 0.$$

We write $V[D \xrightarrow{+} F] := V[D \xrightarrow{-} F](D)$ for the positive modification of V along D towards F.

We will primarily work with elementary modifications of the normal bundle of a curve $C \subset \mathbb{P}^r$ towards pointing bundles, whose definition we now recall. Given any linear space $\Lambda \subset \mathbb{P}^r$, the projection π from Λ, when restricted to C, is unramified on an open $U_\Lambda \subset C$. If U_Λ is dense in C and contains C^{sing}, then the normal sheaf of the map π uniquely extends to a rank $(\dim \Lambda + 1)$ subbundle of N_C, which we denote by $N_{C\to\Lambda}$ and call the pointing bundle towards Λ. The pointing bundle exact sequence is

$$0 \to N_{C\to\Lambda} \to N_C \to \pi^* N_{\pi(C)}(C \cap \Lambda) \to 0.$$

We abbreviate and write $N_C[p \xrightarrow{+} \Lambda] := N_C[p \xrightarrow{+} N_{C\to\Lambda}]$ for modifications towards pointing bundles.

Suppose that M is any subvariety of $\mathbb{P}^r$ meeting C quasi-transversely at a point p. Let $\Lambda \subset T_p M$ be a complementary linear space to p. Then we write

$$N_C[p \overset{+}{\rightsquigarrow} M] := N_C[p \xrightarrow{+} \Lambda].$$

If $M \cap C = \{p_1, p_2, \dots, p_n\}$, with all points of intersection quasi-transverse, then we write

$$N_C[\overset{+}{\rightsquigarrow} M] := N_C[p_1 \overset{+}{\rightsquigarrow} M]\cdots[p_n \overset{+}{\rightsquigarrow} M].$$

Our interest in modifications towards pointing bundles is rooted in the following result of Hartshorne–Hirschowitz, describing the normal bundle of a nodal curve.

Lemma 2.4 ([HH83, Corollary 3.2]). *Let $X \cup Y$ be a connected nodal curve in $\mathbb{P}^r$. Then*

$$N_{X\cup Y}|_X \simeq N_X[\overset{+}{\rightsquigarrow} Y].$$

2.3. **The Farey sequence.** Recall that the N-Farey sequence is the sequence of fractions whose denominators are bounded by N in lowest terms. We refer the reader to [HW79] for the properties of the Farey sequence.

Lemma 2.5. *Let V be a vector bundle of slope $\mu(V) = \frac{p}{q}$ in lowest terms and suppose that*

$$0 \to S \to V \to Q \to 0$$

is an exact sequence of vector bundles such that either $\mu(S)$ is an adjacent q-Farey fraction to $\mu(V)$ with $\gcd(\deg S, \operatorname{rk} S) = 1$, or similarly for Q. If both S and Q are stable, then any destabilizing subbundle of V is isomorphic to either S or Q.

Proof. Suppose that V has degree ep and rank eq for some $e \geq 1$. Then the slope of the other bundle ($\mu(Q)$ or $\mu(S)$ respectively) is an adjacent eq-Farey fraction; this can be seen using the following two standard properties of adjacent Farey fractions:

- Two rational numbers in lowest terms, p_1/q_1 and p_2/q_2, are adjacent in the $\max(q_1, q_2)$-Farey sequence if and only if
$$\det \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} = \pm 1.$$
- In this case, they are adjacent in the q-Farey sequence for any $\max(q_1, q_2) \leq q < q_1 + q_2$, and the next fraction appearing between them is
$$\frac{p_1 + p_2}{q_1 + q_2}.$$

There are four cases to consider: $\mu(S)$ or $\mu(Q)$ is the next or previous *eq*-Farey fraction. Up to replacing the sequence with its dual, it suffices to consider the two cases that $\mu(S)$ or $\mu(Q)$ is the next Farey fraction. Let F be any subbundle of V. Then F has a filtration

$$0 \to F \cap S \to F \to \operatorname{Im}(F \to Q) \to 0.$$

If $\mu(S)$ is the next Farey fraction: Since $F \cap S$ is a subsheaf of S, we have $\mu(F \cap S) \leq \mu(S)$ with equality only if F contains S. Since $\mu(V)$ is the previous *eq*-Farey fraction to $\mu(S)$, if equality does not hold, then $\mu(F \cap S) \leq \mu(V)$. Similarly $\mu(\operatorname{Im}(F \to Q)) \leq \mu(Q) < \mu(V)$. Hence, $\mu(F) \leq \mu(V)$ unless $F = S$.

If $\mu(Q)$ is the next Farey fraction: Similarly, $\mu(\operatorname{Im}(F \to Q)) \leq \mu(Q)$ with equality only if $F \to Q$ is surjective. Since $\mu(V)$ is the previous *eq*-Farey fraction to $\mu(Q)$, if equality does not hold, then $\mu(\operatorname{Im}(F \to Q)) \leq \mu(V)$. Similarly $\mu(F \cap S) \leq \mu(S) < \mu(V)$. Hence, $\mu(F) \leq \mu(V)$ unless $F \to Q$ is an isomorphism. □

Lemma 2.6. *Suppose that $\mathscr{V}$ is a family of vector bundles on a positive-genus curve C parameterized by a rational base B. Suppose that, for $b_1, b_2 \in B$, the specializations $\mathscr{V}|_{b_i}$ fit into exact sequences*

$$0 \to S_i \to \mathscr{V}|_{b_i} \to Q_i \to 0$$

satisfying the hypotheses of Lemma 2.5 with $\mu(S_1) = \mu(S_2)$. If $c_1(S_1) \neq c_1(S_2)$, then the general fiber of $\mathscr{V}$ is semistable.

Proof. Suppose that $\mathscr{V}|_b$ is unstable for $b \in B$ general. Then there exists a destabilizing subbundle $\mathscr{F} \subset \mathscr{V}|_b$. Consider the rational map

$$c_1(\mathscr{F}) \colon B \dashrightarrow \operatorname{Pic} C.$$

Since B is rational, this map is constant.

On the other hand, we may specialize to the fiber over b_i. As we approach along any arc, $\mathscr{F}|_b$ limits to one of S_i or Q_i (based on which one has slope greater than $\mu(\mathscr{V})$) by Lemma 2.5. Therefore, $c_1(\mathscr{F})$ extends to a regular map in a neighborhood of b_i. Our assumption that $c_1(S_1) \neq c_1(S_2)$ (and so also $c_1(Q_1) \neq c_1(Q_2)$) then gives a contradiction. □

2.4. **Natural bundles on a genus 1 curve.** Let E be a genus 1 curve. We say that a map $f \colon \operatorname{Pic}^a E \to \operatorname{Pic}^b E$ is natural if for any automorphism $\theta \colon E \to E$, the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Pic}^a E & \xrightarrow{f} & \operatorname{Pic}^b E \\ \downarrow \theta^* & & \downarrow \theta^* \\ \operatorname{Pic}^a E & \xrightarrow{f} & \operatorname{Pic}^b E \end{array}$$

Lemma 2.7. *If $f \colon \operatorname{Pic}^a E \to \operatorname{Pic}^b E$ is natural, then a divides b.*

Proof. Translation by a point of order a is the identity on $\operatorname{Pic}^a E$, and so must also be on $\operatorname{Pic}^b E$. □

3. Our degeneration

Let $E \subset \mathbb{P}^{g-1}$ be an elliptic normal curve. Let $H \simeq \mathbb{P}^{g-2}$ be a general hyperplane and let $\Gamma := E \cap H$ be the hyperplane section of E. Let R be a general rational curve of degree $g-2$ in H, meeting E quasi-transversely at the points of Γ. Then by [LV22, Lemma 5.7], the curve $E \cup R$ is a Brill–Noether curve of degree $2g-2$ and genus g; i.e., it is a degeneration of a canonical curve.

Lemma 3.1 ([LV22, Lemma 5.8 and Proposition 13.7])**.** *We have*

$$N_{E \cup R}|_R \simeq \begin{cases} \mathcal{O}(g+1)^{\oplus(g-2)} & : g \text{ odd} \\ \mathcal{O}(g) \oplus \mathcal{O}(g+1)^{\oplus(g-4)} \oplus \mathcal{O}(g+2) & : g \text{ even.} \end{cases}$$

By [CLV22, Lemma 4.1], when g is odd, it suffices to show that $N_{E\cup R}|_E$ is semistable to conclude that the normal bundle of a general canonical curve is semistable. This is addressed in Section 4. When g is even, we will need to know that $N_{E\cup R}|_E$ is semistable, and also that certain modifications of $N_{E\cup R}|_E$, related to the Harder–Narasimhan (HN) filtration of $N_{E\cup R}|_R$, are semistable. We conclude this section with a brief geometric description of the HN-filtration, expanding on [LV22, Section 13].

3.1. **The HN-filtration when g is even.** In this section, we suppose that $g = 2n+2$ is even. We first recall some results we will need from [LV22, Section 13]. Suppose that $E \subset \mathbb{P}^{2n+1}$ is an elliptic normal curve. Let $p_1 + \cdots + p_{2n+2}$ be a general section of $O_E(1)$. Let $q_1, \dots, q_{2n+2}$ be general points on $\mathbb{P}^1$. By [LV22, Lemma 13.1], there are exactly two degree $n+1$ maps

$$f_i \colon E \to \mathbb{P}^1$$

sending p_j to q_j for all $1 \le j \le 2n+2$. Together, these define a map

$$\overline{f} \colon E \to \mathbb{P}^1 \times \mathbb{P}^1,$$

which is birational onto an (n^2-1)-nodal curve of bidegree $(n+1, n+1)$ [LV22, Lemma 13.2], none of whose nodes lie on the diagonal.

Let S denote the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the n^2-1 nodes of $\overline{f}(E)$, with total exceptional divisor F, and write $f \colon E \hookrightarrow S$ for the resulting embedding. Let R denote the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$, viewed as a divisor on S. By construction, R meets E at $p_1, \dots, p_{2n+2}$. By [LV22, Lemma 13.3], the line bundle $L = \mathcal{O}_S(n,n)(-F) = K_S(1,1)(E)$ on S restricts to

$$L|_E \simeq \mathcal{O}_E(1,1) \simeq \mathcal{O}_E(p_1 + \cdots + p_{2n+2}).$$

Furthermore, $H^0(L(-E)) = 0$, and so $H^0(L) \simeq H^0(L|_E)$. The complete linear system $|L|$ defines an embedding of S in $\mathbb{P}H^0(L|_E) \simeq \mathbb{P}^{2n+1}$. Along R, the bundle $L|_R$ has degree $2n$, and hence maps R into a hyperplane in $\mathbb{P}^{2n+1}$. The reducible curve $E \cup R$ is a degeneration of a canonical curve. Write $\pi_i \colon S \to \mathbb{P}^1$ for the two projections onto each factor of $\mathbb{P}^1 \times \mathbb{P}^1$. By [LV22, Equation (196)],

$$(\pi_i)_* L \simeq (f_i)_* \mathcal{O}_E(p_1 + \cdots p_{2n+2}) \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n+1)}.$$

The embedding $S \to \mathbb{P}^{2n+1}$ given by $|L|$ thus factors through the balanced scrolls

$$\Sigma_i = \mathbb{P}[(\pi_i)_* L] \simeq \mathbb{P}^1 \times \mathbb{P}^n$$

embedded by the relative $\mathcal{O}(1)$.

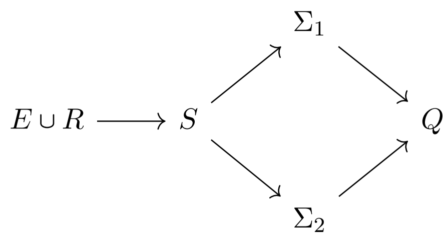
Finally, we recall a construction of Zamora [Z, Lemma 1.1] of a rank 4 quadric in $\mathbb{P}^{2n+1}$ containing E, and show that it also contains the scrolls Σ_1 and Σ_2. Let s_1, s_2 be a basis for the linear system giving rise to the first map $f_1 \colon E \to \mathbb{P}^1$ and let t_1, t_2 be a basis for the linear system giving rise to the second map $f_2 \colon E \to \mathbb{P}^1$. Then the $s_i \otimes t_j$ are sections of $\mathcal{O}(1,1)|_E = L|_E$, and we may therefore view them as linear functions on the $\mathbb{P}^{2n+1}$. Furthermore, as a section of $L|_E^{\otimes 2}$,

$$\det \begin{pmatrix} s_1 \otimes t_1 & s_2 \otimes t_1 \\ s_1 \otimes t_2 & s_2 \otimes t_2 \end{pmatrix} = 0$$

This determinant defines a rank 4 quadric $Q \subset \mathbb{P}^{2n+1}$ containing E. Changing the bases s_1, s_2 or t_1, t_2 corresponds to a row/column operation, so this quadric is independent of the choice of basis.

To see that the quadric contains Σ_1, we will show that it contains every fiber $\mathbb{P}^n = \operatorname{Span}(f_1^{-1}(x))$ for $x \in \mathbb{P}^1$. Choose a basis so that the first element s_1 vanishes on $f_1^{-1}(x)$. Thus the linear functions corresponding to $s_1 \otimes t_1$ and $s_1 \otimes t_2$ vanish along $\operatorname{Span}(f_1^{-1}(x))$ in $\mathbb{P}^{2n+1}$, and hence the quadric Q contains this plane. Varying x, we see that Q contains Σ_1. Similarly Q contains Σ_2. Putting all of this together, we can summarize this situation with the following setup:

Setup 3.2. Given an elliptic $E \subset \mathbb{P}^{2n+1}$ and two maps $f_i \colon E \to \mathbb{P}^1$, we obtain the following inclusions:



The maps in Setup 3.2 give rise to a filtration of $N_{E\cup R}$,

$$N_{E\cup R/S} \subset N_{E\cup R/Q} \subset N_{E\cup R}. \tag{1}$$

Proposition 3.3 ([LV22, Proposition 13.7])**.** *The restriction of* (1) *to R is the HN-filtration of $N_{E\cup R}|_R$.*

Remark 2. In [LV22, Proposition 13.7], the middle piece of the filtration does not have a geometric description. Instead, it is described as "$N_{E\cup R/\Sigma_1} + N_{E\cup R/\Sigma_2}$" — which is equal to $N_{E\cup R/Q}$ since it is contained in it, and has the same rank and degree.

Proposition 3.4. *Fixing a line bundle $\mathcal{O}_E(1)$ of degree $2n+2$ on an elliptic curve E, the set of possible $S, R, \Sigma_1, \Sigma_2, Q$ in Setup 3.2 varies in a rational base.*

Proof. The data in (3.2) is determined by the following choices:

(1) A basis (up to common scaling) for $H^0(\mathcal{O}_E(1))$, which determines the embedding $E \subset \mathbb{P}^{2n+2}$. The choice of a basis of a vector space depends on a rational base.
(2) An unordered pair of line bundles $f_i^*\mathcal{O}_{\mathbb{P}^1}(1)$ that sum to $\mathcal{O}_E(1)$. The space of line bundles of a fixed degree on E can be identified with E. This choice corresponds to the fiber of the map $a\colon (E\times E)/\mathfrak{S}_2 \to E$ given by addition over $\mathcal{O}_E(1)$. The surface $(E\times E)/\mathfrak{S}_2$ is a ruled surface over E, so this choice is rational.
(3) Two sections (up to common scaling) of each of these line bundles (defining $f_i\colon E \to \mathbb{P}^1$). As in (1), this choice depends on a rational base.

We conclude that the set of possible $S, R, \Sigma_1, \Sigma_2, Q$ in Setup 3.2 varies in a rational base. □

4. Semistability of the restriction to E

In this section, we show that the restricted normal bundle $N_{E\cup R}|_E$, where $E \cup R \subset \mathbb{P}^{g-1}$ is the degenerate canonical curve introduced in Section 3, is semistable.

Theorem 4.1. *If $g \notin \{4, 6\}$, then $N_{E\cup R}|_E$ is semistable.*

We will first show that $N_{E\cup R}|_E$ is "close-enough-to-semistable" that no naturally defined destabilizing subbundles could exist. We have that

$$\mu(N_{E\cup R}|_E) = \frac{g(g+1)}{g-2} = g + 3 + \frac{6}{g-2}.$$

The fractional part of the slope depends on g modulo 6. Write

$$g - 2 = 6k + \epsilon \quad \text{where} \quad 0 \le \epsilon < 6.$$

Lemma 4.2. *The bundle $N_{E\cup R}|_E$ has no subbundles of slope greater than $g+3+\frac{1}{k}$ and no quotient bundles of slope less than $g+3+\frac{1}{k+1}$.*

We will deduce this from taking $m = 0$ in the following more general statement.

Lemma 4.3. *Suppose that $E \subseteq \mathbb{P}^{g-1}$ is an elliptic normal curve and R is a general g-secant rational curve of degree $g-2$. Let $0 \le m \le 5$. Write $g - 2 = (6-m)k + \epsilon$ for $0 \le \epsilon < 6 - m$. Then*

$$N_E[p_1 + \cdots + p_{g-m} \overset{+}{\rightsquigarrow} R]$$

has no subbundles of slope greater than $g+3+\frac{1}{k}$, and no quotient bundles of slope less than $g+3+\frac{1}{k+1}$.

In the course of proving Lemma 4.3, we will need the following result.

Lemma 4.4. *Let $n \geq 2$ be an integer and suppose that $\Lambda \subset \mathbb{P}^{g-1}$ is a quasi-transverse n-secant $(n-1)$-plane to E. Suppose that $R \subset \Lambda$ is a general rational curve of degree $n-1$ through $E \cap \Lambda$ and $q \in R$ is another general point. Let y be a general point on E. Then the modified pointing bundle*

$$N_{E\to\Lambda}[\overset{+}{\rightsquigarrow} R][y \overset{+}{\to} q]$$

is stable of slope $g+3+\frac{1}{n}$.

Proof. We will prove this by induction on n. Specialize q to one of the points p where R meets E. If $n > 2$, then the pointing bundle exact sequence towards p is

$$0 \to N_{E\to p}(y) \to N_{E\to\Lambda}[\overset{+}{\rightsquigarrow} R][y \overset{+}{\to} p] \to N_{\overline{E}\to\overline{\Lambda}}[\overset{+}{\rightsquigarrow} \overline{R}](p) \to 0.$$

The subbundle $N_{E\to p}(y)$ is isomorphic to $\mathcal{O}_E(1)(2p+y)$, which is stable of slope $g+3$. The quotient is a twist of another instance of our problem in $\mathbb{P}^{g-2}$. We may therefore assume by induction that it is stable of slope $g+3+\frac{1}{n-1}$. Since $c_1(\mathcal{O}_E(1)(2p+y))$ depends on the choice of the point p, we conclude by Lemma 2.6 that the general fiber is semistable (and hence stable) as desired.

It suffices, therefore, to treat the base case of $n = 2$. In this case, $R = \Lambda$ is a 2-secant line $\overline{pp'}$, and after specializing as above, the pointing bundle exact sequence towards p is

$$0 \to N_{E\to p}(y+p') \to N_{E\to\Lambda}[p \overset{+}{\to} p'][p' \overset{+}{\to} p][y \overset{+}{\to} p] \to N_{\overline{E}\to p'}(2p) \to 0.$$

In this case the subbundle and quotient bundles are stable line bundles of slopes $g+4$ and $g+3$ respectively. Again, applying Lemma 2.6, we conclude that the general fiber is semistable (and hence stable) as desired. □

Proof of Lemma 4.3. Our argument will be by backwards induction on m. The base case of $m = 5$ is Lemma 4.5 below, so we suppose $m \leq 4$.

We first prove the upper bound on the slope of a subbundle by exhibiting a degeneration that lies in an exact sequence with a subbundle that is stable of slope exactly $g+3+\frac{1}{k}$ and quotient which satisfies our inductive hypothesis. Let $\Lambda_1 \simeq \mathbb{P}^{k-1} \subset \mathbb{P}^{g-1}$ be the span of the first k points $p_1, \dots, p_k$ of $E \cap R$. Let $\Lambda_2 \simeq \mathbb{P}^{g-k-2}$ be the span of the last $g-1-k$ points $p_{k+2}, \dots p_g$. Since the remaining point p_{k+1} is constrained to lie in the hyperplane spanned by the other points, there is a unique line L through p_{k+1} that meets both Λ_1 and Λ_2.

Let x_1 and x_2 denote the points where L meets Λ_1 and Λ_2, respectively. Let R_1 be a general rational curve in Λ_1 of degree $k-1$ through $p_1, \dots, p_k, x_1$, and let R_2 be a general rational curve in Λ_2 of degree $g-k-2$ through $p_{k+2}, \dots, p_g, x_2$. Then

$$R^\circ := L \cup R_1 \cup R_2$$

is a degeneration of R. It suffices to prove that $N_E[p_1 + \cdots + p_{g-m} \overset{+}{\rightsquigarrow} R^\circ]$ has no subbundles of slope greater than $g+3+\frac{1}{k}$ to prove the lemma. Consider the pointing bundle exact sequence for pointing towards the subspace Λ_1:

$$0 \to N_{E\to\Lambda_1}[\overset{+}{\rightsquigarrow} R_1][p_{k+1} \overset{+}{\to} x_1] \to N_E[p_1+\cdots+p_{g-m} \overset{+}{\rightsquigarrow} R^\circ] \to N_{\overline{E}}[p_{k+2}+\cdots+p_{g-m} \overset{+}{\rightsquigarrow} \overline{R_2}](p_1+\cdots+p_k) \to 0.$$

In order to use Lemma 4.4 to show that $N_{E\to\Lambda_1}[\overset{+}{\rightsquigarrow} R_1][p_{k+1} \overset{+}{\to} x_1]$ is stable of slope $g+3+\frac{1}{k}$, we need that, as the points $p_{k+1}, \dots, p_g$ vary, the point x_1 is general in Λ_1. That is, there are no obstructions to lifting a deformation of the point x_1 to a deformation of the plane $\Lambda_2' := \overline{\Lambda_2, x_1}$ (maintaining the necessary incidences with E). These obstructions live in $H^1(\Lambda_2', N)$, where the bundle N is the kernel of the map

$$N_{\Lambda_2'} \to N_{\Lambda_2'}\big|_{x_1} \oplus \bigoplus_{i=k+1}^{g} N_{\overline{\Lambda_2' T_{p_i} E}}\big|_{p_i}.$$

The key numerical input is $2k \leq g$, which follows from $m \leq 4$. Since Λ_2' is the complete intersection of the k hyperplanes spanned by Λ_2' and all but one of the tangent lines $T_{p_i}E$ for $k+1 \leq i \leq 2k \leq g$, the bundle N sits in the exact sequence

$$0 \to \bigoplus_{i=k+1}^{2k} N_{\Lambda_2'/\overline{\Lambda_2' T_{p_i} E}} \otimes \mathscr{I}_{x_1 \cup p_{k+1} \cup \cdots \cup \widehat{p_i} \cup \cdots \cup p_g} \to N \to P \to 0,$$

where P is a punctual sheaf (and hence $h^1(P) = 0$). Moreover, the evaluation map ev in

$$0 \to N_{\Lambda_2'/\overline{\Lambda_2' T_{p_i} E}} \otimes \mathscr{I}_{x_1 \cup p_{k+1} \cup \cdots \cup \widehat{p_i} \cup \cdots \cup p_g} \to \mathcal{O}_{\Lambda_2'}(1) \xrightarrow{\text{ev}} \mathcal{O}_{\Lambda_2'}(1)|_{x_1 \cup p_{k+1} \cup \cdots \cup \widehat{p_i} \cup \cdots \cup p_g} \to 0$$

is surjective on global sections, since the points $x_1 \cup p_{k+1} \cup \cdots \cup \widehat{p_i} \cup \cdots \cup p_g$ form a basis for the plane Λ_2', and $h^1(\mathcal{O}_{\Lambda_2'}(1)) = 0$. Therefore

$$H^1\left(N_{\Lambda_2'/\overline{\Lambda_2' T_{p_i} E}} \otimes \mathscr{I}_{x_1 \cup p_{k+1} \cup \cdots \cup \widehat{p_i} \cup \cdots \cup p_g}\right) = 0,$$

and hence $H^1(N) = 0$.

In the quotient, $N_{\overline{E}}[p_{k+2} + \cdots + p_{g-m} \overset{+}{\rightsquigarrow} \overline{R_2}]$ is another case of our inductive hypothesis with one fewer modification occurring at the points of incidence of $\overline{R_2}$ with $\overline{E}$. The result now follows from our inductive hypothesis.

Now we turn to the lower bound on the slope of any quotient. We will exhibit a specialization that lies in an exact sequence with a subbundle that is stable of slope exactly $g + 3 + \frac{1}{k+1}$ and a quotient bundle which satisfies our inductive hypothesis. We will modify the same argument by letting Λ_1 be the k-dimensional span of $p_1, \ldots, p_{k+1}$ and letting Λ_2 be a the $(g-k-3)$-dimensional span of $p_{k+3}, \ldots, p_g$. As above, there is a unique line L through the remaining point p_{k+2} that meets both Λ_1 (at a point x_1) and Λ_2 (at a point x_2). We define R_1 and R_2 analogously to above. In the pointing bundle exact sequence towards Λ_1:

$$0 \to N_{E \to \Lambda_1}[\overset{+}{\rightsquigarrow} R_1][p_{k+2} \overset{+}{\to} x_1] \to N_E[p_1 + \cdots + p_{g-m} \overset{+}{\rightsquigarrow} R^\circ] \to N_{\overline{E}}[p_{k+3} + \cdots + p_{g-m} \overset{+}{\rightsquigarrow} \overline{R_2}](p_1 + \cdots + p_{k+1}) \to 0,$$

the subbundle is stable of slope $g + 3 + \frac{1}{k+1}$ by Lemma 4.4 (using the same argument to ensure generality of x_1), and the quotient is a twist of another case of our inductive hypothesis in $\mathbb{P}^{g-k-2}$ (with a smaller value of k if $\epsilon = 0$), with one fewer modification occurring along $\overline{R_2}$.

This completes the inductive step. All that remains is therefore to verify the base case, which is Lemma 4.5 below. ☐

Lemma 4.5. *Suppose that $E \subset \mathbb{P}^{g-1}$ is an elliptic normal curve, and R is a degree $g-2$ rational curve meeting E at $p_1, \ldots, p_g$ quasi-transversely. Then*

$$N_E' := N_E[p_1 + \cdots + p_{g-5} \overset{+}{\rightsquigarrow} R]$$

is stable of slope $g + 3 + \frac{1}{g-2}$.

Proof. We will prove this by induction on g. The base case is $g = 5$, in which case $N_E' = N_E$ is stable by [EiL92]. Otherwise, when $g \geq 6$, the bundle N_E' is modified at p_1. Let $\Lambda \simeq \mathbb{P}^{g-3}$ be the span of $p_2, \ldots, p_{g-1}$. Let L be the line through p_1 and p_g that meets Λ at a point x. Let R' be a rational curve of degree $g-3$ through $p_2, \ldots, p_{g-1}, x$. Then $R^\circ = R' \cup L$ is a degeneration of R. Consider the specialization

$$N_E[p_1 + \cdots + p_{g-5} \overset{+}{\rightsquigarrow} R^\circ]$$

of N_E'. Consider the pointing bundle exact sequence for pointing towards p_g:

$$0 \to N_{E \to p_g}(p_1) \to N_E[p_1 + \cdots + p_{g-5} \overset{+}{\rightsquigarrow} R^\circ] \to N_{\overline{E}}(p_g)[p_2 + \cdots + p_{g-5} \overset{+}{\rightsquigarrow} \overline{R'}] \to 0.$$

The subbundle has slope $g+3$ exactly. Since $\overline{R'}$ is a rational curve of degree $g-3$ meeting $\overline{E}$ at $p_2, \ldots, p_{g-1}, x$, the quotient bundle is a twist of an instance the same problem in $\mathbb{P}^{g-2}$. By induction it is stable. Moreover, $c_1(N_{E\to p_g}(p_1))$ depends on the ordering of $p_1, p_2, \ldots, p_g$. Hence by Lemma 2.6, the general fiber N'_E is semistable (thus stable) as desired. □

We complete the proof by appealing to the naturality of the maximal destabilizing subbundle, and using the following purely combinatorial lemma.

Lemma 4.6. *Let $k \geq 0$ and $0 \leq \epsilon < 6$ be integers with*

$$(k, \epsilon) \notin \{(0,2),(0,4)\}.$$

Then there are no integers r, d satisfying

$$1 \leq r < 6k + \epsilon, \text{ and} \tag{2}$$

$$6k + 5 + \epsilon + \frac{6}{6k+\epsilon} < \frac{d}{r} \leq 6k + 5 + \epsilon + \frac{1}{k}, \text{ and} \tag{3}$$

$$6k + 5 + \epsilon + \frac{1}{k+1} \leq \frac{(6k+5+\epsilon)(6k+\epsilon) + 6 - d}{6k + \epsilon - r}, \text{ and} \tag{4}$$

$$(6k + 2 + \epsilon) \mid d. \tag{5}$$

Proof. Suppose such integers d and r exist. Clearing denominators, (3) and (4) yield:

$$(6k+\epsilon)d - (6k+\epsilon+2)(6k+\epsilon+3)r > 0$$
$$-kd + (6k^2 + k\epsilon + 5k + 1)r \geq 0$$
$$-(k+1)d + (6k^2 + k\epsilon + 11k + \epsilon + 6)r \geq \epsilon - 6.$$

Adding $6 - \epsilon$ times the second of these inequalities to ϵ times the third yields

$$(6k+\epsilon)d - (6k+\epsilon+2)(6k+\epsilon+3)r \leq \epsilon(6-\epsilon).$$

Combined with the first, we learn that the integer

$$X := (6k+\epsilon)d - (6k+\epsilon+2)(6k+\epsilon+3)r$$

satisfies

$$0 < X \leq \epsilon(6-\epsilon).$$

On the other hand, by (5),

$$X = (6k+\epsilon)d - (6k+\epsilon+2)(6k+\epsilon+3)r \equiv 0 \mod 6k+\epsilon+2.$$

It follows that $\epsilon(6-\epsilon) \geq 6k+\epsilon+2$, or upon rearrangement, $6k+2 \leq \epsilon(5-\epsilon)$. Since ϵ is an integer with $0 \leq \epsilon \leq 5$, we have $\epsilon(5-\epsilon) \leq 6$, and so $6k+2 \leq 6$, which implies $k = 0$. Moreover, if $\epsilon = 0$ or $\epsilon = 5$, then $\epsilon(5-\epsilon) = 0$, in violation of $6k+2 \leq \epsilon(5-\epsilon)$. The cases $(k,\epsilon) = (0,2)$ and $(0,4)$ are excluded by assumption, so the only remaining cases are $(k,\epsilon) = (0,1)$ and $(0,3)$:

- When $(k,\epsilon) = (0,1)$, we have $1 \leq r < 6k+\epsilon = 1$, which is a contradiction.
- When $(k,\epsilon) = (0,3)$, we have $0 < X \leq 9$ and $X \equiv 0 \bmod 5$. Therefore $3d - 30r = X = 5$, which is a contradiction by looking mod 3.

This completes the proof. □

Proof of Theorem 4.1. Let (d,r) be the degree and rank of the maximal destabilizing subbundle of $N_{E\cup R}|_E$. Since this naturally-defined bundle depends only on the choice of $\mathcal{O}_E(1)$ plus choices varying in a rational base, its determinant gives a natural map $\operatorname{Pic}^g E \to \operatorname{Pic}^d E$. By Lemma 2.7, the degree d is divisible by g. By Lemma 4.2, the slope d/r is at most $g + 3 + \frac{1}{k}$, with quotient bundle having slope at least $g + 3 + \frac{1}{k+1}$. By Lemma 4.6, no such integers d and r exist, and hence no destabilizing bundles exist, when $g \notin \{4, 6\}$. □

Proof of Theorem 1.1 in odd genus. Let C be a general canonical curve of odd genus $g \geq 3$. By [CLV22, Lemma 4.1], semistability of N_C follows from the semistability of $N_{E\cup R}|_E$ and $N_{E\cup R}|_R$. The first of these is Theorem 4.1; the second is Lemma 3.1. □

Proof of Theorem 1.1 in even genus using the Strong Franchetta Conjecture. The proof of Theorem 1.1 in even genus is considerably harder. Here we will give an argument using the Strong Franchetta Conjecture proved by Harer [H83] and Arbarello and Cornalba [AC87, AC98] in characteristic 0 and Schröer [S03] in characteristic p. In next two sections, we will give an elementary proof.

Suppose that the normal bundle of the general canonical curve is unstable. Specialize to $E \cup R$ as in Section 3. If $g \geq 8$, then $N_{E\cup R}|_E$ is semistable by Theorem 4.1, and any destabilizing subbundle of $N_{E\cup R}|_R$ of rank r has slope at most $\mu(N_{E\cup R}|_R) + \frac{1}{r}$ by Lemma 3.1. Consequently, if $g \geq 8$, then the maximal destabilizing subbundle F of N_C would satisfy

$$\mu(N_C) < \mu(F) \leq \mu(N_C) + \frac{1}{r} \quad \text{where} \quad r = \operatorname{rk} F.$$

On the other hand, by the Strong Franchetta Conjecture, $\det F$ is a multiple of the canonical bundle. We conclude that the degree of F is $s(2g-2)$ for some integer s. Since the slope of the normal bundle of a canonical curve is $(g+1)(2g-2)/(g-2)$, we obtain the inequality

$$\frac{(g+1)(2g-2)}{g-2} < \frac{s(2g-2)}{r} \leq \frac{(g+1)(2g-2)}{g-2} + \frac{1}{r},$$

or upon rearrangement,

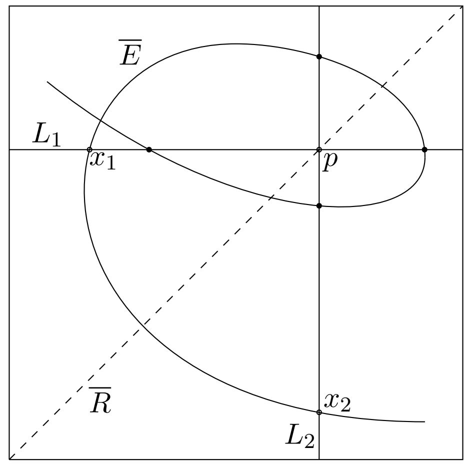
$$0 < (s-r)(g-2) - 3r \leq \frac{g-2}{2g-2} < 1.$$

Since $(s-r)(g-2)-3r$ is an integer, this is a contradiction. Hence, N_C is semistable for the general canonical curve.

5. Degeneration so that Q_{sing} meets E

In order to give an elementary proof of Theorem 1.1 in the even genus case using the explicit description of the HN-filtration given in Section 3.1, it will suffice to bound the slopes of subbundles of $N_{E/Q}[2\Gamma \overset{+}{\to} N_{E/S}]$. To achieve this, we will utilize a further degeneration in which E meets the singular locus Q_{sing} of the rank 4 quadric Q described in Section 3.1 in two points $\{x_1, x_2\}$. The basic inductive strategy will be to degenerate in this way, and then examine the sequence obtained by projection from the line $\overline{x_1 x_2}$. If we do this carefully, the quotient will be another instance of our Setup 3.2 in $\mathbb{P}^{2n-1}$. In this section, we construct this degeneration and prove that the projection exact sequence behaves as desired. In the next section, we will use this to complete our inductive proof of Theorem 1.1 in the even genus case.

To construct this degeneration, we start with an instance $(\overline{E}, \overline{R}, \overline{S}, \overline{Q})$ of Setup 3.2 in $\mathbb{P}^{2n-1}$. Write $\overline{\Gamma} = \overline{E} \cap \overline{R}$. Recall that via the given maps $\overline{f}_1$ and $\overline{f}_2$, respectively the diagonal, the curves $\overline{E}$ and $\overline{R}$ map to $\mathbb{P}^1 \times \mathbb{P}^1$, as pictured below.



We take $x_1, x_2 \in \overline{E}$ so that $\overline{f}_1(x_1) = \overline{f}_2(x_2)$, and write $p = (\overline{f}_1(x_1), \overline{f}_2(x_2))$. Let $\overline{L}_1 = \overline{f}_1(x_1) \times \mathbb{P}^1$ and $\overline{L}_2 = \mathbb{P}^1 \times \overline{f}_2(x_2)$ denote the corresponding lines of the ruling (which meet at p). Let Δ denote the remaining set (not including $\{x_1, x_2\}$) of points where one of the $\overline{L}_i$ meets $\overline{E}$, together with p and the nodes of $\overline{E}$. Construct the blowup S° of $\mathbb{P}^1 \times \mathbb{P}^1$ at Δ, and write

$$R^\circ = \text{proper transform of } \overline{R}$$
$$E^\circ = \text{proper transform of } \overline{E}$$
$$L_i = \text{proper transform of } \overline{L}_i$$

For $q \in \Delta$, write F_q for the exceptional divisor over q. Set $p_i = L_i \cap F_p$.

The images of $E^\circ \cup L_1 \cup L_2$, $R^\circ \cup F_p$, and S°, under the complete linear series $|\mathcal{O}_{S^\circ}(n,n)(-\sum_{q\in\Delta} F_q)|$, along with the cone Q° over $\overline{Q}$ with vertex $\overline{x_1 x_2}$, give a degeneration of (E,R,S,Q) in our Setup 3.2 in $\mathbb{P}^{2n+1}$. The image of $E^\circ \cup L_1 \cup L_2$ coincides with the image of just E°, because L_1 and L_2 are contracted to points x_1 and x_2 on E°, respectively. The image of R° is of degree $2n-1$, and F_p is mapped to the line which meets E° at x_1 and x_2 (and which also meets the other component R°).

Consider (E,R,S,Q) limiting to $(E^\circ, R^\circ \cup F_p, S^\circ, Q^\circ)$. The above description shows that x_1 and x_2 are limits of points $\widehat{x}_1, \widehat{x}_2 \in \Gamma := E \cap R$. Write $\Gamma_- = \Gamma \smallsetminus \{\widehat{x}_1, \widehat{x}_2\}$. The limit of Γ_- is identified with $\overline{\Gamma}$. Our next task is to determine the flat limit of the bundles

$$N_{E/Q}[2\Gamma_- + \widehat{x}_1 + \widehat{x}_2 \overset{+}{\to} N_{E/S}] = N_{E/Q}[2\Gamma_- \overset{+}{\to} N_{E/S}][\widehat{x}_1 + \widehat{x}_2 \overset{+}{\rightsquigarrow} R].$$

This is subtle precisely because E° passes through Q°_{sing} (in particular the flat limit is not just $N_{E^\circ/Q^\circ}[2\overline{\Gamma} \overset{+}{\to} N_{E^\circ/S^\circ}][x_1 + x_2 \overset{+}{\rightsquigarrow} R^\circ]$). To do this, define

$$B^\circ := \mathrm{Bl}_{Q^\circ_{\text{sing}}} Q^\circ.$$

Explicitly, B° is the graph of the rational map given by projection: $Q^\circ \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. In particular, the exceptional divisor of B° is isomorphic to

$$[Q^\circ_{\text{sing}} \simeq \mathbb{P}^{2n-3}] \times \mathbb{P}^1 \times \mathbb{P}^1.$$

The line $\overline{x_1 x_2}$ naturally embeds in Q°_{sing} and the lines L_i naturally embed in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 5.1. *Let (E,R,S,Q) be an instance of Setup 3.2 in $\mathbb{P}^{2n+1}$. Then*

$$N_{E/Q}[2\Gamma_- \overset{+}{\to} N_{E/S}][\widehat{x}_1 + \widehat{x}_2 \overset{+}{\rightsquigarrow} R]$$

admits a specialization to

$$N^\circ := N_{E^\circ/B^\circ}[2\overline{\Gamma} \overset{+}{\to} N_{E^\circ/S^\circ}][x_1 \overset{+}{\rightsquigarrow} \overline{x_1 x_2} \times L_1][x_2 \overset{+}{\rightsquigarrow} \overline{x_1 x_2} \times L_2].$$

Proof. Since E does not meet Q_{sing}, we have

$$N_{E/Q}[2\Gamma_- \xrightarrow{+} N_{E/S}][\widehat{x}_1 + \widehat{x}_2 \overset{+}{\rightsquigarrow} R] \simeq N_{E/B}[2\Gamma_- \xrightarrow{+} N_{E/S}][\widehat{x}_1 + \widehat{x}_2 \overset{+}{\rightsquigarrow} R] \quad \text{where} \quad B := \mathrm{Bl}_{Q_{\text{sing}}} Q.$$

This bundle fits into a flat family $\mathcal{N}$ whose central fiber is

$$N := \mathcal{N}|_0 = N_{E^\circ \cup L_1 \cup L_2/B^\circ}[2\overline{\Gamma} \xrightarrow{+} N_{E^\circ/S^\circ}][\overset{+}{\rightsquigarrow} F_p] \quad \text{where} \quad B^\circ := \mathrm{Bl}_{Q^\circ_{\text{sing}}} Q^\circ.$$

The L_i are sent to lines in the exceptional divisor of types $(0,1,0)$ and $(0,0,1)$, respectively. In particular, their normal bundles in the exceptional divisor are trivial, and so their normal bundles in B° are $\mathcal{O}_{L_i}^{\oplus(r-2)} \oplus \mathcal{O}_{L_i}(-1)$. The restriction

$$N|_{L_i} \simeq N_{L_i/B^\circ}[x_i \overset{+}{\rightsquigarrow} E^\circ][p_i \overset{+}{\rightsquigarrow} F_p]$$

is obtained by making two positive modifications. Since E° is transverse to the exceptional divisor at the x_i, the positive modification at x_i is transverse to $\mathcal{O}_{L_i}^{\oplus(r-2)}$. Therefore

$$N|_{L_i} \simeq \mathcal{O}_{L_i}^{\oplus(r-2)} \oplus \mathcal{O}_{L_i}(1).$$

To identify the positive subbundle, note that there is a unique subbundle of $N|_{L_i}$ that is isomorphic to $\mathcal{O}_{L_i}(1)$, and that one such subbundle is $N_{L_i/\overline{x_1x_2}\times L_i}(p_i)$. Consider the modification

$$\mathcal{N}' := \mathcal{N}[L_1 \xrightarrow{+} N_{L_1/\overline{x_1x_2}\times L_1}(p_1)][L_2 \xrightarrow{+} N_{L_2/\overline{x_1x_2}\times L_2}(p_2)].$$

Away from the central fiber, we have $\mathcal{N}' \simeq \mathcal{N}$. The central fiber $\mathcal{N}'|_0$ therefore gives another flat limit of the bundle $N_{E/Q}[2\Gamma_- \xrightarrow{+} N_{E/S}][\widehat{x}_1 + \widehat{x}_2 \overset{+}{\rightsquigarrow} R]$. But by construction, $\mathcal{N}'|_0$ has trivial restriction to L_1 and L_2. Blowing down L_1 and L_2, we conclude that a flat limit of the bundles $N_{E/Q}[2\Gamma_- \xrightarrow{+} N_{E/S}][\widehat{x}_1 + \widehat{x}_2 \overset{+}{\rightsquigarrow} R]$ is therefore

$$N^\circ = \mathcal{N}'|_{E^\circ} \simeq N_{E^\circ/B^\circ}[2\overline{\Gamma} \xrightarrow{+} N_{E^\circ/S^\circ}][x_1 \overset{+}{\rightsquigarrow} \overline{x_1x_2} \times L_1][x_2 \overset{+}{\rightsquigarrow} \overline{x_1x_2} \times L_2]. \qquad \square$$

Our final goal is to relate this to projection from the line $\overline{x_1x_2}$. By construction, this projection map sends $(E^\circ, R^\circ, S^\circ Q^\circ)$ in $\mathbb{P}^{2n+1}$ to $(\overline{E}, \overline{R}, \overline{S}, \overline{Q})$ in $\mathbb{P}^{2n-1}$. We accomplish this by rewriting N° in terms of the normal bundle of the proper transform of E° in

$$B^\circ_- := \mathrm{Bl}_{\overline{x_1x_2}} Q^\circ.$$

Namely, write M_i for the preimage of $\overline{x_1x_2} \times L_i$ in the exceptional divisor of B°_- (under the natural rational map from the exceptional divisor of B°_- to the exceptional divisor of B°). Then

$$N^\circ \simeq N_{E^\circ/B^\circ_-}[2\overline{\Gamma} \xrightarrow{+} N_{E^\circ/S^\circ}][x_1 \overset{+}{\rightsquigarrow} M_1][x_2 \overset{+}{\rightsquigarrow} M_2].$$

Explicitly, the exceptional divisor of B°_- is isomorphic to $\overline{x_1x_2} \times \overline{Q}$, with $M_i = \overline{x_1x_2} \times \overline{M}_i$, where the $\overline{M}_i$ are the $(2n-3)$-planes of the rulings of $\overline{Q}$ corresponding to L_i.

Note that $\overline{x_1x_2} \times p$ is contained in M_1 and M_2, and is contracted to the point $p \in \overline{Q}$ under projection. Moreover, $\overline{M}_i$ is transverse (not just quasi-transverse!) to $\overline{E}$ at x_i. Projection from $\overline{x_1x_2}$ therefore induces the exact sequence

$$0 \to \mathcal{O}_{E^\circ}(1)(x_1 + x_2)^{\oplus 2} \to N^\circ \to N_{\overline{E}/\overline{Q}}[2\overline{\Gamma} \xrightarrow{+} N_{\overline{E}/\overline{S}}](x_1 + x_2) \to 0. \tag{6}$$

6. Completing the proof in even genus

Let $g = 2n + 2$ be even. We consider the degenerate canonical curve $E \cup R \subset \mathbb{P}^{2n+1}$ introduced in Section 3. In this section, we leverage the geometric description of the HN-filtration of $N_{E\cup R}|_R$ given in Section 3.1 and the semistability of $N_{E\cup R}|_E$ proved in Section 4 to prove that the normal bundle of a general canonical curve of even genus is semistable.

Let S, Σ_1, Σ_2, and Q be as in Setup 3.2. We first reduce to proving a bound on the slopes of certain subbundles of $N_{E\cup R/Q}|_E$.

Condition 6.1. For a general $E \cup R \subset \mathbb{P}^{2n+1}$, every subbundle $F \subseteq N_{E\cup R/Q}|_E$ with

$$N_{E\cup R/S}|_\Gamma \subset F|_\Gamma$$

satisfies

$$\mu(F) \leq 2n + 5 + \frac{3}{n} - \frac{1}{\operatorname{rk} F}.$$

Condition 6.2. For a general $E \cup R \subset \mathbb{P}^{2n+1}$, every subbundle $F \subseteq N_{E/Q}[2\Gamma \overset{+}{\to} N_{E/S}]$ satisfies

$$\mu(F) \leq 2n + 5 + \frac{3}{n} + \frac{2n+1}{\operatorname{rk} F}.$$

Lemma 6.3. *Condition 6.2 implies Condition 6.1.*

Proof. If F is a subbundle of $N_{E\cup R/Q}|_E = N_{E/Q}[\Gamma \overset{+}{\to} N_{E/S}]$ with $N_{E\cup R/S}|_\Gamma \subset F|_\Gamma$, then the modification $F[\Gamma \overset{+}{\to} N_{E/S}]$ is a subbundle of $N_{E/Q}[2\Gamma \overset{+}{\to} N_{E/S}]$ with

$$\mu(F[\Gamma \overset{+}{\to} N_{E/S}]) = \mu(F) + \frac{2n+2}{\operatorname{rk} F}. \qquad \square$$

Proposition 6.4. *Suppose that Condition 6.1 is satisfied. Then $N_{E\cup R}$ is semistable.*

Proof. Let $\nu \colon E \sqcup R \to E \cup R$ denote the normalization, and $G \subseteq \nu^* N_{E\cup R}$ be any subbundle. By Lemma 3.1 and Theorem 4.1, we have

$$\mu(G|_R) \leq 2n + 3 + \frac{1}{\operatorname{rk} G} \quad \text{and} \quad \mu(G|_E) \leq \mu(N_{E\cup R}|_E) = 2n + 5 + \frac{3}{n}.$$

Combining these, we have

$$\mu^{\mathrm{adj}}(G) \leq \mu(G) = \mu(G|_R) + \mu(G|_E) \leq 4n + 8 + \frac{3}{n} + \frac{1}{\operatorname{rk} G},$$

with the stronger bound

$$\mu^{\mathrm{adj}}(G) \leq 4n + 8 + \frac{3}{n} = \mu(N_{E\cup R})$$

unless G is actually a subbundle of $N_{E\cup R}$ and $G|_R \supset N_{E\cup R/S}|_R$. We therefore assume that both of these hold. The restriction $G|_E$ is thus a subbundle of $N_{E\cup R}|_E$ with $N_{E\cup R/S}|_\Gamma \subset G|_\Gamma$. Write G' for the kernel of the map from $G|_E$ to $N_Q|_E$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{E\cup R/Q}|_E & \longrightarrow & N_{E\cup R}|_E & \longrightarrow & N_Q|_E \simeq \mathcal{O}_E(2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & \nearrow & & & \\
 & & \ker(G|_E \to N_Q|_E) = G' & \longrightarrow & G|_E & & & &
\end{array}$$

If $G|_E \neq G'$, then the map $G|_E/G' \to \mathcal{O}_E(2)$ factors through $\mathcal{O}_E(2)(-\Gamma)$, which is stable of slope $2n+2$. On the other hand, the kernel $G' \subseteq N_{E\cup R/Q}|_E$ has slope

$$\mu(G') \leq 2n + 5 + \frac{3}{n} - \frac{1}{\operatorname{rk} G'} \leq 2n + 5 + \frac{3}{n} - \frac{1}{\operatorname{rk} G},$$

by assumption. Thus $G|_E$ also has slope bounded by $2n + 5 + \frac{3}{n} - \frac{1}{\operatorname{rk} G}$. Hence

$$\mu(G) = \mu(G|_R) + \mu(G|_E) \leq \left(2n + 3 + \frac{1}{\operatorname{rk} G}\right) + \left(2n + 5 + \frac{3}{n} - \frac{1}{\operatorname{rk} G}\right) = \mu(N_{E\cup R}),$$

and $N_{E\cup R}$ is semistable. $\square$

Our goal is therefore to prove that Condition 6.1 holds for all $n \geq 3$. In fact, we will prove that Condition 6.2 holds for all $n \geq 3$, since this implies that Condition 6.1 holds. While Condition 6.2 is stated for *all* subbundles of $N_{E/Q}[2\Gamma \overset{+}{\to} N_{E/S}]$, it suffices to check the slope bound for the finitely many Harder–Narasimhan pieces.

Lemma 6.5. *Let N be a vector bundle on an irreducible curve C with HN-filtration*

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = N.$$

Let $B(r,d)$ be any (affine) linear function whose coefficient of d is nonnegative. If $B(0,0) \le 0$ and $B(\operatorname{rk} V_i, \deg V_i) \le 0$ for all i, then $B(\operatorname{rk} F, \deg F) \le 0$ for all subbundles $F \subset N$.

Proof. The points

$$\{(0,-\infty),(0,0),(\operatorname{rk} V_1, \deg V_1), \dots, (\operatorname{rk} V_m, \deg V_m)\}$$

form the vertices of a convex polygon in the (r,d) plane. For any subbundle $F \subseteq N$, the pair $(\operatorname{rk} F, \deg F)$ is in this polygon. The assumption that $B(r,d) \le 0$ for all vertices implies that it is also true for any point of the convex polygon. □

Corollary 6.6. *Suppose that for each HN-piece V of $N_{E/Q}[2\Gamma \xrightarrow{+} N_{E/S}]$ we have*

$$\mu(V) \le 2n + 5 + \frac{3}{n} + \frac{2n+1}{\operatorname{rk} V}.$$

Then Condition 6.2 holds.

Proof. Apply Lemma 6.5 with

$$B(r,d) = d - \left(2n + 5 + \frac{3}{n}\right) r - (2n+1).$$ □

The final input is the specialization of (E,R,S,Q) constructed in Section 5, giving rise to the exact sequence (6). Using this, we will prove the following numerical proposition, which is the heart of our inductive proof.

Proposition 6.7. *Let $n > 3$. If Condition 6.2 holds in $\mathbb{P}^{2n-1}$, then it holds in $\mathbb{P}^{2n+1}$.*

Proof. We will use the notation and results of Section 5. In particular, let $(\overline{E}, \overline{R}, \overline{S}, \overline{Q})$ be a general instance of Setup 3.2 in $\mathbb{P}^{2n-1}$. Let x_1, x_2 be points on $\overline{E}$ such that $\overline{f}_1(x_1) = \overline{f}_2(x_2)$. Then in Section 5 we constructed a specialization $(E^\circ, R^\circ \cup F_p, S^\circ, Q^\circ)$ of a general instance (E,R,S,Q) of Setup 3.2 in $\mathbb{P}^{2n+1}$, such that E° meets Q°_{sing} in the points x_1, x_2. We write $\widehat{x}_1, \widehat{x}_2$ for points on E limiting to x_1, x_2.

Applying Corollary 6.6, it suffices to check that Condition 6.2 holds for each piece of the HN-filtration of $N_{E/Q}[2\Gamma \xrightarrow{+} N_{E/S}]$. Since the HN-pieces are natural, their degrees are multiples of $2n+2$ by Lemma 2.7. Let $F_0 \subset N_{E/Q}[2\Gamma \xrightarrow{+} N_{E/S}]$ be any such subbundle of rank r and degree a multiple of $2n+2$. Let $F \subset N_{E/Q}[2\Gamma - \widehat{x}_1 - \widehat{x}_2 \xrightarrow{+} N_{E/S}]$ be the intersection of F_0 with $N_{E/Q}[2\Gamma - \widehat{x}_1 - \widehat{x}_2 \xrightarrow{+} N_{E/S}]$. We have

$$\mu(F_0) \le \mu(F) + \frac{2}{\operatorname{rk} F},$$

so it suffices to show

$$\mu(F) \le 2n + 5 + \frac{3}{n} + \frac{2n-1}{\operatorname{rk} F}.$$

We now utilize the specialization constructed in Section 5. Write N° for the bundle appearing in Lemma 5.1, which is a flat limit of the bundle $N_{E/Q}[2\Gamma - \widehat{x}_1 - \widehat{x}_2 \xrightarrow{+} N_{E/S}]$. This bundle sits in the exact sequence

$$0 \to \mathcal{O}_{E^\circ}(1)(x_1 + x_2)^{\oplus 2} \to N^\circ \xrightarrow{\phi} N_{\overline{E}/\overline{Q}}[2\overline{\Gamma} \xrightarrow{+} N_{\overline{E}/\overline{S}}](x_1 + x_2) \to 0.$$

Let $F^\circ \subseteq N^\circ$ be the saturation of the flat limit of F. Then $\operatorname{rk} F^\circ = \operatorname{rk} F = r$ and $\deg F^\circ \ge \deg F$.

Case 1: F° intersects $\ker \phi$ nontrivially. We obtain an exact sequence

$$0 \to F' \to F^\circ \to F'' \to 0,$$

where $F' \subset \mathcal{O}_E(1)(x_1+x_2)^{\oplus 2}$ and $F'' \subset N_{\overline{E}/\overline{Q}}[2\overline{\Gamma} \overset{+}{\to} N_{\overline{E}/\overline{S}}](x_1+x_2)$. Since $\mathcal{O}_E(1)(x+y)^{\oplus 2}$ is semistable, we have that

$$\mu(F') \leq \mu(\mathcal{O}_E(1)(x_1+x_2)^{\oplus 2}) = 2n+4.$$

By our inductive hypothesis, we have that

$$\mu(F'') \leq 2n+5+\frac{3}{n-1}+\frac{2n-1}{\operatorname{rk} F''}.$$

We conclude that

$$\mu(F^\circ) \leq \frac{1}{r}\left(2n+4\right)+\frac{r-1}{r}\left(2n+5+\frac{3}{n-1}+\frac{2n-1}{r-1}\right) = 2n+5+\frac{3r-2-n}{r(n-1)}+\frac{2n-1}{r}. \tag{7}$$

If $n \geq 3$, then $6n-3 \leq n^2+2n$. Since $r \leq \operatorname{rk}(N_{E/Q}[2\Gamma \overset{+}{\to} N_{E/S}]) = 2n-1$, we have the inequality $3r \leq n^2+2n$, which implies that

$$\frac{3r-2-n}{r(n-1)} \leq \frac{3}{n} \quad \text{if} \quad n \geq 3.$$

Substituting into (7), we get

$$\mu(F) \leq \mu(F^\circ) \leq 2n+5+\frac{3}{n}+\frac{2n-1}{r},$$

which proves the proposition when F intersects the kernel of ϕ nontrivially.

Case 2: F° is isomorphic to its image under ϕ. Identifying F° with its image under ϕ, we have

$$F^\circ \subset N_{\overline{E}/\overline{Q}}[2\overline{\Gamma} \overset{+}{\to} N_{\overline{E}/\overline{S}}](x_1+x_2) \quad \text{and} \quad r \leq 2n-2.$$

By the inductive hypothesis, we have that

$$\mu(F^\circ) \leq 2n+5+\frac{3}{n-1}+\frac{2n-1}{r},$$

and hence for the general fiber we have

$$\deg(F_0) \leq \deg(F)+2 \leq \deg(F^\circ)+2 \leq (2n+5)r+\frac{3r}{n-1}+2n+1.$$

We would instead like to show the stronger inequality

$$\deg(F_0) \leq (2n+5)r+\frac{3r}{n}+2n+1. \tag{8}$$

To do this, we will use the fact that naturality of the HN-pieces implies that $2n+2$ divides $\deg(F_0)$. If there are no integers k satisfying the inequality

$$\frac{3r}{n} < k \leq \frac{3r}{n-1}, \tag{9}$$

then (8) holds, as $\deg(F_0)$ is an integer. Hence, we assume that there is an integer k satisfying (9).

First, suppose $n \geq 7$. We claim that the width of the interval (9) is strictly less than 1. Indeed, since $r \leq 2n-2$ and $n \geq 7$,

$$\frac{3r}{n-1}-\frac{3r}{n} = \frac{3r}{n^2-n} \leq \frac{6n-6}{n^2-n} < 1.$$

If (8) does not hold, then

$$3r+k-1 \equiv (2n+5)r+k+2n+1 = \deg(F_0) \equiv 0 \pmod{2n+2},$$

i.e., we may write $3r+k-1 = (2n+2)\ell$ for an integer ℓ. Plugging this back into (9), and subtracting 2ℓ from each term, we obtain

$$\frac{2\ell-k+1}{n} < k-2\ell \leq \frac{4\ell-k+1}{n-1}.$$

If $k-2\ell \leq 0$, the left inequality is violated. If $k-2\ell \geq 1$, then the left fraction is nonpositive, which contradicts our observation that the width of this interval is strictly less than 1.

For $4 \leq n \leq 6$, we complete the proof by checking directly that there are no integers k satisfying the conditions

$$\frac{3r}{n} < k \leq \frac{3r}{n-1}, \quad 0 < r \leq 2n \quad \text{and} \quad 3r + k - 1 \equiv 0 \pmod{2n-2}. \tag{10}$$

The following three tables summarize the possible values of k for each value of r and compute $3r + k - 1 \pmod{2n-2}$ in the three cases $n = 4, 5$ and 6, respectively.

$n = 4$

r	1	2	3	4	5	6
k	1	2	3	4	4 or 5	5 or 6
$3r + k - 1 \pmod{10}$	3	7	1	5	8 or 9	2 or 3

$n = 5$

r	1	2	3	4	5	6	7	8
k	$\varnothing$	$\varnothing$	2	3	$\varnothing$	4	5	5 or 6
$3r + k - 1 \pmod{12}$			10	2		9	1	4 or 5

$n = 6$

r	1	2	3	4	5	6	7	8	9	10
k	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	3	$\varnothing$	4	$\varnothing$	5	6
$3r + k - 1 \pmod{14}$					3		10		3	7

We see that when $n > 3$, there are no integers k satisfying (10). □

To finish, it suffices to deal with the base case:

Proposition 6.8. *Condition 6.2 holds in* $\mathbb{P}^7$.

Proof. In this case $n = 3$ and we want that every subbundle F of $N_{E/Q}[2\Gamma \overset{+}{\to} N_{E/S}]$ has slope at most $12 + \frac{7}{\operatorname{rk} F}$. To prove this, we use the normal bundle exact sequence for $E \subset S \subset Q$. Since S is the complete intersection of Σ_1 and Σ_2 in Q, we have that

$$N_{S/Q}|_E \simeq N_{S/\Sigma_1}|_E \oplus N_{S/\Sigma_2}|_E.$$

Since Σ_1 and Σ_2 are exchanged by monodromy, the two bundles $N_{S/\Sigma_i}|_E$ have degree 24 and the same profile of Jordan–Holder factors. We will first show that $N_{S/\Sigma_i}|_E$ is semistable of slope 12.

The bundle $N_E[\Gamma \overset{+}{\to} N_{E/S}] = N_E[\overset{+}{\rightsquigarrow} R]$ has slope 12, and is hence semistable because it satisfies interpolation by [LV22] (in the language of that paper, this is the inductive hypothesis $I(8, 1, 7, 0, 1)$ and the tuple $(8, 1, 7, 0, 1)$ is good). Consider the normal bundle exact sequence

$$0 \to N_{E/S}(\Gamma) \to N_{E/Q}[\Gamma \to N_{E/S}] \to \left[N_{S/Q}|_E \simeq N_{S/\Sigma_1}|_E \oplus N_{S/\Sigma_2}|_E\right] \to 0.$$

The line subbundle $N_{E/S}(\Gamma)$ has degree 8. First suppose that one (and hence both) of $N_{S/\Sigma_i}|_E$ had a line subbundle of slope at least 15. Then the full preimage in $N_{E/Q}[\Gamma \overset{+}{\to} N_{E/S}]$ would be a bundle of slope at least $38/3 = 12 + 2/3$. Since this is also a subbundle of $N_E[\Gamma \overset{+}{\to} N_{E/S}]$, it contradicts the semistability of $N_E[\Gamma \overset{+}{\to} N_{E/S}]$. Hence every line subbundle of $N_{S/\Sigma_i}|_E$ is of degree at most 14. It suffices, therefore, to rule out the possibility that $N_{S/\Sigma_i}|_E$ is a direct sum of line bundles of degrees $14, 10$ or $13, 11$. In either of these cases, the sum of the two positive subbundles would be of degree 28 or 26. Since 8 does not divide 28 or 26, this is impossible by Lemma 2.7. Hence $N_{S/\Sigma_i}|_E$ is semistable.

We now turn to the normal bundle exact sequence involving the double modification

$$0 \to N_{E/S}(2\Gamma) \to N_{E/Q}[2\Gamma \overset{+}{\to} N_{E/S}] \to \left[N_{S/Q}|_E \simeq N_{S/\Sigma_1}|_E \oplus N_{S/\Sigma_2}|_E\right] \to 0,$$

and consider how F sits with respect to this sequence. If F does not contain $N_{E/S}(2\Gamma)$, then $\mu(F) \leq 12 \leq 12 + \frac{7}{\operatorname{rk} F}$. If F contains $N_{E/S}(2\Gamma)$, then

$$\mu(F) \leq 16\left(\frac{1}{\operatorname{rk} F}\right) + 12\left(\frac{\operatorname{rk} F - 1}{\operatorname{rk} F}\right) \leq 12 + \frac{4}{\operatorname{rk} F} \leq 12 + \frac{7}{\operatorname{rk} F}.$$

□

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