1. 516 Final Exam

This is the take-home final for Math 516. It is due Wednesday December 3. This is an open book exam. You may discuss the problems with people in the class and you may consult books. However, the final write-up must be yours. You may not collaborate while writing the solutions and you may not use anything other than the text book and your course notes.

Problem 1. Do problems 7.15, 7.16, 7.17, 7.18, 7.19 and 7.20 on page 383 of Aluffi.

Problem 2. Let R be a commutative ring with unit. If for every $r \in R$, there exists an integer $n_r > 1$ such that $r^{n_r} = r$, prove that every prime ideal of R is maximal.

Problem 3. Let $\phi : \mathbb{Z}^3 \to \mathbb{Z}^4$ be the $\mathbb{Z}$-module homomorphism given by the matrix

$$\begin{pmatrix} 2 & 4 & 6 \\ 4 & 8 & 10 \\ 3 & 5 & 7 \\ 6 & 10 & 12 \end{pmatrix}.$$

Determine the cokernel of ϕ up to isomorphism.

Problem 4. Let R be a commutative ring with unit. Show that an ideal I is prime if and only if I satisfies the following two conditions

(1) If $I = I_1 \cap I_2$ for two ideals I_1, I_2 in R,then $I = I_1$ or $I = I_2$.
(2) If $a \in R$ and $a^n \in I$ for some positive integer n, then $a \in I$.

Problem 5. Prove that if I is an ideal in a Noetherian ring R, then there exists finitely many prime ideals $P_1, \dots, P_m$ such that $P_1 P_2 \cdots P_m \subset I$.

Problem 6. Describe the spectrum of $\mathbb{Z}[i]$ as explicitly as you can.

Problem 7. Show that the ring $\mathbb{Z}[i\sqrt{2}] := \{a + b\ i\sqrt{2} | a, b \in \mathbb{Z}\}$ is a Euclidean domain. (Hint: use the norm $a^2 + 2b^2$). Using this fact determine all integer solutions of the equation $y^2 + 2 = x^3$.

Problem 8. Let $A \in \mathrm{GL}_n(\mathbb{C})$ be an element of finite order. Show that A is diagonalizable.

Problem 9 (Warings problem for quadratic polynomials)**.** Let $f = \sum_{i,j} a_{i,j} x_i x_j$ be a real quadratic form in $\mathbb{R}[x_1, \dots, x_n]$.

(1) Show that there is a one-to-one correspondence between real quadratic forms f and real symmetric matrices M such that $f = x^T M x$, where x is the column vector with entries consisting of the variables. For example, $2x^2 + 4xy + y^2$ can be expressed as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(2) Show that f can be written as a linear combination of squares of linear forms.
(3) Show that the minimal number of squares that are needed to express f is equal to the rank of the corresponding symmetric matrix M.

Problem 10. Compute the character table of $\mathfrak{S}_5$.

Problem 11. (1) Show that $M_{2\times 2}(\mathbb{C})$ has the structure of a $\mathbb{C}$ vector space and is isomorphic to $\mathbb{C}^4$. Hence, endow $M_{2\times 2}(\mathbb{C})$ with the Euclidean topology.

(2) Show that $GL_2(\mathbb{C})$ is an open subset in $\mathbb{C}^4$. Hence, it inherits the Euclidean topology from $\mathbb{C}^4$.
(3) Let $O \subset M_{2\times 2}(\mathbb{C})$ be an orbit of the $GL_2(\mathbb{C})$ acting by conjugation. Classify all the orbits. (Hint: Think Jordan canonical form.)
(4) Show that an orbit O is closed in $M_{2\times 2}(\mathbb{C})$ if and only if every $A \in O$ is diagonalizable.
(5) Generalize to the conjugation action of $GL_n(\mathbb{C})$ on $M_{n\times n}(\mathbb{C})$.