1. Exercises for lecture 1

Exercise 1.1. Find generators for the ideal of $G(2,5)$ and $G(3,6)$ in the Plücker embedding.

Exercise 1.2. Calculate the following products in the cohomology ring of $G(4,8)$.

$$\sigma_1 \cdot \sigma_{3,2,1},\ \sigma_2 \cdot \sigma_{3,2,1},\ \sigma_3 \cdot \sigma_{3,2,1}.$$

Exercise 1.3. Describe the products $\sigma_{n-k,0,\dots,0} \cdot \sigma_\lambda$ and $\sigma_{1^k} \cdot \sigma_\lambda$ in the cohomology ring of $G(k,n)$.

Exercise 1.4. Describe the Bruhat order for $G(2,4)$ and $G(3,6)$.

Exercise 1.5. Express $\sigma_{3,2,1}$ in $G(4,8)$ in terms of Pieri classes using the Giambelli formula.

Exercise 1.6. Calculate all the Littlewood-Richardson coefficients of $G(2,4)$ and $G(2,5)$.

Exercise 1.7. Calculate $\sigma_{4,3,3,1} \cdot \sigma_{5,3,2,1}$ in $G(5,10)$.

2. Exercises for Lecture 2

Exercise 2.1. Write down the equations that cut out the Schubert varieties $\Sigma_{3,1}(F_\bullet)$, $\Sigma_{3,0}(F_\bullet)$ and $\Sigma_{2,1}(F_\bullet)$ on $G(2,6)$, where $F_\bullet$ is the standard flag $F_i = Span(e_1,\dots,e_i)$. Show that these Schubert varieties are singular along $\Sigma_{4,4}(F_\bullet)$, $\Sigma_{4,4}(F_\bullet)$ and $\Sigma_{3,3}(F_\bullet)$, respectively. Similarly, write down the equations for $\Sigma_{2,1,0}(F_\bullet)$ on $G(3,6)$. Show that the singular locus of $\Sigma_{2,1,0}(F_\bullet)$ is reducible consisting of a union of two $\mathbb{P}^3$'s $\Sigma_{2,2,2}(F_\bullet)$ and $\Sigma_{3,3,0}(F_\bullet)$.

Exercise 2.2. Show that the degree of the Grassmannian $G(2,n)$ is the $(n-2)$-nd Catalan number. Remember the Catalan numbers are $1,2,5,14,\dots$, where the n-th Catalan number counts the number of ways of placing parentheses. E.g., when $n=1$, we only have (). When $n=2$, we can have ()() or (()), etc. If you are feeling energetic, compute the degree of $G(k,n)$ in general.

Exercise 2.3. Prove the formula

$$(-1)^k \sigma_{\lambda_1,\dots,\lambda_k} = \sum_{j=1}^{k} (-1)^j \sigma_{\lambda_1,\dots,\lambda_{j-1},\lambda_{j+1}-1,\dots,\lambda_k-1} \cdot \sigma_{\lambda_j+k-j}$$

using the Pieri rule.

Exercise 2.4. Show that the Schubert variety $\Sigma_{n-3,i}(F_\bullet)$ in $G(2,n)$ is isomorphic to a cone over the rational normal scroll $S_{1,1,\dots,1}$, where the number of ones is $n-2-i$.

Exercise 2.5. Let $2k \le n$. Show that the projective tangent spaces to $G(k,n)$ in the Plücker embedding at two distinct points $[\Lambda_1]$ and $[\Lambda_2]$ do not intersect unless $\dim(\Lambda_1 \cap \Lambda_2) = k-1$. If $\dim(\Lambda_1 \cap \Lambda_2) = k-1$, the two projective tangent spaces intersect in a $\mathbb{P}^{n-1}$. Using this exercise, show that the dual of $G(k,n)$ (the hyperplanes in $\mathbb{P}^*(\bigwedge^k V)$ that are tangent to $G(k,n)$) form a hypersurface provided $k > 3$. Show that the dual of $G(2,n)$ is a hypersurface if n is even. The dual of $G(2,n)$ is defective when n is odd. It is a codimension 3 subvariety of $\mathbb{P}^*(\bigwedge^2 V)$. In particular, when $n=5$, show that the Grassmannian is self-dual.

Exercise 2.6. Show that $G(k,n)$ is not a complete intersection except when $k=1$, $k=n-1$ and $2k=n=4$. (Hint: Show that the Plücker relations are independent and count them to see that their number is larger than the codimension.)

Exercise 2.7. Let V_n denote an n-dimensional vector space. Show that every vector in $\bigwedge^2 V_4$ can be written as a linear combination of two decomposable wedges. Show that not every vector in $\bigwedge^2 V_n$ can be written as a linear combination of two decomposable wedges when $n \ge 6$.

3. Exercises for Lecture 3

Exercise 3.1. Calculate the following products in the cohomology of $G(3,8)$ using the Littlewood-Richardson rule.

$$\sigma_{4,2,0} \cdot \sigma_{4,2,0}, \ \sigma_{3,2,1} \cdot \sigma_{4,2,0}, \ \sigma_{3,2,1} \cdot \sigma_{3,2,1}.$$

Exercise 3.2. Use the Littlewood-Richardson rule to compute $\sigma_{3,2,1} \cdot \sigma_{3,2,1}$ in the cohomology ring of $G(4,8)$. Compute the same product using the Giambelli and Pieri rules.

Exercise 3.3. Compute the class in the Grassmannian $G(2, n+1)$ of the space of lines contained in a smooth quadric hypersurface in $\mathbb{P}^n$. Compute the class in $G(2, n+1)$ of the space of lines contained in a general cubic hypersurface.

Exercise 3.4. Show that a general quartic surface in $\mathbb{P}^3$ does not contain any lines. Let $Q_1 + tQ_2$ be a general pencil of quartic surfaces. How many quartic surfaces in this pencil contain a line?

Exercise 3.5. Show that when one of the classes is a Pieri class, we recover the Pieri rule from the geometric Littlewood-Richarson rule.

Exercise 3.6. Show that if the set S of boxes has the property that the southwest and northeast corners of the boxes are distinct and $B_i \not\subset B_j$ for any two $B_i, B_j \in S$ with $i \neq j$, then S represents a Richardson variety in the Grassmannian. Conversely, show that every Richardson variety arises this way. Compute the dimension of the Richardson variety and verify directly the dimension formula given in class in this case.

Exercise 3.7. If S is a set of boxes such that $B_i \cap B_j = \emptyset$ for $i \neq j$, show that the variety represented by S is a product of projective spaces. Prove that under the Plücker embedding the image of this variety is the Segre embedding of the corresponding product of projective spaces.

Exercise 3.8. If S is a set of boxes with the property that the southwest and northeast corners of the boxes are distinct, then show that the variety represented by S can be realized as the projection of a Richardson variety in the flag variety $F(k_1, \dots, k_r; n)$ under the natural projection to $G(k_r, n)$. What is the relation between r and the number of neighbors that the boxes have? Conversely, show that every projection of a Richardson variety in $F(k_1, \dots, k_r; n)$ to $G(k_r, n)$ is a variety corresponding to a set of boxes.

4. Exercises for Lecture 4

Exercise 4.1. Calculate the canonical class of $G(k,n)$. Show that a general codimension 4 linear section of $G(2,5)$ is a Del Pezzo surface of degree 5 (isomorphic to the blow-up of the plane in 4 points). How many lines does such a surface have? Determine the number using Schubert calculus. Show that a codimension 7 linear section of $G(2,7)$ is a Calabi-Yau threefold. Compute the Hodge numbers of this threefold.

Exercise 4.2. Determine the multiplicity of the singular locus of $\Sigma_{a,b}$ in $G(2,n)$. More generally, there is a formula for expressing the multiplicity of each singular point on a Schubert variety. Derive this formula.

Exercise 4.3. Use the Kleiman Transversality Theorem to determine the singular locus of a Richardson variety in $G(k,n)$. Explicitly describe the singular locus of $\Sigma_{2,1,0} \cap \Sigma_{2,1,0}$ in $G(3,7)$.

Exercise 4.4. Show that a Richardson variety is smooth if and only if it is isomorphic to a product of Grassmannians.

Exercise 4.5. Show that the multiplicity of a singular point on a Richardson variety $\Sigma_\lambda(F_\bullet) \cap \Sigma_\mu(G_\bullet)$ in $G(k,n)$ is the product of the multiplicities of the point in $\Sigma_\lambda(F_\bullet)$ and $\Sigma_\mu(G_\bullet)$.

Exercise 4.6. Show that the singular locus of a Richardson variety does not have to contain a fixed point under the maximal torus. Characterize the Richardson varieties in $G(k,n)$ for which the singular locus contains a torus fixed singular point.

5. Exercises for Lecture 5

Exercise 5.1. List all the Schubert varieties in $OG(2,5)$, $OG(2,6)$ and $OG(3,7)$. Determine the multiplication table of these orthogonal Grassmannians.

Exercise 5.2. Calculate the classes of the moduli space of vector bundles of rank two with fixed odd-degree determinant on a hyperelliptic curve of genus g for $g = 2,3,4,5$.

Exercise 5.3. Describe the cohomology ring of the moduli space of rank two vector bundles with fixed odd-degree determinant on a curve of genus two. Develop a Schubert calculus for this moduli space.

Exercise 5.4. Calculate the classes of the restriction varieties corresponding to the Schubert classes $\sigma_{4,2,0}$ and $\sigma_{5,3,1}$ in $G(3,9)$.

Exercise 5.5. Express $\sigma_{3,1}^3$ in $OG(3,11)$ as a linear combination of restriction of Schubert varieties in $G(3,11)$.

Exercise 5.6. Find the dimension of the isotropic Grassmannian $SG(k,n)$ for the symplectic group. Describe the Schubert varieties. Calculate the multiplication table of $SG(2,4)$ and $SG(3,6)$.

Exercise 5.7. Show that the orthogonal Grassmannian $OG(2,5)$ is isomorphic to $\mathbb{P}^3$. Show that the inclusion of $OG(2,5)$ in $G(2,5)$ followed by the Plücker map embeds the $\mathbb{P}^3$ in $\mathbb{P}(\bigwedge^2 V)$ by the 2-uple Veronese embedding. Show that $OG(2,6)$ is isomorphic to the flag variety $F(1,3;4)$. Interpret $OG(2,6)$ as lines on the Plücker embedding of $G(2,4)$. Calculate the cohomology ring of $OG(2,6)$ directly and using the fact that it is $F(1,3;4)$. Show that each of the two components of $OG(3,6)$ is isomorphic to $\mathbb{P}^3$. Show that the inclusion of $OG(3,6)$ in $G(3,6)$ followed by the Plücker map embeds each $\mathbb{P}^3$ by the 2-uple Veronese embedding.

6. Exercises for Lecture 6

Exercise 6.1. Determine the number of conics that intersect 8 general lines in $\mathbb{P}^3$. Determine the number of conics that are tangent to 5 general conics in $\mathbb{P}^2$.

Exercise 6.2. Determine the number of twisted cubics that contain 6 general points in $\mathbb{P}^3$. More generally, find the number of rational normal curves of degree d in $\mathbb{P}^d$ that contain $d+3$ general points. Determine the number of twisted cubics in $\mathbb{P}^3$ that contain 5 general points and intersect 2 general lines. More generally, determine the number of rational normal curves of degree d in $\mathbb{P}^d$ that contain $d+2$ general points and intersect a general line and a general linear space of codimension 2.

Exercise 6.3. Show that $\mathbb{P}GL(4)$ does not act transitively on the space of smooth conics in $G(2,4)$. Show that there are three orbits and describe each orbit. Describe the surface in $\mathbb{P}^3$ swept out by the lines parameterized by a conic in each orbit.

Exercise 6.4. Determine the number of orbits of the $\mathbb{P}GL(n)$ action on the set of smooth, non-degenerate rational curves of degree $d \leq \min(k, n-k)$ in $G(k,n)$. What distinguishes these orbits?

Exercise 6.5. Show that the Gromov-Witten invariants of partial flag varieties can be used to count the number of rational scrolls of any splitting type satisfying incidence constraints with respect to general linear spaces.

Exercise 6.6. Find a recursive formula for the number of rational curves of degree d in $\mathbb{P}^2$ containing $3d-1$ general points.

Exercise 6.7. Find the number of quadric surfaces in $\mathbb{P}^3$ that contain 9 general points. Find the number of quadric cones in $\mathbb{P}^3$ that contain 8 general points. Find the number of quadric surfaces in $\mathbb{P}^4$ that contain 3 general points and intersect 7 general lines. Find the number of quadric surface cones in $\mathbb{P}^4$ that contain 3 general points and intersect 6 general lines. For a challenge find the number of cubic scrolls in $\mathbb{P}^4$ that contain 9 general points.