SYMPLECTIC RESTRICTION VARIETIES AND GEOMETRIC BRANCHING RULES

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To Joe, with gratitude, in celebration of his sixtieth birthday

Abstract. In this paper, we introduce new, combinatorially defined subvarieties of isotropic Grassmannians called *symplectic restriction varieties*. We study their geometric properties and compute their cohomology classes. In particular, we give a positive, combinatorial, geometric branching rule for computing the map in cohomology induced by the inclusion $i : SG(k,n) \to G(k,n)$. This rule has many applications in algebraic geometry, symplectic geometry, combinatorics, and representation theory. In the final section of the paper, we discuss the rigidity of Schubert classes in the cohomology of $SG(k,n)$. Symplectic restriction varieties, in certain instances, give explicit deformations of Schubert varieties, thereby showing that the corresponding classes are not rigid.

Contents

1. Introduction

Specialization has been a fruitful technique since the beginning of enumerative geometry. Enumerative geometry studies the problem of determining the number of geometric objects (such as curves or linear spaces) satisfying constraints (such as being incident to general linear spaces). Determining these invariants is often very hard. However, if the constraints are in a special position, the problem may become easier. The specialization technique consists of finding a special configuration of constraints for which the answer to the enumerative problem becomes evident and then relating the original problem to this simpler problem.

In the last three decades, Joe Harris has been a master at using specialization to answer long standing problems of algebraic geometry. For example, Griffiths and Harris in their celebrated paper [GH1], using an ingenious specialization, proved the Brill-Noether Theorem by showing that Schubert cycles, which parameterize linear spaces that intersect general secant lines of a rational normal curve, intersect dimensionally properly. Later, Eisenbud and Harris, by specializing to a g-cuspidal rational curve, gave a simple proof of the Gieseker-Petri Theorem [EH1], [EH2]. More importantly, they developed the theory of limit linear series to systematically study limits of linear systems under certain specializations [EH3].

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The theory led to rapid advances in Brill-Noether theory and our understanding of the moduli space of curves [EH4]. In a parallel development, Griffiths and Harris used specializations to give a new proof of the Noether-Lefschetz Theorem independent of Hodge theory [GH2]. Harris also effectively used specialization to study the geometry of the Severi varieties of nodal plane curves of degree d and genus g. He proved their irreducibility [H] and, in joint work with Caporaso, computed their degrees [CH].

Inspired by Griffiths and Harris' proof of the Brill-Noether Theorem, in the last decade, Vakil and the author have used similar specializations to systematically compute the structure constants of the cohomology of Grassmannians and flag varieties [V], [C3], [C5], [CV]. Specializations have also been successfully applied to study the cohomology of other homogeneous varieties. For example, the author has computed the restriction coefficients and proved geometric branching rules for Type B and D flag varieties [C2]. Given the prominent role that the specialization technique has played in the work of Joe Harris and his students, it seems fitting to include a paper calculating enumerative invariants by specialization in a volume celebrating Joe Harris and his work.

The purpose of this paper is to compute the restriction coefficients and prove geometric branching rules for Type C Grassmannians using specializations. The extension to Type C flag varieties is straightforward, but in order to keep the exposition in this paper short, we postpone the discussion to the companion paper [C4].

Let V be an n-dimensional vector space over the complex numbers $\mathbb{C}$. Let Q be a non-degenerate skew-symmetric form on V. Since Q is non-degenerate, n must be even, say $n = 2m$. A linear space $W \subset V$ is called *isotropic* with respect to Q if for every $w_1, w_2 \in W$, $w_1^T Q w_2 = 0$. The *symplectic isotropic Grassmannian* $SG(k,n)$ parameterizes k-dimensional subspaces of V that are isotropic with respect to Q.

The isotropic Grassmannian $SG(k,n)$ naturally includes in the Grassmannian $G(k,n)$. This inclusion i induces a map on cohomology

$$i^* : H^*(G(k,n),\mathbb{Z}) \to H^*(SG(k,n),\mathbb{Z}).$$

The cohomology groups of both $G(k,n)$ and $SG(k,n)$ have integral bases given by Schubert classes. Given a Schubert class σ_κ in $H^*(G(k,n),\mathbb{Z})$, $i^*\sigma_\kappa$ can be expressed as a non-negative linear combination

$$i^*\sigma_\kappa = \sum_{\lambda,\mu} c^\kappa_{\lambda;\mu}\sigma_{\lambda;\mu}$$

of the Schubert classes $\sigma_{\lambda;\mu}$ in $H^*(SG(k,n),\mathbb{Z})$. The coefficients $c^\kappa_{\lambda;\mu}$ are called *symplectic restriction* or *branching coefficients*. These coefficients carry a lot of geometric, combinatorial and representation theoretic information. For example, they are closely related to computing moment polytopes and restrictions of representations of $SL(n)$ to $Sp(n)$ (see [BS], [C2], [GS], [He], and [P]). The main technical theorem of this paper gives a positive, geometric rule for computing restriction coefficients.

Theorem 1.1. *Algorithm 3.29 gives a positive, geometric rule for computing the symplectic restriction coefficients.*

More importantly, we will introduce a new set of varieties called *symplectic restriction varieties*. These varieties parameterize isotropic subspaces that satisfy rank conditions with respect to a not-necessarily isotropic flag. In Section 4, we will specify the conditions that these flags need to satisfy and carefully define these varieties. The reader may informally think of these varieties as varieties that interpolate between the restrictions of general Schubert varieties in $G(k,n)$ to $SG(k,n)$ and Schubert varieties in $SG(k,n)$.

The proof of Theorem 1.1 will proceed by a specialization. We will specialize the flag defining a Schubert variety in $G(k,n)$ successively until we arrive at an isotropic flag. We will show that at each stage of the specialization, the corresponding restriction varieties break into a union of restriction varieties, each occurring with multiplicity one. In Section 3, we will develop combinatorial objects called *symplectic diagrams* to record the result of these specializations.

In earlier work, Pragacz gave a positive rule for computing restriction coefficients for Lagrangian Grassmannians [Pr1], [Pr2]. It is also possible to compute restriction coefficients (in a non-positive way) by first computing the pullbacks of the tautological bundles from $G(k,n)$ to $SG(k,n)$ and then using localization or the theory of Schubert polynomials to express the Chern classes of these bundles in terms of Schubert classes. To the best of the author's knowledge, Algorithm 3.29 is the first positive, geometric rule for computing the restriction coefficients for all isotropic Grassmannians $SG(k,n)$.

While the combinatorics of symplectic restriction coefficients can be very complicated, the beauty of the approach is that the computation depends on four very simple geometric principles. We now explain these principles. Let Q_d^r denote a d-dimensional vector space such that the restriction of Q has corank r. Let $\mathrm{Ker}(Q_d^r)$ denote the kernel of the restriction of Q to Q_d^r. Let L_j denote an isotropic subspace of dimension j with respect to Q. Let $L_j^{\perp}$ denote the set of $w \in V$ such that $w^TQv = 0$ for all $v \in L_j$.

Evenness of rank. The rank of a non-degenerate skew-symmetric form is even. Hence, $d-r$ is even for Q_d^r. Furthermore, if $d=r$, then Q_d^r is isotropic.

The corank bound. Let $Q_{d_1}^{r_1} \subset Q_{d_2}^{r_2}$ and let $r_2' = \dim(\mathrm{Ker}(Q_{d_2}^{r_2}) \cap Q_{d_1}^{r_1})$. Then $r_1 - r_2' \leq d_2 - d_1$. In particular, $d + r \leq n$ for Q_d^r.

The linear space bound. The dimension of an isotropic subspace of Q_d^r is bounded above by $\lfloor \frac{d+r}{2} \rfloor$. Furthermore, an m-dimensional linear space L satisfies $\dim(L \cap \mathrm{Ker}(Q_d^r)) \geq m - \lfloor \frac{d-r}{2} \rfloor$.

The kernel bound. Let L be an $(s+1)$-dimensional isotropic space such that $\dim(L \cap \mathrm{Ker}(Q_d^r)) = s$. If an isotropic linear subspace M of Q_d^r intersects $L - \mathrm{Ker}(Q_d^r)$, then M is contained in $L^{\perp}$.

These four principles dictate the order of the specialization and determine the limits that occur. Given a flag, we will specialize the smallest dimensional non-isotropic subspace Q_d^r, whose corank can be increased subject to the corank bound, keeping all other flag elements unchanged. We will replace Q_d^r with $\tilde{Q}_d^{r+2}$. The branching rule simply says that under this specialization, the limit L' of a linear space L satisfying rank conditions with respect to the original flag satisfies the same rank conditions with the unchanged flag elements and either $\dim(L' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(L \cap \mathrm{Ker}(Q_d^r))$ or $\dim(L' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(L \cap \mathrm{Ker}(Q_d^r)) + 1$. Furthermore, both of these cases occur with multiplicity one unless the latter leads to a smaller dimensional variety or the former violates the linear space bound. See Sections 3 and 5 for an explicit statement of the rule and for examples.

The organization of this paper is as follows. In Section 2, we will recall basic facts concerning the geometry of isotropic Grassmannians. In Section 3, we will introduce the algorithm in combinatorial terms without reference to geometry. In Section 4, we will define symplectic restriction varieties and explain the combinatorics in geometric terms. In Section 5, we will describe the specialization and prove that the combinatorial game introduced in Section 3 computes the restriction coefficients. In the last section, we will give an application of symplectic restriction varieties to questions of rigidity.

Acknowledgements: The germs of the ideas in this paper date back to my conversations with Joe Harris while I was in graduate school. I would like to thank him for his guidance and unfailing support.

2. Preliminaries

In this section, we recall basic facts concerning the geometry of isotropic Grassmannians.

Let $n = 2m$ be a positive, even integer. Let V be an n-dimensional vector space over $\mathbb{C}$. Let Q be a non-degenerate, skew-symmetric form on V. By Darboux's Theorem, we can choose a basis for V such that in this basis Q is expressed as $\sum_{i=1}^{m} x_i \wedge y_i$. A subspace W of V is called *isotropic* if $w^T Q v = 0$ for any two vectors $v, w \in W$. The dimension of an isotropic subspace of V is at most m. Given a vector space W, the orthogonal complement $W^\perp$ of W is defined as the set of $v \in V$ such that $v^T Q w = 0$ for every $w \in W$. If the dimension of W is k, then the dimension of $W^\perp$ is $n-k$ and the restriction of Q to $W^\perp$ has rank $n-2k$ (or, equivalently, corank k).

The Grassmannian $SG(k,n)$ parameterizing k-dimensional isotropic subspaces of V is a homogeneous variety for the symplectic group $Sp(n)$. The Grassmannian $SG(m,n)$ parameterizing maximal isotropic subspaces has dimension

$$\dim(SG(m,n)) = \frac{m(m+1)}{2}.$$

This can be seen inductively. The dimension of $SG(1,2) \cong \mathbb{P}^1$ is one since every vector is isotropic with respect to Q. Consider the incidence correspondence

$$I = \{(w, W) \mid w \in \mathbb{P}(W) \text{ and } [W] \in SG(m,n)\}$$

parameterizing a pair of a maximal isotropic subspace W and a point w of $\mathbb{P}(W)$. The first projection of the incidence correspondence I maps to $\mathbb{P}(V)$ with fibers isomorphic to $SG(m-1, n-2)$. The second projection maps the incidence correspondence to $SG(m,n)$ with fibers isomorphic to $\mathbb{P}(W)$. By the Theorem on the Dimension of Fibers [S, I.6.7] and induction, we conclude that the dimension of $SG(m,n)$ is $\frac{m(m+1)}{2}$.

The dimension of the isotropic Grassmannian $SG(k,n)$ is

$$\dim SG(k,n) = \frac{m(m+1)}{2} + \frac{(m-k)(3k-m-1)}{2} = nk - \frac{3k^2-k}{2}.$$

To see this, consider the incidence correspondence

$$I = \{(W_1, W_2) \mid W_1 \in SG(k,n), W_2 \in SG(m,n), W_1 \subset W_2\}$$

parameterizing two-step flags consisting of a k-dimensional isotropic space contained in a maximal isotropic space. Since every k-dimensional isotropic space can be completed to a maximal isotropic space, the first projection is onto $SG(k,n)$. The fibers of the first projection are isomorphic to the isotropic Grassmannian $SG(m-k, n-2k)$. The second projection is onto $SG(m,n)$ with fibers isomorphic to $G(k,m)$. The Theorem on the Dimension of Fibers [S, I.6.7] and the previous paragraph imply the claim.

More generally, we will need to study spaces parameterizing k-dimensional linear spaces isotropic with respect to a degenerate skew form Q_n^r of corank r on an n-dimensional vector space. Naturally, $n-r$ needs to be even. Since the restriction of Q_n^r to a linear space complementary to its kernel is non-degenerate, we conclude that the largest dimensional isotropic subspace has dimension $r + \frac{n-r}{2}$. Set $h = \frac{n-r}{2}$. Then the space of $(r+h)$-dimensional isotropic linear spaces with respect to Q_n^r is isomorphic to $SG(h, 2h)$ and has dimension $\frac{h(h+1)}{2}$. Considering the incidence correspondence

$$I = \{(W_1, W_2) \mid W_1 \subset W_2 \text{ isotropic with respect to } Q_n^r, \dim(W_1) = k, \dim(W_2) = h + r\},$$

we see that the space of k-dimensional isotropic subspaces of Q_n^r has dimension $\frac{h(h+1)}{2} + k(h+r-k)$ if $k \geq h$ and $\frac{h(h+1)}{2} + k(h+r-k) - \frac{(h-k)(h-k+1)}{2}$ if $k < h$.

By Ehresmann's Theorem [E] (see [Bo, IV.14.12]), the cohomology of $SG(k,n)$ is generated by the classes of Schubert varieties. Let $0 \leq s \leq k$ be a non-negative integer. Let $\lambda_\bullet : 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq m$ be a sequence of increasing positive integers. Let $\mu_\bullet : m > \mu_{s+1} > \mu_{s+2} > \cdots > \mu_k \geq 0$ be a sequence of decreasing non-negative integers such that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq k$. Then the

Schubert varieties in $SG(k,n)$ may be indexed by pairs of admissible sequences $(\lambda_\bullet;\mu_\bullet)$. Fix an isotropic flag

$$F_\bullet = F_1 \subset F_2 \subset \cdots F_m \subset F_{m-1}^\perp \subset \cdots F_1^\perp \subset V.$$

The Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet}(F_\bullet)$ is defined as the Zariski closure of the set of linear spaces

$$\{W \in SG(k,n) \mid \dim(W \cap F_{\lambda_i}) = i \text{ for } 1 \leq i \leq s, \dim(W \cap F_{\mu_j}^\perp) = j \text{ for } s < j \leq k\}.$$

In the literature, it is customary to denote Schubert classes in the cohomology of $SG(m,n)$ by strictly decreasing partitions $m \geq a_1 > a_2 > \cdots > a_s > 0$ of length $s \leq m$. In our notation, the sequence $a_\bullet$ translates to the sequence $\lambda_\bullet$ by setting $a_i = m+1-\lambda_i$. Note that when $n = 2m$, the sequence $\lambda_\bullet$ determines the sequence $\mu_\bullet$ by the requirement that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq m$. Therefore, it is common to omit the sequence $\mu_\bullet$ from the notation. We will not follow this convention. In Schubert calculus, many authors prefer to record Schubert classes so that the codimension will be easily accessible. Our notation has the advantage that it is preserved under natural maps between Grassmannians arising from linear embeddings between ambient vector spaces.

We will index Schubert classes in the cohomology of the Grassmannian $G(k,n)$ by increasing sequences of non-negative integers $a_\bullet : 0 < a_1 < a_2 < \cdots < a_k \leq n$. The Schubert variety $\Sigma_{a_\bullet}(F_\bullet)$ with respect to a flag $F_\bullet$ parameterizes k-dimensional subspaces W of V that satisfy $\dim(W \cap F_{a_i}) \geq i$ for $1 \leq i \leq k$.

3. A combinatorial game

In this section, we will introduce a combinatorial game that computes the symplectic restriction coefficients. The purpose of this section is to explain the mechanics of the rule without reference to geometry. In the next two sections, we will interpret the game in geometric terms and prove that it computes the symplectic restriction coefficients. The geometrically minded reader may wish to look ahead at the next two sections.

Notation 3.1. Let $0 \leq s \leq k$ be an integer. A *sequence of n natural numbers of type s* for $SG(k,n)$ is a sequence of n natural numbers such that every number is less than or equal to $k-s$. We write the sequence from left to right with a small gap to the right of each number in the sequence. We refer to the gap after the i-th number in the sequence as the *i-th position*. For example, 1 1 2 0 0 0 0 0 and 3 0 0 2 0 1 0 0 are two sequences of 8 natural numbers of types 1 and 0, respectively, for $SG(3,8)$.

Definition 3.2. Let $0 \leq s \leq k$ be an integer. A *sequence of brackets and braces of type s* for $SG(k,n)$ consists of a sequence of n natural numbers of type s, s brackets] ordered from left to right and $k-s$ braces } ordered from right to left such that:

1. (1) Every bracket or brace occupies a position and each position is occupied by at most one bracket or brace.
2. (2) Every bracket is to the left of every brace.
3. (3) Every positive integer greater than or equal to i is to the left of the i-th brace.
4. (4) The total number of integers equal to zero or greater than i to the left of the i-th brace is even.

Example 3.3. 11]200}0}00 and 300}20}10}0 are typical examples of sequences of brackets and braces for $SG(3,8)$ that have the two examples from Notation 3.1 as their sequences of natural numbers. When writing a sequence of brackets and braces, we often omit the gaps not occupied by a bracket or a brace.

Example 3.4. Let us give several non-examples to clarify Definition 3.2. The first condition disallows diagrams such as]0000} (the first bracket is not in a position), 0]]000, 000}}0, 00]}00 (two brackets, two braces, or a bracket and a brace occupy the same position, respectively). The second condition disallows diagrams such as 00}0]000 (a brace cannot be to the left of a bracket). The third condition disallows diagrams such as 100}30}20}0 (3 is to the right of the third brace and 2 is to the right of the second brace). The fourth condition disallows diagrams such as 1]2000}0}00 (the number of zeros to the left of the second brace, and the number of zeros and twos to the left of the first brace are odd).

Notation 3.5. By convention, the brackets are indexed from left to right and the braces are indexed from right to left. We write $]^i$ and $\}^i$ to denote the i-th bracket and i-th brace, respectively. Their positions are denoted by $p(]^i)$ and $p(\}^i)$. The position of a bracket or a brace is equal to the number of integers to its left. For notational convenience, we declare that, in a sequence of brackets and braces of type s for $SG(k,n)$, the brace $\}^{k-s+1}$ denotes $]^s$ and an integer in the sequence equal to $k-s+1$ should be read as 0. Let $l(i)$ denote the number of integers in the sequence that are equal to i. Let r_i be the total number of positive integers less than or equal to i that are to the left of $\}^i$. For $0<j<i$, let $\rho(i,j) = p(\}^j) - p(\}^i)$ and let $\rho(i,0) = n - p(\}^i)$. Equivalently, $\rho(i,0)$ (respectively, $\rho(i,j)$) denotes the number of integers to the right of the i-th brace (respectively, to the right of the i-th brace and to the left of the j-th brace).

Example 3.6. For the sequence of brackets and braces 300}20}10}0 for $SG(3,8)$, the positions are $p(\}^3) = 3, p(\}^2) = 5, p(\}^1) = 7$. We have $r_i = l(i) = 1$, for $1 \leq i \leq 3$, $\rho(i,i-1) = 2$, for $2 \leq i \leq 3$, and $\rho(1,0) = 1$.

Example 3.7. For the sequence of brackets and braces 1]22]00}00}0 for $SG(4,8)$, the positions are $p(]^1) = 1, p(]^2) = 3, p(\}^2) = 5, p(\}^1) = 7$. We have $r_1 = l(1) = 1$, $l(2) = 2$, and $r_2 = 3$. Moreover, $\rho(2,1) = 2$ and $\rho(1,0) = 1$.

Definition 3.8. Two sequences of brackets and braces are *equivalent* if the lengths of their sequence of numbers are equal, the brackets and braces occur at the same positions, and the collection of digits that occur between any consecutive brackets and/or braces are the same up to reordering.

Example 3.9. The sequences 1221]00200}000}00, 1122]20000}000}00 and 003}02}01}0, 300}20}10}0 are equivalent pairs of sequences. We can depict an equivalence class of sequences by the representative where the digits are listed so that between any two consecutive brackets and/or braces the positive integers precede the zeros and are listed in non-decreasing order. We will always use this *canonical representative* and often blur the distinction between the equivalence class and this representative.

Definition 3.10. A sequence of brackets and braces is *in order* if the sequence of numbers consists of a sequence of non-decreasing positive integers followed by zeros except possibly for one i immediately to the right of $\}^{i+1}$ for $1 \leq i < k-s$. Otherwise, we say that the sequence is *not in order*. A sequence is *in perfect order* if the sequence of numbers consists of non-decreasing positive integers followed by zeros.

Example 3.11. The sequences 300}20}10}000, 11]22]00}00}00, 1]33]0000}200}0}0 are in order. Furthermore, 11]22]00}00}00 is in perfect order. The sequences 11]00]100}000, 1]20000}1}0}00, 122]100}00}00 are not in order.

Definition 3.12. A sequence of brackets and braces is *saturated* if $l(i) = \rho(i,i-1)$ for $1 \leq i \leq k-s$.

Example 3.13. The sequences 11]22]00}00}00 and 1]22]100}00}00 are saturated, whereas, 22]00}00}00 and 1]0000}00}000 are not.

The next definition is a technical definition that plays a role in the proof and is a consequence of the order in which the game is played. The reader can define a symplectic diagram as a sequence of brackets and braces that occurs in the game and refer to the conditions only when necessary.

Definition 3.14. A *symplectic diagram* for $SG(k,n)$ is a sequence of brackets and braces of type s for $SG(k,n)$ for some $0 \leq s \leq k$ such that:

(S1) $l(i) \leq \rho(i,i-1)$ for $1 \leq i \leq k-s$.

(S2) Let τ_i be the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Then

$$2\tau_i \leq p(\}^i) + r_i.$$

(S3) Either the sequence is in order or there exists at most one integer $1 \leq \eta \leq k-s$ such that the sequence of integers is non-decreasing followed by a sequence of zeros except for at most one occurrence of η between $]^s$ and $\}^{\eta+1}$ and at most one occurrence of $i < \eta$ after $\}^{i+1}$.

(S4) Let ξ_j denote the number of positive integers between $\}^j$ and $\}^{j-1}$. If an integer i occurs to the left of all the zeros, then either $i = 1$ and there is a bracket in the position following it, or there exists at most one index j_0 such that $\rho(j, j-1) = l(j)$ for $j_0 \neq j > \min(i, \eta)$ and $\rho(j_0, j_0 - 1) \leq l(j_0) + 2 - \xi_{j_0}$. Moreover, any integer η violating order occurs to the right of $\}^{j_0}$.

Remark 3.15. Conditions (S1) and (S2) are necessary to guarantee that symplectic diagrams represent geometrically meaningful objects. Conditions (S3) and (S4) are consequences of the order the game is played and describe the most complicated possible diagrams that can occur. The reader can ignore these conditions. They are necessary to carry out the dimension counts and to prove that the algorithm is defined at each step. They are not needed in order to run the algorithm.

Example 3.16. Let us give some examples to clarify Definition 3.14. Condition (S1) allows for diagrams such as 11]22]2]00}000}00 but disallows 22]3300}2}00}000 (there are two 3's and three 2's in the sequence but $\rho(2,3) = 1$ and $\rho(1,2) = 2$). Condition (S2) disallows diagrams such as 000]10}0 ($r_1 = 1$, $\tau_1 = 4$, but $2 \cdot 4 > 5 + 1$). Condition (S3) allows for 2344]300}00}00}10}0 (a non-decreasing sequence of positive integers 2344 followed by a sequence consisting of one 3, one 1 and zeros), but disallows 22]110000}2200}0000}00 (there are two 1s and two 2s following the non-decreasing sequence 22) or 22]133]00}00}00}0 (there are two 3s following the non-decreasing sequence 22). Condition (S4) allows for diagrams such as 11]3300}00}1}000, 1]1]33]00}00}00}00, however, it disallows diagrams such as 144]00}00}00}00}0 (1 occurs in the initial non-decreasing part of the sequence, but 2 and 3 do not occur. 1 is not followed by a bracket and $l(3) = 0 \neq \rho(3,2) = 2$, $l(2) = 0 \neq \rho(2,1) = 2$).

The next definition is crucial for the game and the reader should remember these conditions.

Definition 3.17. A symplectic diagram is called *admissible* if it satisfies the following additional conditions.

(A1) The two integers to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k$, then the integer is one.

(A2) Let x_i be the number of brackets $]^h$ such that every integer to the left of $]^h$ is positive and less than or equal to i. Then

$$x_i \geq k - i + 1 - \frac{p(\}^i) - r_i}{2}.$$

Example 3.18. Condition (A1) disallows diagrams such as 11]23]00}00}00}00 (the digits preceding the second bracket are not equal), 2]200}00}00 (there is a bracket in position 1, but the first digit is not 1). Condition (A2) is hard to visualize without resorting to counting. Let p be the position of the rightmost bracket such that every digit to the left of p is positive and less than or equal to i. In words, condition (A2) says that the total number of zeros and integers greater than i in the sequence is at least twice the number of brackets and braces in positions $p+1$ through $p(\}^i)$. The following diagrams violate condition (A2): 22}00}00 ($x_2 = 0$, $p(\}^2) = r_2 = 2$, but $0 < 1$), 200}2}00} (the number of braces up to $p(\}^2) = 4$ is 2; the number of zeros is 2, but $2 < 2 \cdot 2$), 11]33]00}00}1}000 (the total number of brackets and braces between positions 3 and $9 = p(\}^1)$ is 4. The number of zeros and integers greater than 1 is 6, but $2 \cdot 4 > 6$).

Remark 3.19. The admissible symplectic diagrams are the main combinatorial objects in this paper. They represent symplectic restriction varieties, which are the main geometric objects of the paper and will be defined in the next section. The symplectic diagram records a non-necessarily isotropic flag.

The corresponding symplectic restriction variety parameterizes isotropic spaces that satisfy certain rank conditions with respect to this flag. The definition of an admissible symplectic diagram reflects the basic facts about isotropic subspaces discussed in the introduction, as we will see in the next section.

Definition 3.20. The *symplectic diagram* $D(\sigma_{\lambda;\mu})$ *associated to the Schubert class* $\sigma_{\lambda;\mu}$ in $SG(k,n)$ is the saturated symplectic diagram in perfect order, where the brackets occur at positions $\lambda_1, \dots, \lambda_s$ and the braces occur at positions $n-\mu_{s+1}, \cdots, n-\mu_k$.

Example 3.21. The symplectic diagram associated to $\sigma_{2,4;4,2}$ in $SG(4,10)$ is 11]22]00}00}00.

Lemma 3.22. *The diagram* $D(\sigma_{\lambda;\mu})$ *is an admissible symplectic diagram.*

Proof. Let $n=2m$. Since $0<\lambda_1<\cdots<\lambda_s\leq m<n-\mu_{s+1}<\cdots<n-\mu_k$, the brackets and braces occur in different positions and the brackets are to the left of the braces. Since the sequence is saturated and in perfect order, the number of integers in the sequence equal to i is $\mu_{k-i+1}-\mu_{k-i+2}\leq \mu_{s+1}<m$ (with the convention that $\mu_{k+1}=0$), for $1\leq i\leq k-s$ and occur to the left of $\}^{k-s}$. Finally, the number of integers equal to zero or greater than or equal to i to the left of $\}^i$ is $n-2\mu_{k-i+1}=2(m-\mu_{k-i+1})$. Therefore, $D(\sigma_{\lambda;\mu})$ satisfies all 4 conditions in Definition 3.2.

By definition, $D(\sigma_{\lambda;\mu})$ is saturated, so $l(i)=\rho(i,i-1)$ and conditions (S1) and (S4) hold. Since the diagram is in perfect order, (S3) holds and $\tau_i=\max(\lambda_s,\mu_{s+1})\leq m$. On the other hand, $p(\}^i)+r_i = n-\mu_{k-i+1}+\mu_{k-i+1}=n=2m\geq 2\tau_i$. Therefore, $D(\sigma_{\lambda;\mu})$ satisfies all the conditions in Definition 3.14.

Finally, since $\lambda_j\neq\mu_i+1$ for any i,j, the two integers preceding a bracket must be equal. Furthermore, if $\lambda_1=1$, $\mu_1\geq 1$. Hence, condition (A1) holds. For $1\leq i\leq k-s$, $k-i+1-(p(\}^i)-r_i)/2 = k-i+1+\mu_{k-i+1}-m$. From the sequence $0,1,\dots,m-1$, remove the integers $\lambda_1-1,\lambda_2-1,\dots,\lambda_s-1$ to obtain a sequence $\alpha_m<\alpha_{m-1}<\cdots<\alpha_{s+1}$. By assumption $\mu_{k-i+1}=\alpha_j$ for some $j\geq k-i+1$. Hence, $k-i+1+\mu_{k-i+1}-m\leq \alpha_j-(m-j)=x_i$. To see the last equality, observe that x_i is the number of integers λ_h that are less than or equal to $\mu_{k-i+1}=\alpha_j$. This number is equal to the number of integers $(\alpha_j-(m-j))$ between 0 and α_j that do not occur in the sequence $\alpha_m,\dots,\alpha_j$. Hence, condition (A2) holds. We conclude that $D(\sigma_{\lambda;\mu})$ is an admissible symplectic diagram. □

The game is defined on admissible symplectic diagrams. We will see in the next section that saturated admissible diagrams in perfect order represent Schubert varieties in $SG(k,n)$. The goal of the algorithm is to transform every admissible symplectic diagram to a collection of saturated admissible diagrams in perfect order. Given an admissible symplectic diagram D, we will associate to it one or two sequences D^a and/or D^b of brackets and braces. Initially, neither D^a nor D^b has to be admissible. We will shortly describe an algorithm that modifies D^a and D^b so that they become admissible. The game records a degeneration of the flag elements represented by D.

Definition 3.23. Let D be an admissible symplectic diagram of type s for $SG(k,n)$. For the purposes of this definition, read any mention of $k-s+1$ as 0 and any mention of $\}^{k-s+1}$ as $]^s$.

(1) If D is not in order, let η be the integer in condition (S3) violating the order.

- (i) If every integer $\eta<i\leq k-s$ occurs to the left of η, let ν be the leftmost integer equal to $\eta+1$ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by interchanging η and ν.
- (ii) If an integer $\eta<i\leq k-s$ does not occur to the left of η, let ν be the leftmost integer equal to $i+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{i+1}$ not equal to ν and changing ν to i.

(2) If D is in order but is not a saturated admissible diagram in perfect order, let κ be the largest index for which $l(i)<\rho(i,i-1)$.

(i) If $l(\kappa) < \rho(\kappa, \kappa-1) - 1$, let ν be the leftmost digit equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by changing ν and the leftmost 0 to the right of $\}^{k+1}$ not equal to ν to κ.

(ii) If $l(\kappa) = \rho(\kappa, \kappa-1) - 1$, let η be the integer equal to $\kappa - 1$ immediately to the right of $\}^{\kappa}$.

(a) If κ occurs to the left of η, let ν be the leftmost integer equal to κ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by changing ν to $\kappa-1$ and η to zero.

(b) If κ does not occur to the left of η, let ν be the leftmost integer equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{\kappa+1}$ not equal to ν and changing ν to κ.

Let p be the position in D immediately to the right of ν. If there exists a bracket at a position $p' > p$ in D^a, let $q > p$ be the minimal position occupied by a bracket in D^a. Let D^b be the diagram obtained from D^a by moving the bracket at position q to position p. Otherwise, D^b is not defined.

Example 3.24. Let $D = 2300\}10\}0\}0$, then $\eta = 1$ violates the order and $\nu = 2$ and 3 occur to the left of it. Hence, we are in case (1)(i) and $D^a = 1300\}20\}0\}0$ is obtained by swapping 1 and 2. Similarly, let $D = 200]200\}00\}$, then the second 2 violates the order and $D^a = 220]000\}00\}$, $D^b = 22]0000\}00\}$.

Let $D = 124400\}00\}1\}0\}00$, the 1 in the ninth place violates the order and 3 does not occur to its left, so we are in case (1)(ii) and $D^a = 123400\}10\}0\}0\}00$.

Let $D = 22]00\}00\}00$, then D is in order and $\kappa = 1$. Since $l(1) = 0 < \rho(1, 0) - 1$, we are in case (2)(i) and $D^a = 12]00\}10\}00$ and $D^b = 1]200\}10\}00$.

Let $D = 3300\}200\}0\}$, then D is in order and $\kappa = 3$. Since $l(3) = 2 = \rho(3, 2) - 1$, we are in case (2)(ii)(a) and $D^a = 2300\}000\}0\}$.

Finally, let $D = 330000\}00\}1\}0$, then D is in order and $\kappa = 2$. Since $l(2) = 0 = \rho(2, 1) - 1$ and 2 does not occur in the sequence, we are in case (2)(ii)(b) and $D^a = 230000\}10\}0\}0$.

We will soon check that both D^a and D^b are symplectic diagrams; however, they do not have to be admissible. We now describe algorithms for turning them into admissible diagrams.

Algorithm 3.25. If D^a is not an admissible symplectic diagram, perform the following steps to turn it into an admissible diagram.

Step 1. If D^a does not satisfy condition (A2), let i be the maximal index for which condition (A2) fails. Define a new diagram D^c as follows. Let the two rightmost integers equal to i in D^a be in the places $\pi_1 < \pi_2$. Delete $\}^i$ and move the i in place π_2 to place $\pi_1 + 1$. Slide the integers in places $\pi_1 < \pi < \pi_2$ and brackets and braces in positions $\pi_1 < p < \pi_2$ one to the right. Add a bracket at position $\pi_1 + 1$. Subtract one from the integers $i < h \leq k - s$; and if $i = k - s$, change the integers equal to $k - s$ to 0. Let D^c be the resulting diagram and replace D^a with D^c. If D^a satisfies condition (A2), proceed to the next step.

Step 2. If D^a fails condition (A1), let $]^j$ be the smallest index bracket for which it fails and let i be the integer preceding $]^j$. Change this i to $i-1$ and move $\}^{i-1}$ one position to the left. Repeat this procedure until the sequence of brackets and braces satisfies condition (A1). Let the resulting sequence be D^c. In both steps, we refer to D^c as a *quadric diagram derived from* D^a.

Algorithm 3.26. If D^b does not satisfy condition (A1), run Step 2 of Algorithm 3.25 on D^b. Explicitly, let $]^j$ be the minimal index bracket for which (A1) fails. Let i be the integer immediately to the left of $]^j$. Replace i with $i-1$ and move $\}^{i-1}$ one position to the left. As long as the resulting sequence does not satisfy condition (A1), repeat this process either until the resulting sequence is an admissible symplectic diagram (in which case, this is *the symplectic diagram derived from* D^b) or two braces occupy the same position. In the latter case, no admissible symplectic diagrams are derived from D^b.

Example 3.27 (Examples of Algorithm 3.25). Let $D = 22]33]00\}00\}00\}00$. Then the diagram $D^a = 12]33]00\}00\}10\}00$ fails condition (A2) since $x_1 = 0 < 1 = 5 - (10-2)/2$. Hence, according to Step 1 of Algorithm 3.25, we replace D^a with $11]1]22]00\}00\}000$ (delete $\}^1$, move the 1 in position 9 to position 2 and slide everything in positions 2-8 one position to the right, add a bracket in position 2, and subtract 1 from the integers greater than 1). The latter is an admissible diagram.

Let $D = 00\}00\}00$. Then $D^a = 22\}00\}00$ fails condition (A2) since $x_2 = 0 < 1 - (2-2)/2$. Hence, Step 1 of Algorithm 3.25 replaces D^a with $00]00\}00$ (delete $\}^2$ and add a bracket in position 2), which is admissible.

Similarly, if $D = 11]33]00\}00\}00\}00$, then the diagram $D^a = 11]23]00\}20\}00\}00$ fails condition (A2) since $x_2 = 1 < 2$. Hence, according to Step 1 of Algorithm 3.25, we replace D^a with $11]22]2]00\}000\}00$, which is admissible.

If $D = 22]2]200\}0000\}00$, then the diagram $D^a = 12]2]200\}1000\}00$ is not admissible since it fails condition (A1) for $]^1$. Step 2 of Algorithm 3.25 replaces D^a first with $11]2]200\}100\}000$ (change the 2 preceding $]^1$ to 1 and move $\}^1$ one position to the right). Note that this diagram fails condition (A1) for $]^2$. Hence, Step 2 replaces it with $11]1]200\}10\}0000$ (change the 2 preceding $]^2$ to 1 and move $\}^1$ one position to the left). This diagram is admissible, hence it is the diagram derived from D^a.

Example 3.28 (Examples of Algorithm 3.26). Let $D = 11]33]00\}00\}00\}00$, then $D^b = 11]2]300\}20\}00\}00$ fails condition (A1). Algorithm 3.26 replaces it with $11]1]300\}20\}0\}000$, which is admissible.

Let $D = 00]0000\}00\}00\}$, then $D^b = 3]30000\}00\}00\}$ does not satisfy condition (A1) since the digit to the left of $]^1$ has to be 1. Algorithm 3.26 replaces D^b first with $2]30000\}0\}000\}$, which still fails condition (A1). Hence, Algorithm 3.26 replaces this diagram with $1]30000\}0\}00\}0$, which is admissible.

If $D = 00]0000\}2\}0\}$, then $D^a = 30]2000\}0\}0\}$ and $D^b = 3]20000\}0\}0\}$. They both fail condition (A1). When we run Algorithm 3.26 on D^b, we turn the 3 into 2 and slide $\}^2$ one position to the left. In that case, we obtain $1]30000\}\}00\}$. Since two braces occupy the same position, no diagrams are derived from D^b in this case. When we run Algorithm 3.25 on D^a, we obtain the admissible diagram $33]200\}00\}0\}$.

Let D be an admissible symplectic diagram and let ν be as in Definition 3.23. Let $\pi(\nu)$ denote the place of ν in the sequence of integers. If $p(]^s) > \pi(\nu)$, then $]^{x_{\nu-1}+1}$ is the first bracket to the right of ν. If the integer to the immediate left of $]^{x_{\nu-1}+1}$ is positive, let $y_{x_{\nu-1}+1}$ be this integer. Otherwise, let $y_{x_{\nu-1}+1} = k-s+1$. The condition $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ plays an important role. In words, this condition says that the number of values larger than ν or equal to zero that the integers to the left of $]^{x_{\nu-1}+1}$ attain is one more than the cardinality of the set of integers consisting of zero and integers larger than ν occurring to the left of $]^{x_{\nu-1}+1}$. In view of conditions (S3), (S4) and (A1), a sequence satisfying this equality looks like

$$\cdots \nu\ \nu+1 \cdots \nu+l-1\ \nu+l\ \nu+l] \cdots \quad \text{or} \quad \cdots \nu\ \nu+1\ \cdots \nu+l\ 00] \cdots,$$

where we have drawn the part of the sequence starting with the left most ν and ending with $]^{x_{\nu-1}+1}$. We are now ready to state the algorithm.

Algorithm 3.29. Let D be an admissible symplectic diagram of type s for $SG(k,n)$. If D is saturated and in perfect order, return D and stop. Otherwise, let D^a and D^b be defined as in Definition 3.23.

(1) If $p(]^s) \le \pi(\nu)$ or $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 > y_{x_{\nu-1}+1} - \nu$ in D, then return the admissible symplectic diagrams that are derived from D^a.
(2) Otherwise, return the admissible symplectic diagrams that are derived from both D^a and D^b.

We run the algorithm on two symplectic diagrams.

Example 3.30.

$$\begin{array}{ccccc} 00\}00\}00 & \to & 00]00\}00 & \to & 00]0]000 \\ & & \downarrow & & \\ & & 1]100\}00 & & \end{array}$$

In this example, first $D^a = 22\}00\}00$ is not admissible since the diagram fails condition (A2). Therefore, we replace it by $00]00\}00$. Next, $D^a = 10]10\}00$ and $D^b = 1]100\}00$. D^a is not admissible since it does not satisfy condition (A2). Hence, we replace it by the admissible diagram $00]0]000$. D^b is admissible. Note that the last two diagrams are saturated and in perfect order, so the algorithm terminates. We will soon see that this calculation shows $i^*\sigma_{2,4} = \sigma_{2,3;} + \sigma_{1;2}$ in $SG(2,6)$.

Finally, we give a larger example in $SG(3,10)$ that illustrates the inductive structure of the game.

Example 3.31.

$$\begin{array}{cccccc} & & 300\}20\}10\}000 \to 200\}00\}10\}000 \to & 200]00\}10\}000 & \to & 1]0000\}00\}000 \\ & & & \downarrow & & \downarrow \\ & & & 100]00\}00\}000 & & 1]2200\}00\}000 \\ & & \swarrow & & \searrow & \downarrow \\ & & 100]0]000\}000 & & 11]200\}0\}0000 & 1]1200\}10\}000 \\ & \swarrow & \searrow & & \downarrow & \downarrow \\ \underline{000]0]0]00000} & & 11]00]100\}000 & & \underline{11]11]00\}0000} & 1]1100\}00\}000 \\ & \swarrow & \searrow & & & \downarrow \\ \underline{11]1]0000\}000} & & \underline{11]11]00\}0000} & & & \underline{1]1100]00\}000} \end{array}$$

We will see that this calculation shows $i^*\sigma_{3,5,7} = \sigma_{3,4,5;} + \sigma_{2,3;3} + 2\sigma_{2,4;4} + \sigma_{1,5;3}$ in $H^*(SG(3,10),\mathbb{Z})$.

Definition 3.32. A *degeneration path* is a sequence of admissible symplectic diagrams

$$D_1 \to D_2 \to \cdots \to D_r$$

such that D_{i+1} is one of the outcomes of running Algorithm 3.29 on D_i for $1 \leq i < r$.

The main theorem of this paper is the following.

Theorem 3.33. *Let D be an admissible symplectic diagram for $SG(k,n)$. Let $V(D)$ be the symplectic restriction variety associated to D. Then, in terms of the Schubert basis of $SG(k,n)$, the cohomology class $[V(D)]$ can be expressed as*

$$[V(D)] = \sum c_{\lambda;\mu}\sigma_{\lambda;\mu},$$

where $c_{\lambda;\mu}$ is the number of degeneration paths starting with D and ending with the symplectic diagram $D(\sigma_{\lambda;\mu})$.

Theorem 1.1 stated in the introduction is a corollary of Theorem 3.33.

Definition 3.34. Let $\sigma_{a_\bullet}$ be a Schubert class in $G(k,n)$. If $a_j < 2j-1$ for some $1 \leq j \leq k$, then $i^*\sigma_{a_\bullet} = 0$ and we do not associate a symplectic diagram to $\sigma_{a_\bullet}$. Suppose that $a_j \geq 2j-1$ for $1 \leq j \leq k$. Let u be the number of i such that $a_i = 2i-1$. For j such that $a_j \neq 2j-1$, let u_j be the number of integers $i < j$ such that $a_i = 2i-1$. Let v_j be the number of integers $i > j$ such that $a_i = 2i-1$. Then the diagram $D(a_\bullet)$ associated to $i^*\sigma_{a_\bullet}$ is a diagram consisting of u brackets at positions $1, 2 \cdots, u$ and a brace for each $a_j > 2j-1$ at position $a_j - u_j + v_j$. The sequence of integers consists of u integers equal to 1 followed by zeros except for one integer equal to $k-j-v_j+1$ immediately following the first bracket or brace to the right of $\}^{k-j-v_j+1}$ (or in the first position if $j + v_j = 1$) for each odd $a_j > 2j-1$.

Example 3.35. The diagram $D(\sigma_{3,5,7})$ in $SG(3,8)$ is 300}20}10}0. The diagram $D(\sigma_{1,3,6,7,10})$ in $SG(5,10)$ is 1]1]1]00}00}000. The diagram $D(\sigma_{1,3,7,8,9,12})$ in $SG(6,14)$ is 1]1]1]300}0}00}00000

Remark 3.36. The reader will notice that $D(\sigma_{a_\bullet})$ is the diagram obtained by running Algorithm 3.25 on the diagram that has a brace at positions a_j and whose sequence consists of zeros except for one $k-j+1$ immediately to the right of $\}^{k-j+2}$ when a_j is odd.

Lemma 3.37. *If $a_j \geq 2j-1$ for $1 \leq j \leq k$, then $D(a_\bullet)$ is an admissible symplectic diagram.*

Proof. The brackets occur at positions $1, \dots, u$. Let a_j and a_{j+l} be two consecutive integers in the sequence $a_\bullet$ satisfying $a_i > 2i-1$. Then the positions of the corresponding braces are $a_j - u_j + v_j$ and $a_{j+l} - u_{j+l} + v_{j+l}$. Since $u_{j+l} = u_j + l - 1$ and $v_{j+l} = v_j - l + 1$, the positions of the two braces differ by the quantity $\beta = a_{j+l} - a_j - 2l + 2$. If $l = 1$, $\beta > 0$. If $l > 1$, then $a_j < a_{j+1} = 2j+1$. Since $a_{j+l} \geq 2j + 2l$, β is also positive. The first brace corresponds to the smallest index j_0 such that $a_{j_0} > 2j_0 - 1$ and occurs at position $a_{j_0} - (j_0 - 1) + (u - j_0 + 1) = u + a_{j_0} - 2j_0 + 2 \geq u + 2$. The number of positive integers less than or equal to $k - j - v_j + 1$ to the left of $\}^{k-j-v_j+1}$ is u (respectively, $u+1$) if a_j is even (respectively, odd). Hence, the number $a_j - u_j + v_j - u(-1) = a_j - 2u_j(-1)$ (where -1 occurs if a_j is odd) of integers equal to zero or greater $k - j - v_j + 1$ to the left of $\}^{k-j-v_j+1}$ is even. Therefore, conditions (1)-(4) of Definition 3.2 hold.

By construction, $l(i) \leq 1$ for $i > 1$ and $l(1) = u(+1)$ depending on whether the largest $a_j > 2j-1$ is even (or odd). In either case, one easily sees that $l(1) \leq \rho(1,0)$. The number of positive integers to the left of $\}^{k-j-v_j+1}$ is equal to u plus the number o_j of odd $a_l < a_j$ such that $a_l > 2l-1$. Since $2(u_j + o_j) \leq 2j \leq a_j$, we have that $2(u + o_j) \leq a_j - u_j + v_j + u = a_j + 2v_j$ and condition (S2) holds. The sequence is in order and the only integers other than $k-u$ occurring in the initial part of the sequence are ones, which are followed by brackets. We conclude that all the conditions in Definition 3.14 hold.

Since any bracket is preceded by 1, condition (A1) holds. Finally, for $\}^{k-j-v_j+1}$, the quantity $j + v_j - \frac{a_j - u_j + v_j - u(-1)}{2} = j + u - \frac{a_j(-1)}{2} \leq u$ (where -1 occurs if a_j is odd) since $a_j > 2j-1$. We conclude that $D(a_\bullet)$ is an admissible symplectic diagram. □

The precise formulation of Theorem 1.1 is given by the following corollary.

Corollary 3.38. *Let $\sigma_{a_\bullet}$ be a Schubert class in $G(k,n)$. If $a_j < 2j-1$ for some $1 \leq j \leq k$, then set $i^*\sigma_{a_\bullet} = 0$. Otherwise, let $D(\sigma_{a_\bullet})$ be the diagram associated to $\sigma_{a_\bullet}$. Express*

$$i^*\sigma_{a_\bullet} = \sum c_{\lambda;\mu}\sigma_{\lambda;\mu}$$

in terms of the Schubert basis of $SG(k,n)$. Then $c_{\lambda;\mu}$ is the number of degeneration paths starting with $D(\sigma_{a_\bullet})$ and ending with the symplectic diagram $D(\sigma_{\lambda;\mu})$.

Proof. In Lemma 4.20, we will prove that the intersection of a general Schubert variety with class $\sigma_{a_\bullet}$ with $SG(k,n)$ is a restriction variety of the form $V(D(\sigma_{a_\bullet}))$. The corollary is immediate from this lemma and Theorem 3.33. □

We conclude this section by proving that Algorithm 3.29 is well-defined and terminates. The proof of Theorem 3.33 is geometric and will be taken up in the next two sections.

Proposition 3.39. *Algorithm 3.29 replaces an admissible symplectic diagram with one or two admissible symplectic diagrams.*

Proof. If D is a saturated symplectic diagram in perfect order, then the algorithm returns D and there is nothing further to check. We will first check that D^a and D^b are (not necessarily admissible) symplectic diagrams. The diagram D^b is obtained from D^a by moving a bracket to the left. Conditions (2), (3), (4) of Definition 3.2 and conditions (S1), (S2), (S3) and (S4) of Definition 3.14 are preserved under moving a bracket to the left. Since $\nu \neq 1$ is the leftmost integer in D equal to a given integer, by condition (A1) for D, there cannot be a bracket at position p in D or D^a. Hence, condition (1) is satisfied for D^b. We conclude that if D^a is a symplectic diagram, then D^b is also a symplectic diagram. We will now check that D^a is a symplectic diagram in each case.

In case (1)(i), by condition (S3) for D, let η be the unique integer that violates the order. Since η is violating the order, η is to the left of $\}^{\eta+1}$. D^a is obtained by swapping η and ν, the leftmost integer equal to $\eta+1$. This operation does not change the positions of the brackets and braces and keeps $l(i)$ fixed for every i. After the swap, every integer i is still to the left of $\}^i$ for every i since η was to the left of $\}^{\eta+1}$. Furthermore, the operation also preserves or decreases τ_i for every i. We thus conclude that conditions (1) through (4) of Definition 3.2 and condition (S1), (S2) and (S4) of Definition 3.14 hold for the diagram D^a. After the swap, η is part of the non-decreasing initial sequence in D^a. Hence, the diagram D^a is either in order or $\eta+1$ is the only integer violating the order. Condition (S3) holds for D^a. We conclude that D^a is a symplectic diagram.

In case (1)(ii), let η be the unique integer that violates the order. Assume that $\eta < i \leq k-s$ does not occur to the left of η. Then i does not occur anywhere in the sequence and, in condition (S4) for D, $i = j_0$. We claim that the i-th and $(i-1)$-st braces in D must look like $\cdots\}^i\eta\}^{i-1}\cdots$. By conditions (S3) and (S4) for D, η is to the right of $\}^i$ and to the left of $\}^{\eta+1}$. If η is between $\}^{i+h}$ and $\}^{i+h-1}$ for $h \neq -1$, then since $\rho(i+h, i+h-1) = l(i+h)$ by condition (S4), the parity in condition (4) is violated for $\}^{i+h-1}$. We conclude that η is between $\}^i$ and $\}^{i-1}$. Furthermore, $1 \leq \rho(i, i-1) \leq l(i) + 2 - \xi_i = 1$ by condition (S4). The formation of D^a does not affect conditions (1) through (3) in Definition 3.2. Condition (4) holds for D^a since the formation of D^a changes the number of integers that are equal to zero or greater than j to the right of $\}^j$ only when $j = i$ and for $\}^i$ it changes the number by two. Since the formation of D^a only increases $l(i)$ by one and decreases or preserves $l(j)$ for $j \neq i$, D^a satisfies (S1). Similarly, τ_i increases by one and all other τ_j remain fixed or decrease. On the other hand, r_i increases by two, hence D^a satisfies condition (S2). There is one exception. If $i = k-s$ and every integer to the left of $]^s$ is positive, τ_{k-s} increases by two. Then, $\tau_{k-s} = r_{k-s}$, hence $2\tau_{k-s} \leq p(\}^{k-s}) + r_{k-s}$ and D^a satisfies (S2). The diagram D^a is either in order or η is still the only integer violating the order, hence D^a satisfies (S3). Finally, the formation of D^a changes $l(i) = 1$ and decreases $l(i+1)$ by one. Hence, $l(i) = \rho(i, i-1)$ for D^a. By condition (S4) for D, we have that $\rho(j, j-1) = l(j)$ in D^a for any j for which the equality held for D except for $j = i+1$. Furthermore, $\xi_{i+1} = 1$ in D^a, so $\rho(i+1, i) = l(i+1) + 1 = l(i+1) + 2 - \xi_{i+1}$ in D^a. Hence (S4) holds for D^a. We conclude that D^a is a symplectic diagram.

From now on assume that D is in order. Then there cannot be $i \geq \kappa$ such that i is immediately to the right of $\}^{i+1}$. Suppose there exists such an i. The number $\chi(i)$ and $\chi(i+1)$ of zeros and integers greater than i, respectively $i+1$, to the left of $\}^i$, respectively $\}^{i+1}$, has to be even. However, $\chi(i) = \chi(i+1) + l(i+1) + \rho(i+1, i) - 1$. Since by assumption $\rho(i+1, i) = l(i+1)$, we conclude that either $\chi(i)$ or $\chi(i+1)$ cannot be even leading to a contradiction.

In case (2)(i), changing ν to κ and the first zero to the right of $\}^{\kappa+1}$ does not change the positions of brackets and braces, it decreases $l(\kappa+1)$ by one and increases $l(\kappa)$ by two. Furthermore, the sequence D^a is still in order, unless $\kappa = k-s$ and there are zeros to the left of $]^s$. In the latter case, the κ to the right of $]^s$ is the unique integer violating order. Since by assumption $l(\kappa) < \rho(\kappa, \kappa-1) - 1$ in D, $l(\kappa) \leq \rho(\kappa, \kappa-1)$ in D^a. The parity of the integers equal to zero or greater than i also remains constant for all $1 \leq i \leq k-s$. We conclude that conditions (1) through (4) in Definition 3.2 and conditions (S1) and (S3) in Definition 3.14 hold for D^a. The quantity τ_i remains constant for $i > \kappa$ and increases by one for $i \leq \kappa$ unless $\kappa = k-s$, $l(k-s) \geq p(]^s)$ and τ_{k-s} increases by two. In the latter case, τ_{k-s} is less than or equal to both r_{k-s} and $p(\}^{k-s})$ and (S2) holds. In the former case, r_κ increases by two, hence (S2) holds for the index κ. Since $\rho(\kappa, \kappa-1) < l(\kappa) - 1$, (S2) also holds for indices $i < \kappa$. If there exists an index $i < \kappa$ in D such that i is not a 1 followed by a bracket, then in condition (S4) for

D, we have that $j_0 = \kappa$. Furthermore, $\rho(\kappa, \kappa-1) = l(\kappa) + 2$. Hence, the formation of D^a preserves the equalities in condition (S4) except for $j = \kappa$ or $\kappa + 1$. In D^a, we have that $\rho(\kappa, \kappa-1) = l(\kappa)$ and $\rho(\kappa+1, \kappa) = l(\kappa+1) + 1 = l(\kappa+1) + 2 - \xi_{\kappa+1}$. We conclude that condition (S4) holds for D^a. Therefore, D^a is a symplectic diagram.

Finally, the argument showing that D^a is a symplectic diagram in case (2)(ii)(a) is identical to the argument in case (1)(i) and the argument in case (2)(ii)(b) is identical to the case (1)(ii), so we leave them for the reader. We conclude that both D^a and D^b are symplectic diagrams. However, they need not be admissible. We now check that Algorithms 3.25 and 3.26 preserve the fact that the resulting sequences are symplectic diagrams and output admissible symplectic diagrams.

D^a may fail to be admissible either because it fails condition (A1) or (A2) in Definition 3.17. The formation of D^a from D does not change the quantities x_h, $p(\}^h)$. In cases (1)(i) and (2)(ii)(a) the quantity r_h either remains the same or decreases. Hence, in these cases D^a satisfies condition (A2). In case (1)(ii), r_h remains the same or decreases except for r_i, which increases by two. Hence, the inequality in condition (A2) can only be violated for the index i by one. If it is violated, we conclude that in D, we have $x_i = k - i + 1 - \frac{p(\}^i) - r_i}{2}$. Recall that in this case D looks like $\cdots \}^i \eta \}^{i-1} \cdots$. Since i does not appear in D, $x_i = x_{i-1}$. Writing the inequality in (A2) for D and the index $i-1$ and noting that $r_{i-1} = r_i + 1$ and $p(\}^{i-1}) = p(\}^i) + 1$, we see that $x_i = x_{i-1} \geq k - i + 2 - \frac{p(\}^i) - r_i}{2} = x_i + 1$. Since D satisfies (A2), this is a contradiction. We conclude that D^a satisfies (A2) also in the case (1)(ii). By similar reasoning, in cases (2)(i) and (2)(ii)(b), D^a can violate the inequality in (A2) only for the index κ by one. After Step 1 of Algorithm 3.25, all the inequalities in condition (A2) remain unchanged or improve and $\}^\kappa$ is eliminated. We conclude that after Step 1, the resulting diagram satisfies (A2). When the inequality in (A2) is violated for D^a, it is violated for the index κ by at most 1. When we form D^b in cases (2)(i) and (2)(ii)(b), x_κ also increases by one. Hence, D^b, when it exists, always satisfies (A2).

Observe that the operation in Step 1 of Algorithm 3.25 preserves the fact that D^a is a symplectic diagram. By construction, conditions (1)-(4) and (S1) and (S2) hold. The diagram resulting after Step 1 is in order, hence (S3) holds. The operation renames $l(i)$ as $l(i-1)$ for $i > \kappa+1$ and $\rho(i+1, i)$ as $\rho(i, i-1)$ for $i > \kappa + 1$. The operation does not change the quantities $l(i)$ and $\rho(i, i-1)$ when $i < \kappa$ and replaces $l(\kappa)$ and $l(\kappa+1)$ with their sum under the name $l(\kappa)$. The quantities $\rho(\kappa, \kappa-1)$ and $\rho(\kappa+1, \kappa)$ are replaced by $\rho(\kappa, \kappa-1) + \rho(\kappa+1, \kappa) - 1$ and renamed $\rho(\kappa, \kappa-1)$. Hence, the equalities in condition (S4) are preserved. Since (A1) also holds for the resulting diagram D^c, we conclude that if D^a fails condition (A2), then Step 1 of Algorithm 3.25 produces an admissible symplectic diagram.

Observe that changing a digit to the left of a bracket and moving a brace one unit to the left, increases x_i and r_i by one and decreases $p(\}^i)$ by one. Hence, it preserves the inequality in condition (A2). It also preserves the conditions (1) through (4) and (S1) through (S4), with the possible exception of (1) in case $p(\}^{i+1}) = p(\}^i) - 1$. Condition (A1) is violated for D^a when there is a bracket in position $p(\nu) + 1$ and it is violated only for that bracket. After l applications of Step 2 of Algorithm 3.25, Condition (A1) is still violated if there exists brackets at positions $p(\nu) + 1, p(\nu) + 2, \cdots, p(\nu) + l$. Since there are a finite number of brackets, this process stops and the resulting diagram satisfies condition (A1). In this case, the only brace that moves is $\}^{\nu-1}$. Since $l(\nu) \leq \rho(\nu, \nu-1)$ in D, the intermediate sequences and the resulting sequence all satisfy condition (1). If D^b does not satisfy condition (A1), then the only bracket that can violate it is the one in position $p(\nu)$. In this case, Algorithm 3.26 successively decreases the integer to the right of the bracket in $p(\nu)$ by one until it either becomes equal to the integer to its right or to one in case there isn't an integer to its right. Hence, this algorithm terminates in finitely many steps. A diagram might violate condition (1) in the process, but in that case the diagram is discarded. Hence, after finitely many steps either the diagram is discarded or results in an admissible symplectic diagram. We conclude that Algorithm 3.29, replaces D with one or two admissible symplectic diagrams. □

Proposition 3.40. *After finitely many applications of Algorithm 3.29, every admissible symplectic diagram is transformed to a collection of admissible symplectic diagrams in perfect order.*

Proof. If the diagram D is not in order, after one application of the algorithm either the diagram is in order or the integer violating the order increases or the position of the integer violating the order in the sequence decreases. Since these steps cannot go on indefinitely, after finitely many steps, the diagram is in order. Furthermore, during the process either the number of braces decreases or the number of positive integers less than or equal to i, for $1 \leq i \leq k-s$ in the initial part of the sequence remains constant or increases. If the diagram is in order, then at each application of the algorithm either the number of braces decreases or the number of positive integers less than or equal to i, for $1 \leq i \leq k-s$, in the initial part of the sequence increases. Since these cannot go on indefinitely, we conclude that repeated applications of the algorithm transform an admissible symplectic diagram into a collection of admissible symplectic diagrams in perfect order. Hence, the algorithm terminates in finitely many steps. □

4. Symplectic restriction varieties

In this section, we interpret admissible symplectic diagrams geometrically. We introduce symplectic restriction varieties and discuss their basic geometric properties.

Recall that Q denotes a non-degenerate skew-symmetric form on an n-dimensional vector space V. Let L_{n_j} denote an isotropic subspace of Q of dimension n_j. Let $Q^{r_i}_{d_i}$ denote a linear space of dimension d_i such that the restriction of Q to it has corank r_i. Let K_i denote the kernel of the restriction of Q to $Q^{r_i}_{d_i}$.

Definition 4.1. A *sequence* $(L_\bullet, Q_\bullet)$ is a partial flag of linear spaces $L_{n_1} \subsetneq \cdots \subsetneq L_{n_s} \subsetneq Q^{r_{k-s}}_{d_{k-s}} \subsetneq \cdots \subsetneq Q^{r_1}_{d_1}$ such that

- $\dim(K_i \cap K_h) \geq r_i - 1$ for $h > i$.
- $\dim(L_{n_j} \cap K_i) \geq \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}) - 1)$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$.

The main geometric objects of this paper will be sequences satisfying further properties.

Definition 4.2. A sequence is *in order* if

- $K_i \cap K_h = K_i \cap K_{i+1}$, for all $h > i$ and $1 \leq i \leq k-s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}))$, for $1 \leq j \leq s$ and $1 \leq i < k-s$.

A sequence $(L_\bullet, Q_\bullet)$ is *in perfect order* if

- $K_i \subseteq K_{i+1}$, for $1 \leq i < k-s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, r_i)$ for all i and j.

Definition 4.3. A sequence $(L_\bullet, Q_\bullet)$ is called *saturated* if $d_i + r_i = n$, for $1 \leq i \leq k-s$.

The next definition is the analogue of Definition 3.14 and is a consequence of the order of specialization.

Definition 4.4. A sequence $(L_\bullet, Q_\bullet)$ is called a *symplectic sequence* if it satisfies the following properties.

(GS1) The sequence $(L_\bullet, Q_\bullet)$ is either in order or there exists at most one integer $1 \leq \eta \leq k-s$ such that

$$K_i \subseteq K_h \text{ for } h > i > \eta \text{ and } K_i \cap K_h = K_i \cap K_{i+1} \text{ for } i < \eta \text{ and } h > i.$$

Furthermore, if $K_\eta \subseteq K_{k-s}$, then

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}})) \text{ for } i < \eta \text{ and}$$

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}) - 1) \text{ for } i \geq \eta.$$

If $K_\eta \not\subseteq K_{k-s}$, then

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}})) \ \text{ for all } \ i.$$

(GS2) If $\alpha = \dim(K_i \cap Q^{r_{k-s}}_{d_{k-s}}) > 0$, then either $i = 1$ and $n_\alpha = \alpha$ or there exists at most one j_0 such that, for $j_0 \neq j > \min(i, \eta)$, $r_j - r_{j-1} = d_{j-1} - d_j$. Furthermore,

$$d_{j_0-1} - d_{j_0} \leq r_{j_0} - r_{j_0-1} + 2 - \dim(K_{j_0-1}) + \dim(K_{j_0-1} \cap Q^{r_{j_0}}_{d_{j_0}})$$

and $K_\eta \not\subset Q^{r_{j_0}}_{d_{j_0}}$.

Remark 4.5. Given a sequence $(L_\bullet, Q_\bullet)$, the basic principles about skew-symmetric forms imply inequalities among the invariants of a sequence. The evenness of rank implies that $d_i - r_i$ is even for every $1 \leq i \leq k-s$. The corank bound implies that $r_i - \dim(Q^{r_i}_{d_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The linear space bound implies that $2(n_s + r_i - \dim(K_i \cap L_{n_s})) \leq r_i + d_i$ for every $1 \leq i \leq k-s$. These inequalities are implicit in the sequence $(L_\bullet, Q_\bullet)$.

Remark 4.6. For a symplectic sequence $(L_\bullet, Q_\bullet)$, the invariants n_j, r_i, d_i together with the dimensions $\dim(L_{n_j}, K_i)$ and $\dim(Q^{r_h}_{d_h} \cap K_i)$ determine the sequence $(L_\bullet, Q_\bullet)$ up to the action of the symplectic group. This will become obvious when we construct these sequences by choosing bases.

Definition 4.7. A symplectic sequence $(L_\bullet, Q_\bullet)$ is *admissible* if it satisfies the following additional conditions:

(GA1) $n_j \neq \dim(L_{n_j} \cap K_i) + 1$ for any $1 \leq j \leq s$ and $1 \leq i \leq k-s$.
(GA2) Let x_i denote the number of isotropic subspaces L_{n_j} that are contained in K_i. Then

$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}.$$

The translation between sequences and symplectic diagrams. Symplectic sequences can be represented by symplectic diagrams introduced in §3. An isotropic linear space L_{n_j} is represented by a bracket] in position n_j. A linear space $Q^{r_i}_{d_i}$ is represented by a brace } in position d_i such that there are exactly r_i positive integers less than or equal to i to the left of the i-th brace. Finally, $\dim(L_{n_j} \cap K_i)$ and $\dim(Q^{r_h}_{d_h} \cap K_i)$, $h > i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^h$, respectively.

Example 4.8. 11]200}0}00 records a sequence $L_2 \subset Q^3_5 \subset Q^2_6$, where $L_2 \subset \mathrm{Ker}(Q^2_6)$. In the diagram, there is one bracket that occurs in position 2. There are two braces that occur in positions 5 and 6. We thus conclude that the sequence contains one isotropic subspace of dimension 2 (L_2) and two non-isotropic subspaces of dimensions 5 (Q_5) and 6 (Q_6). There are two integers equal to 1 and one integer equal to 2 in the sequence. Hence, the corank of the restriction of Q to the six (respectively, five) dimensional subspace Q^2_6 (Q^3_5) is two (three). Finally, since every integer to the left of the bracket is equal to one, we conclude that $L_2 \subset \mathrm{Ker}(Q^2_6)$.

More explicitly, given a symplectic sequence $(L_\bullet, Q_\bullet)$, the corresponding symplectic diagram $D(L_\bullet, Q_\bullet)$ is determined as follows: The sequence of integers begins with $\dim(L_{n_1} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_1} \cap K_i) - \dim(L_{n_1} \cap K_{i-1})$ integers equal to i, for $2 \leq i \leq k-s$, in increasing order, followed by $n_1 - \dim(L_{n_1} \cap K_{k-s})$ integers equal to 0. The sequence then continues with $\dim(L_{n_j} \cap K_1) - \dim(L_{n_{j-1}} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_j} \cap K_i) - \max(\dim(L_{n_{j-1}} \cap K_i), \dim(L_{n_j} \cap K_{i-1}))$ integers equal to i in increasing order, followed by $n_j - \max(n_{j-1}, \dim(L_{n_j} \cap K_{k-s}))$ zeros for $j = 2, \ldots, s$ in increasing order. The sequence then continues with $\dim(Q^{r_{k-s}}_{d_{k-s}} \cap K_1) - \dim(L_{n_s} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{k-s}}_{d_{k-s}} \cap K_i) - \max(\dim(Q^{r_{k-s}}_{d_{k-s}} \cap K_{i-1}), \dim(L_{n_s} \cap K_i))$ integers equal to i in increasing order, followed by zeros until position d_{k-s}. Between positions d_i and d_{i-1} $(i > k-s)$, the sequence has $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_1) - \dim(Q^{r_i}_{d_i} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_h) - \max(\dim(Q^{r_i}_{d_i} \cap K_h), \dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_{h-1}))$ integers equal to h in increasing order, for $h \leq i-1$, followed by zeros until position d_{i-1}. Finally, the sequence ends with $n - d_1$ zeros. The brackets occur at positions n_j and the braces occur at positions d_i.

Proposition 4.9. *The diagram $D(L_\bullet, Q_\bullet)$ is a symplectic diagram of type s for $SG(k,n)$. Furthermore, if $(L_\bullet, Q_\bullet)$ is admissible, then $D(L_\bullet, Q_\bullet)$ is admissible.*

Proof. By construction each bracket or brace occupies a position. Since $n_1 < n_2 < \cdots < n_s < d_{k-s} < \cdots < d_1$, a position is occupied by at most one bracket or brace. Since $n_j < d_i$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$, every bracket occurs to the left of every brace. By construction, it is clear that $\dim(L_{n_j} \cap K_i)$ and $\dim(Q^{r_h}_{d_h} \cap K_i)$, for $h \geq i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^h$, respectively. Hence, every integer equal to i occurs to the left of $\}^i$. Finally, the total number of integers equal to zero or greater than i to the left of $\}^i$ is equal to the rank of the restriction of Q to $Q^{r_i}_{d_i}$. Since this rank is necessarily even, the total number of integers equal to zero or greater than i to the left of $\}^i$ is even. This shows that we have a sequence of brackets and braces of type s.

The sequence of brackets and braces is a symplectic diagram. The corank bound implies that $r_i - \dim(Q^{r_i}_{d_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The left hand side of the inequality is represented by the number of integers equal to i in the sequence. The right hand side of the inequality is equal to the number of integers between $\}^i$ and $\}^{i-1}$. We thus get the inequality $l(i) \leq \rho(i, i-1)$ required by Condition (S1) in Definition 3.14. By the linear space bound, the largest dimensional linear space contained in $Q^{r_i}_{d_i}$ has dimension bounded by $(d_i + r_i)/2$. The invariant r_i is equal to both the number of positive integers less than or equal to i contained to the left of $\}^i$ and $\dim(K_i)$. The span of L_{n_s} and the kernels K_h for $h \geq i$ is an isotropic subspace of $Q^{r_i}_{d_i}$. The dimension of this subspace is denoted by τ_i and is equal to the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Hence, $2\tau_i \leq p(\}^i) + r_i$ and condition (S2) of Definition 3.14 holds.

If the sequence is in (perfect) order, then the corresponding sequence of brackets and braces is in (perfect) order. Assume the sequence is not in order. The definition of a sequence implies that, for $i < k-s$, there can be at most one i which is not to the left of $\}^{k-s}$. Suppose the sequence satisfies Condition (GS1). Then, there exists an integer η such that for $i > \eta$ those integers that are not to the left of $\}^{k-s}$ are to the immediate left of $\}^{i+1}$. Furthermore, condition (GS1) implies that the positive numbers up to η are in non-decreasing order and η is the only integer violating the order. Thus condition (S3) is satisfied. Finally, condition (GS2) directly translates to condition (S4). We conclude that the sequence of brackets and braces is a symplectic diagram.

If the sequence $(L_\bullet, Q_\bullet)$ is admissible, then the corresponding symplectic diagram is also admissible. Let i be the minimal index such that $L_{n_j} \subset K_i$. If there isn't such an index, let $i = k-s+1$. If $i > 1$, then condition (GA1) implies that $\dim(L_{n_j} \cap K_{i-1}) \leq n_j - 2$. Hence, the two integers preceding $]^j$ are equal to i (or 0 if $i = k-s+1$). If $i = 1$, then all the integers preceding $]^j$ are equal to 1. Furthermore, if $n_j = 1$, condition (GA1) implies that $L_{n_j} \subset K_i$ for all $1 \leq i \leq k-s$. We conclude that condition (A1) holds. The invariant x_i is equal to both the number of isotropic subspaces L_{n_j} contained in K_i and the number of brackets such that every integer to the left of it is positive and less than or equal to i. Since $d_i = p(\}^i)$, conditions (A2) and (GA2) are exactly the same. This concludes the proof of the proposition. □

Remark 4.10. Proposition 4.9 also explains the definition of a symplectic diagram in geometric terms. Condition (4) of Definition 3.2 is implied by the evenness of rank and simply states that $d_i - r_i$ has to be even. As discussed in the proof of Proposition 4.9, condition (S1) is a translation of the corank bound and condition (S2) is implied by the linear space bound.

Conversely, we can associate an admissible sequence to every admissible symplectic diagram. By Darboux's Theorem, we can take the skew-symmetric form to be defined by $\sum_{i=1}^{m} x_i \wedge y_i$. Let the dual basis for x_i, y_i be e_i, f_i such that $x_i(e_j) = \delta_i^j$, $y_i(f_j) = \delta_i^j$ and $x_i(f_j) = y_i(e_j) = 0$. Given an admissible symplectic diagram, we associate $e_1, \dots, e_{p(]^s)}$ to the integers to the left of $]^s$ in order. We then associate $e_{p(]^s)+1}, \dots, e_{r'}$ to the positive integers to the right of $]^s$ and left of $\}^{k-s}$ in order. Let $e_{i_1}, \dots, e_{i_l}$ be vectors that have so far been associated to zeros. Then associate $f_{i_1}, \dots, f_{i_l}$ to the remaining zeros to the left of $\}^{k-s}$ in order. If there are any zeros to the left of $\}^{k-s}$ that have not been assigned a basis vector, assign them $e_{r'+1}, f_{r'+1}, \dots, e_{r''}, f_{r''}$ in pairs in order. Continuing this way, if there is a positive integer between $\}^{i+1}$ and $\}^i$ associate to it the smallest index basis element e_α that has not yet been assigned. Assume that the integers equal to $i+1$ have been assigned the vectors $e_{j_1}, \dots, e_{j_l}$. Assign to the zeros between $\}^{i+1}$ and $\}^i$, the vectors $f_{j_1}, \dots, f_{j_l}$. If there are any zeros between $\}^{i+1}$ and $\}^i$ that have not been assigned a vector, assign them $e_{\alpha+1}, f_{\alpha+1}, \dots, e_\beta, f_\beta$ in pairs until the zeros are exhausted. Let L_{n_j} be the span of the basis elements associated to the integers to the left of $]^j$. Let $Q_{d_i}^{r_i}$ be the span of the basis elements associated to the integers to the left of $\}^i$. We thus obtain a sequence $(L_\bullet, Q_\bullet)$ whose associated symplectic diagram is D.

Example 4.11. To 11]233]0000}00}0}00 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, f_6, e_7, f_7, f_4, f_5, f_3, f_1, f_2.$$

Then L_2 is the span of e_1, e_2, L_5 is the span of e_1 through e_5, Q_9^5 is the span of e_1 through e_7 and f_6, f_7, Q_{11}^3 is the span of e_1 through e_7 and f_4 through f_7. Finally, Q_{12}^2 is the span of Q_{11}^3 and f_3.

To 22]33]0000}00}100}0 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, f_5, e_6, f_6, f_3, f_4, e_7, f_1, f_2, f_7.$$

L_2 is the span of e_1, e_2, L_4 is the span of e_1 through e_4, Q_8^4 is the span of e_1 through e_6 and f_5, f_6, Q_{10}^2 is the span of Q_8^4 and f_3, f_4 and Q_{13}^1 is the span of Q_{10}^2 and e_7, f_1, f_2.

Finally, to 22]300]300}00}100}0 we associate the sequence of vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, f_4, f_5, f_3, f_6, e_7, f_1, f_2, f_7.$$

Then L_2 is the span of e_1 and e_2, L_5 is the span of e_1 through e_5, Q_8^4 is the span of e_1 through e_5 and f_4, f_5, Q_{10}^2 is the span of Q_8^4 and f_3, f_6. Finally, Q_{13}^1 is the span of all the vectors but f_7.

Remark 4.12. Notice that equivalent symplectic diagrams correspond to permutations of the basis elements that do not change the vector spaces in $(L_\bullet, Q_\bullet)$.

Remark 4.13. The construction of a symplectic sequence $(L_\bullet, Q_\bullet)$ from a symplectic diagram D is well-defined. By condition (S2), the number of zeros to the left of $]^s$ is less than or equal to the number of zeros between $]^s$ and $\}^{k-s}$. Hence, we can choose vectors $f_{i_1}, \dots, f_{i_l}$ corresponding to the vectors $e_{i_1}, \dots, e_{i_l}$. Similarly, if there does not exist a positive integer between $\}^{i+1}$ and $\}^i$, then by condition (S1), $l(i+1) \leq \rho(i+1, i)$. We can, therefore, associate vectors $f_{j_1}, \dots, f_{j_l}$ to the zeros between $\}^{i+1}$ and $\}^i$. If there exists a positive integer between $\}^{i+1}$ and $\}^i$, then there is only one positive integer between them by condition (S3). If $l(i+1) = \rho(i+1, i)$, then condition (4) is violated. Hence, $l(i+1) < \rho(i+1, i)$ and we can associate vectors $f_{j_1}, \dots, f_{j_l}$ to the zeros between $\}^{i+1}$ and $\}^i$. Thus the construction of the sequence makes sense. It is now straightforward to check that the sequence associated to an admissible symplectic diagram is an admissible sequence. Furthermore, the two constructions are inverses of each other.

We are now ready to define symplectic restriction varieties.

Definition 4.14. Let $(L_\bullet, Q_\bullet)$ be an admissible sequence for $SG(k,n)$. Then the *symplectic restriction variety* $V(L_\bullet, Q_\bullet)$ is the Zariski closure of the locus in $SG(k,n)$ parameterizing

$$\{W \in SG(k,n) \mid \dim(W \cap L_{n_j}) = j \text{ for } 1 \leq j \leq s, \dim(W \cap Q_{d_i}^{r_i}) = k-i+1 \text{ and}$$
$$\dim(W \cap K_i) = x_i \text{ for } 1 \leq i \leq k-s\}.$$

Remark 4.15. The geometric reasons for imposing conditions (A1) and (A2) in Definition 3.17 are now clear. Condition (A1) is an immediate consequence of the kernel bound. If $\dim(L_{n_j} \cap K_i) = n_j - 1$ and a linear space of dimension $k-i+1$ intersects n_j in dimension j and K_i in dimension $j-1$, then the linear space is contained in $L_{n_j}^{\perp}$. Hence, we need to impose condition (A1).

The inequality

$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}$$

is an immediate consequence of the linear space bound. We require the k-dimensional isotropic subspaces to intersect $Q_{d_i}^{r_i}$ in a subspace of dimension $k-i+1$ and to intersect the singular locus of $Q_{d_i}^{r_i}$ in a subspace of dimension x_i. By the linear space bound, any linear space of dimension $k-i+1$ has to intersect the singular locus in a subspace of dimension at least $k-i+1-\frac{d_i-r_i}{2}$, hence the inequality in condition (A2).

Example 4.16. The two most basic examples of symplectic restriction varieties are:

1. A Schubert variety $\Sigma_{\lambda;\mu}$ in $SG(k,n)$, which is the restriction variety associated to a symplectic diagram $D(\sigma_{\lambda;\mu})$, and
2. The intersection $\Sigma_{a_\bullet} \cap SG(k,n)$ of a general Schubert variety in $G(k,n)$ with $SG(k,n)$, which is the restriction variety associated to $D(a_\bullet)$.

In general, symplectic restriction varieties interpolate between these two examples.

Lemma 4.17. *A symplectic restriction variety corresponding to a saturated and perfectly ordered admissible sequence is a Schubert variety in $SG(k,n)$. Conversely, every Schubert variety in $SG(k,n)$ can be represented by such a sequence.*

Proof. Let $F_1 \subset \cdots \subset F_1^{\perp} \subset V$ be an isotropic flag. If $\Sigma_{\lambda,\mu}$ is a Schubert variety defined with respect to this flag, then the symplectic restriction variety defined with respect to the sequence $L_{n_j} = F_{\lambda_j}$ and $Q_{d_i}^{r_i} = F_{\mu_{k-i+1}}^{\perp}$ is a saturated and perfectly ordered admissible sequence.

Conversely, suppose that the sequence $(L_\bullet, Q_\bullet)$ is a saturated and perfectly ordered admissible sequence. Since the sequence is saturated, we have that $Q_{d_i}^{r_i} = K_i^{\perp}$. Since the sequence is in perfect order, we have that $\dim(L_{n_j} \cap \mathrm{Ker}(Q_{d_i}^{r_i})) = \min(r_i, n_j)$. Consequently, the set of linear spaces $\{L_{n_j}, \mathrm{Ker}(Q_{d_i}^{r_i})\}$ can be ordered by inclusion, or equivalently, by dimension. Then the resulting partial flag can be extended to an isotropic flag. By condition (GA1) of the definition of an admissible sequence, we have that $n_j \neq r_i + 1$ for any i, j. Hence, the symplectic restriction variety defined with respect to $(L_\bullet, Q_\bullet)$ is the Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet}$, where $\lambda_j = n_j$, for $1 \leq j \leq s$, and $\mu_i = r_{k-i+1}$, for $s < i \leq k$. □

Remark 4.18. By Lemma 4.17, the saturated symplectic diagrams in perfect order represent Schubert varieties.

Next, we show that the intersection of a general Schubert variety Σ with the symplectic Grassmannian $SG(k,n)$ (when non-empty) is a restriction variety.

Lemma 4.19. *Let Σ be a general Schubert variety defined with respect to a partial flag $F_{a_1} \subset \cdots \subset F_{a_k}$. Then $\Sigma \cap SG(k,n) \neq \emptyset$ if and only if $a_i \geq 2i-1$ for $1 \leq i \leq k$.*

Proof. Suppose $a_i < 2i-1$ for some i. If $[W] \in \Sigma \cap SG(k,n)$, then $W \cap F_{a_i}$ is an isotropic subspace of $Q \cap F_{a_i}$ of dimension at least i. Since F_{a_i} is general, the corank of $Q \cap F_{a_i}$ is 0 or 1 and equal to a_i modulo

2. By the linear space bound, the largest dimensional isotropic subspace of $Q \cap F_{a_i}$ has dimension less than or equal to $i-1$. Therefore, W cannot exist and $\Sigma \cap SG(k,n) = \emptyset$.

Conversely, let $a_i = 2i-1$ for every i. Then $G_1 = F_1$ is isotropic, $G_2 = F_1^{\perp}$ in F_3 is the unique two-dimensional isotropic subspace of $Q \cap F_3$ containing G_1. By induction, we see that $G_i = G_{i-1}^{\perp}$ is the unique subspace of dimension i isotropic with respect to $Q \cap F_{2i-1}$ that contains G_{i-1}. Continuing this way, we construct a unique isotropic subspace W of dimension k contained in $\Sigma \cap SG(k,n)$. If $a_i \geq 2i-1$, the vector space W just constructed is still contained in $\Sigma \cap SG(k,n)$, hence this intersection is non-empty. □

Lemma 4.20. *Let Σ be a general Schubert variety defined with respect to a partial flag $F_{a_1} \subset \cdots \subset F_{a_k}$ such that $a_i \geq 2i-1$. Then $\Sigma \cap SG(k,n) = V(D(a_\bullet))$.*

Proof. Let $a_i = 2i-1$, then since F_{a_i} is general, the restriction of Q to F_{a_i} has a one-dimensional kernel K_i. By the linear space bound, any i-dimensional isotropic subspace W contained in F_{a_i} contains K_i. For each j such that $a_j > 2j-1$, recall that u_j is the number of $i < j$ such that $a_i = 2i-1$ and v_j is the number of $i > j$ such that $a_i = 2i-1$. Let K be the span of one-dimensional kernels K_i for each $a_i = 2i-1$. Then $\dim(K) = u$ and any k-dimensional subspace W contained in $\Sigma \cap SG(k,n)$ contains K. For j such that $a_j > 2j-1$, let $G_{j+v_j} = \mathrm{Span}(F_{a_j}, K) \cap K^{\perp}$. The dimension of G_{j+v_j} is $a_j - u_j + v_j$. The corank of the restriction of Q to G_{j+v_j} is $u + \delta(a_j)$, where $\delta(a_j) = 0(1)$ if a_j is even (odd). Furthermore, any isotropic linear space contained in $\Sigma \cap SG(k,n)$ intersects G_{j+v_j} in a subspace of dimension at least $j + v_j$. From this description and the definition of $V(D(a_\bullet))$, it is now clear that $\Sigma \cap SG(k,n) = V(D(a_\bullet))$. □

Proposition 4.21. *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. Then $V(L_\bullet, Q_\bullet)$ is an irreducible subvariety of $SG(k,n)$ of dimension*

$$\dim(V(L_\bullet, Q_\bullet)) = \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i - 1 - 2k + 2i + x_i). \tag{1}$$

Proof. The proof is by induction on k. When $k=1$, if the sequence consists of an isotropic linear space L_{n_1}, then the corresponding symplectic restriction variety is $\mathbb{P}L_{n_1}$ hence it is irreducible of dimension $n_1 - 1$. If the sequence consists of one non-isotropic subspace $Q_{d_1}^{r_1}$, then the corresponding symplectic restriction variety is also projective space of dimension $d_1 - 1$. In both cases, the varieties are irreducible of the claimed dimension. This proves the base case of the induction.

If the sequence does not contain any skew-symmetric forms, then the corresponding restriction variety is isomorphic to a Schubert variety in the ordinary Grassmannian $G(k,n)$. In that case, it is well known that Schubert varieties are irreducible and have dimension $\sum_{j=1}^{k}(n_j - j)$ [C3].

Observe that omitting $Q_{d_1}^{r_1}$ from an admissible sequence $(L_\bullet, Q_\bullet)$ for $SG(k,n)$ gives rise to an admissible sequence $(L'_\bullet, Q'_\bullet)$ for $SG(k-1,n)$. There is a natural surjective morphism $f: V^0(L_\bullet, Q_\bullet) \to V^0(L'_\bullet, Q'_\bullet)$ that sends a vector space W to $W \cap Q_{d_2}^{r_2}$ (or $W \cap L_{n_{k-1}}$ if $s = k-1$). By induction, $V(L'_\bullet, Q'_\bullet)$ is irreducible of dimension $\sum_{j=1}^{s}(n_j - j) + \sum_{i=2}^{k-s}(d_i - 1 - 2k + 2i + x_i)$. The fibers of the morphism f over a point W' correspond to choices of isotropic k-planes W that contain W' and are contained in $Q_{d_1}^{r_1}$. This is a Zariski dense open subset of projective space of dimension $d_1 - 2(k-1) - 1 + x_1$. Hence, by the Theorem on the Dimension of Fibers [S, I.6.7], $V(L_\bullet, Q_\bullet)$ is irreducible of the claimed dimension. This concludes the proof of the proposition. □

5. The geometric explanation of the combinatorial game

In this section, we will prove the combinatorial rule by interpreting it geometrically. The transformation from an admissible diagram D to D^a records a one-parameter specialization of the restriction variety $V(D)$. The algorithm describes the flat limit of this specialization.

The specialization. We now explain the specialization. There are several cases depending on whether D is in order and whether $l(\kappa) < \rho(\kappa, \kappa-1) - 1$ or not. In the previous section, given an admissible quadric diagram D, we associated an admissible sequence by defining each of the vector spaces $(L_\bullet, Q_\bullet)$ as a union of basis elements that diagonalize the skew-symmetric form Q. All our specializations will replace exactly one of the basis elements $v = e_u$ or $v = f_u$ for some $1 \leq u \leq m$ with a vector $v(t) = e_u(t)$ or $v(t) = f_u(t)$ varying in a one-parameter family. For $t \neq 0$, the resulting set of vectors will be a new basis for V, but when $t = 0$ two of the basis elements will become equal. Since each linear space in $(L_\bullet, Q_\bullet)$ is a union of basis elements, we get a one-parameter family of vector spaces $(L_\bullet(t), Q_\bullet(t))$ by replacing every occurrence of the vector v with $v(t)$ for $t \neq 0$. Correspondingly, we have a one-parameter family of restriction varieties $V(L_\bullet(t), Q_\bullet(t))$. Since these varieties are projectively equivalent as long as $t \neq 0$, we obtain a flat one-parameter family. Our task is to describe the limit when $t = 0$.

In case (1)(i), D is not in order, η is the unique integer violating the order, and ν is the leftmost integer equal to $\eta + 1$. Suppose that under the translation between symplectic diagrams and sequences of vector spaces, e_u is the vector associated to η and e_v is the vector associated to ν. Then consider the one-parameter family obtained by changing e_v to $e_v(t) = te_v + (1-t)e_u$ and keeping every other vector fixed. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains e_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is the span of the same basis elements except that e_v is replaced with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (1)(ii), D is not in order, η is the unique integer violating the order, $i > \eta$ does not occur in the sequence to the left of η and ν is the leftmost integer equal to $i+1$. Let e_u be the vector associated to η and let e_v be the vector associated to ν. Consider the one-parameter family obtained by changing f_v to $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is the span of the same basis elements except that f_v is replaced with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (2)(i), D is in order and $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Suppose that e_v is the vector associated to ν, the leftmost $\kappa+1$. Let e_u and f_u be two vectors associated to zeros between $\}^\kappa$ and $\}^{\kappa-1}$. These exist since $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Consider the one-parameter specialization replacing f_v with $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (2)(ii)(a), D is in order and $l(\kappa) = \rho(\kappa, \kappa-1) - 1$. Let ν be the leftmost integer equal to κ and suppose that e_v is the vector associated to ν. Let e_u be the vector associated to the $\kappa - 1$ following $\}^\kappa$. Then let $e_v(t) = te_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains e_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing e_v with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

Finally, in case (2)(ii)(b), D is in order, $l(\kappa) = \rho(\kappa, \kappa-1) - 1$ and there does not exist an integer equal to κ to the left of κ. Let e_v be the vector associated to ν, the leftmost integer equal to $\kappa+1$ and let e_u be the vector associated to $\kappa - 1$ to the right of $\}^\kappa$. Then let $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

The flat limits of the vector spaces are easy to describe. If L_{n_j} or $Q^{r_i}_{d_i}$ does not contain the vector v, then $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$ for all $t \neq 0$. Hence, the flat limit $L_{n_j}(0) = L_{n_j}$ and $Q^{r_i}_{d_i}(0) = Q^{r_i}_{d_i}$. Similarly, if L_{n_j} or $Q^{r_i}_{d_i}$ contains both of the basis elements spanning $v(t)$, then $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$ for all $t \neq 0$. Then in the limit $L_{n_j}(0) = L_{n_j}$ and $Q^{r_i}_{d_i}(0) = Q^{r_i}_{d_i}$. A vector space changes under the specialization only when it contains the vector with coefficient t and does not contain the vector with coefficient $(1-t)$. In this case, in the limit $t = 0$, the flat limit $L_{n_j}(0)$ or $Q^{r_i}_{d_i}(0)$ is obtained by replacing in L_{n_j} or Q_{d_i} the basis element with coefficient t with the basis element with coefficient $(1-t)$.

Notice that in each of these cases, the set of limiting vector spaces is depicted by the symplectic diagram D^a. In case (1)(i), if η is between $\}^a$ and $\}^{a-1}$ and ν is between $]^b$ and $]^{b+1}$ (respectively, between $]^s$ and $\}^{k-s}$), the vector spaces L_{n_j} for $j \leq b$ (respectively, $j \leq s$) and $Q^{r_i}_{d_i}$ for $i < a$ are unaffected. In all the other vector spaces, e_v is replaced by e_u. The effect on symplectic diagrams is to switch η and ν as in the definition of D^a. In case (1)(ii), assume that η is between $\}^i$ and $\}^{i-1}$. The linear spaces other than $Q^{r_i}_{d_i}$ remain unchanged under the degeneration. In $Q^{r_i}_{d_i}$ the vector f_v is replaced by e_u. Note that this increases the corank of the restriction of Q to $Q^{r_i}_{d_i}(0)$ by two since now both vectors e_u and e_v in the kernel. This has the effect of changing ν to i and a zero between $\}^{i+1}$ and $\}^i$ to η as in the definition of D^a. In case (2)(i), all the vector spaces but $Q^{r_\kappa}_{d_\kappa}$ remain unchanged. The degeneration replaces f_v in $Q^{r_\kappa}_{d_\kappa}$ by e_u. This increases the corank of the restriction of Q to $Q^{r_\kappa}_{d_\kappa}(0)$ by two since both e_u and e_v are now contained in the kernel of the restriction. The corresponding symplectic diagram is obtained by changing ν and a zero between $\}^{\kappa+1}$ and $\}^\kappa$ to κ as in the definition of D^a. The cases (2)(ii)(a) and (b) are analogous to the cases (1)(i) and (1)(ii), respectively.

For the rest of the paper, we use the specialization just described.

Example 5.1. For concreteness, consider the restriction variety associated to 200}000}00 in $SG(2,8)$ parameterizing isotropic subspaces that intersect $A = \text{Span}(e_1, e_2, f_2)$ and are contained in $B = \text{Span}(e_i, f_i)$, $1 \leq i \leq 3$. The first specialization is given by $tf_2 + (1-t)e_3$. In the limit, $A_1 = A(0) = \text{Span}(e_1, e_2, e_3)$ and $B(0) = B$. This changes the diagram to 000]000}00. The corresponding restriction variety parameterizes linear spaces that intersect $A(0)$ and are contained in B. The next specialization is given by $tf_1 + (1-t)e_4$. In the limit, $A_1(0) = A_1$ and $B_1 = B(0) = \text{Span}(e_1, e_2, e_3, e_4, f_2, f_3)$. This changes the diagram to 100]100}00. The corresponding restriction variety parameterizes linear spaces that intersect A_1 and are contained in B_1. The final specialization is given by $te_2 + (1-t)e_4$. In the limit, $A_2 = A_1(0) = \text{Span}(e_1, e_4, e_3)$ and $B_1(0) = B_1$. This changes the diagram to 110]000}00. The flat limit of the restriction varieties has two components. The linear spaces may intersect $\text{Span}(e_1, e_4)$, in which case we get the restriction variety associated to the diagram 11]0000}00. Otherwise, by the kernel bound, the linear spaces have to be contained in $A_2^\perp$. In this case, we get the restriction variety associated to the diagram 111]00}000. The reader should convince themselves that this is precisely the outcome of Algorithm 3.29.

We are now ready to state and prove the main geometric theorem.

Theorem 5.2. *(The Geometric Branching Rule) The flat limit of the specialization of $V(D)$ is supported along $\bigcup V(D_i)$, where $V(D_i)$ is a symplectic restriction variety associated to a diagram D_i obtained by running Algorithm 3.29 on D. Furthermore, the flat limit is generically reduced along each $V(D_i)$. In particular, the equality*

$$[V(D)] = \sum [V(D_i)]$$

holds between the cohomology classes of symplectic restriction varieties.

Proof of Theorem 3.33 assuming Theorem 5.2. By Proposition 3.39, Algorithm 3.29 replaces each admissible symplectic diagram by one or two admissible symplectic diagrams. Hence, the algorithm can be repeated. By Proposition 3.40, after finitely many steps, the algorithm terminates leading to a collection of saturated admissible symplectic diagrams in perfect order. By Lemma 4.17, each of these diagrams represent a Schubert variety. Therefore, Theorem 3.33 is an immediate corollary of Theorem 5.2. □

Proof of Theorem 5.2. The proof of Theorem 5.2 has two steps. First, we interpret the algorithm as the specialization described in the beginning of this section. Let $V(D)$ denote the initial symplectic restriction variety. Let $V(D(t))$ denote the one-parameter family of restriction varieties described in the specialization and let $V(D(0))$ be the flat limit at $t = 0$. We show that $V(D(0))$ is supported along the union of restriction varieties $V(D_i)$, where D_i are the admissible symplectic diagrams derived from D via Algorithm 3.29. In the second step, we verify that the support of the flat limit contains each $V(D_i)$ and the flat limit is generically reduced along each $V(D_i)$. This suffices to prove the theorem.

We now analyze the specialization to conclude that the support of $V(D(0))$ is the union of symplectic restriction varieties $V(D_i)$. The proof is by a dimension count. In order to restrict the possible irreducible components of $V(D(0))$, we find conditions that the linear spaces parameterized by $V(D(0))$ have to satisfy. We then observe that these conditions already cut out the symplectic varieties $V(D_i)$ and that each $V(D_i)$ has the same dimension as $V(D)$. The following observation puts strong restrictions on the support of the flat limit.

Observation 5.3. The linear spaces parameterized by $V(D(t))$ intersect the linear spaces $L_{n_j}(t)$ (respectively, $Q^{r_i}_{d_i}(t)$) in a subspace of dimension at least j (respectively, $k - i + 1$). Similarly, they intersect $\mathrm{Ker}(Q^{r_i}_{d_i}(t))$ in a linear space of dimension at least x_i. Since intersecting a proper variety in at least a given dimension is a closed condition, the linear spaces parameterized by $V(D(0))$ have to intersect the linear spaces $L_{n_j}(0)$ (respectively, $Q^{r_i}_{d_i}(0)$) in a subspace of dimension at least j (respectively, $k - i + 1$). Furthermore, they intersect $\mathrm{Ker}(Q^{r_i}_{d_i}(0))$ in a subspace of dimension at least x_i.

Let Y be an irreducible component of $V(D(0))$. We can construct a sequence of vector spaces $F_{u_1} \subset \cdots \subset F_{u_k}$ such that the locus Z parameterizing linear spaces with $\dim(W \cap F_{u_j}) \geq j$ contains Y. We have already seen that the linear spaces $L_{n_j}(0)$ and $Q^{r_i}_{d_i}(0)$ are the linear spaces recorded by the symplectic diagram D^a. Let $z_1, \ldots, z_n$ be the ordered basis of V obtained by listing the basis elements associated to D^a from left to right. Let F_u be the linear space spanned by the basis elements $z_1, \cdots, z_u$. Let $F_{u_1} \subset \cdots \subset F_{u_k}$ be the jumping linear spaces for Y, that is the linear spaces of the form F_u such that $\dim(W \cap F_u) > \dim(W \cap F_{u-1})$ for the general isotropic space W parameterized by Y. Observation 5.3 translates to the inequalities $u_j \leq n_j$ for $j \leq s$ and $u_i \leq d_{k-i+1}$ for $s < i \leq k$. Hence, we can obtain a sequence depicting the linear spaces $F_{u_1}, \ldots, F_{u_k}$ by moving the braces and brackets in the diagram D^a to the left one at a time. By the proof of Proposition 4.21, Equation (1) gives an upper bound on the dimension of the locus Z (note that we used the fact that the sequence is admissible in the proof only to deduce the equality).

We now estimate the dimension of Z. Let $(L^a_\bullet, Q^a_\bullet)$ denote the linear spaces depicted by the diagram D^a. We obtain the sequence defining Z by replacing linear spaces in $(L^a_\bullet, Q^a_\bullet)$ by smaller dimensional ones.

- If we replace a linear space $L^a_{n_i}$ of dimension n_i in $(L^a_\bullet, Q^a_\bullet)$ with a linear space F_{u_i} not contained in $(L^a_\bullet, Q^a_\bullet)$ but containing $L^a_{n_{i-1}}$, then according to Equation (1) the dimension changes as follows. Let y^a_i be the index of the smallest index linear space $Q^{r_l}_{d_l}$ such that $L^a_{n_i} \subset K_l$. Similarly, let y^u_i be the smallest l such that $F_{u_i} \subset K_l$. The left sum in Equation (1) changes by $u_i - n^a_i$. The quantities x_l increase by one for $y^u_i \leq l < y^a_i$. Hence, the sum on the right increases by $y^a_i - y^u_i$. Hence, the total change in dimension is $u_i - n^a_i + y^a_i - y^u_i$. By condition (S4) of Definition 3.14 for D^a and condition (A1) for D, in D^a, there is at most one missing integer among the positive integers to the left of the brackets and the two integers preceding all brackets but possibly $]^{x_{\nu-1}+1}$ are equal.

We conclude that if we move any bracket to the left except for $]^{x_{\nu-1}+1}$, we strictly decrease the dimension. Furthermore, if we move $]^{x_{\nu-1}+1}$ to the left, we strictly decrease the dimension unless in D we have the equality $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$, so that the decrease in the position resulting by shifting the bracket in D^a is equal to the increase in the number of linear spaces $Q^{r_l}_{d_l}$ containing F_{u_i} in their kernel.

- If we replace the linear space $Q^{r_i,a}_{d_i}$ of dimension d^a_i in $(L^a_\bullet, Q^a_\bullet)$ with a non-isotropic linear space $F_{u_{k-i+1}}$ of dimension d^u_i containing $Q^{r_{i-1},a}_{d_{i-1}}$, then, by Equation (1), the dimension changes as follows. Let x^u_i be the number of linear spaces that are contained in the kernel of the restriction of Q to $F_{u_{k-i+1}}$. Then the dimension changes by $d^u_i - d^a_i - x^a_i + x^u_i$. We have that $d^u_i - d^a_i - x^a_i + x^u_i \leq 0$ with strict inequality unless the number of linear spaces contained in the kernel of $F_{u_{k-i+1}}$ increases by an amount equal to $d^a_i - d^u_i$. The latter can only happen if condition (A1) is violated for the diagram so that increasing the dimension of the kernel by one can increase the number of linear spaces contained in the kernel.
- Finally, if we replace the linear space $Q^{r_{k-s},a}_{d_{k-s}}$ of dimension d^a_{k-s} in $(L^a_\bullet, Q^a_\bullet)$ with an isotropic linear space $F_{u_{s+1}}$ containing L_{n_s}, then the first sum in Equation (1) changes by $u_{s+1} - s - 1$. The second sum changes by $-d^a_{k-s} + y^u_{s+1} - x^a_{k-s} + (2s+1)$, where y^u_{s+1} denotes the number of non-isotropic subspaces containing $F_{u_{s+1}}$ in the kernel of the restriction of Q. Hence, the total change is
$$-d^a_{k-s} + u_{s+1} - x^a_{k-s} + y^u_{s+1} + s.$$
If $x^a_{k-s} = s - j < s$, then $y^u_{s+1} = 0$. Since by the linear space bound $u_{s+1} + j + 1 \leq d_{k-s}$, we conclude that the dimension strictly decreases. If $x_{k-s} = s$, then the change is strictly negative unless $r_{k-s} = d_{k-s}$ and $d_{k-s} = u_{s+1}$.

The dimension count shows that $V(D)$ and $V(D^a)$ have the same dimension. When $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ in D, $V(D^b)$ and $V(D^a)$ have the same dimension. Furthermore, Step 2 of Algorithm 3.25 and Algorithm 3.26 preserve the dimension of the variety. By Equation (1), Step 1 of Algorithm 3.25 also preserves the dimension. If condition (A2) is violated for D^a for the index i, then by Proposition 3.39, we have that $2x_i = 2k - 2i - d_i + r_i$. On the other hand, the operation in Step 1 of Algorithm 3.25 changes the left sum in Equation (1) by $r_i + (s - x_i) - s - 1 = r_i - x_i - 1$, since it adds a new bracket of size r_i and increases the positions of the brackets with index $x_i + 1, \dots, s$. It changes the left sum by $-d_i + 1 - x_i + 2k - 2(k-s) + 2(k-s-i)$ since it removes the brace with index i and increases the positions and x_l for the braces with indices $l = i+1, \dots, k-s$. We conclude that the change in dimension is $r_i - 2x_i - d_i + 2k - 2i = 0$. We conclude that every variety $V(D_i)$ associated to $V(D)$ by Algorithm 3.29 has the same dimension as $V(D)$.

We can now determine the support of the flat limit of the specialization. Since in flat families the dimension of the fibers are preserved, Y has the same dimension as $V(D)$. Hence, our dimension calculation puts very strong restrictions on Z. First, suppose that either $x_{\nu-1} = s$ or $p(\}^{x_{\nu-1}+1}) - \pi(\nu) - 1 > y_{x_{\nu-1}+1} - \nu$ in D. If D^a is admissible, then by our dimension counts, replacing an isotropic or non-isotropic linear space in $(L^a_\bullet, Q^a_\bullet)$ with a smaller dimensional linear space produces a strictly smaller dimensional locus. We conclude that the general linear space parameterized by Y satisfies exactly the rank conditions imposed by $(L^a_\bullet, Q^a_\bullet)$. Hence, Y is contained in $V(D^a)$. Since both are irreducible varieties of the same dimension, we conclude that $Y = V(D^a)$. If D^a is not admissible, then it either violates condition (A1) or (A2). If D^a fails condition (A2), then $x_i < k - i + 1 - \frac{d_i - r_i}{2}$ for some i. Since the linear spaces parameterized by Y have to intersect $Q^{r_i}_{d_i}$ in a subspace of dimension $k - i + 1$, by the linear space bound, we conclude that these linear spaces have to intersect K_i in a subspace of dimension at least $x_i + 1$. In D^a, there is only one integer i that is not in the beginning non-decreasing part of the sequence of integers. Geometrically, the linear spaces $L^a_{n_j}$ or $Q^{r_j,a}_{d_j}$ either contain or are contained in K_i or intersect K_i in a codimension one linear space. Let $F_{a_1} \subset F_{a_2} \subset \cdots \subset F_{a_l}$ be a partial flag such that F_{a_h} intersects M in a codimension one subspace of M. Let $M = G_{a_0+1} \subset G_{a_1+1} \subset \cdots \subset G_{a_l+1}$ be the partial flag where G_{a_h+1} is the span of F_{a_h} and M for $h \geq 1$. The locus of linear spaces of dimension $x_i + l + 1$ that intersect F_{a_h} in a subspace of dimension at least $x_i + h$ and intersect M in a subspace of dimension at least $x_i + 1$ is equivalent to the locus of linear spaces that intersect the vector spaces G_{a_h+1} in subspaces of dimension at least $x_i + 1 + h$. Notice that the diagram D^c formed in Step 1 of the Algorithm 3.25 depicts the linear spaces

$$L_{n_1}, \ldots, L_{n_{x_i}}, K_i, \mathrm{Span}(K_i, L_{n_{x_i+1}}), \cdots, \mathrm{Span}(K_i, Q^{r_{i+1}}_{d_{i+1}}), Q^{r_{i-1}}_{d_{i-1}}, \cdots, Q^{r_1}_{d_1}.$$

Hence, by the linear space bound Y must be contained in $V(D^c)$. By Proposition 3.39, D^c is an admissible symplectic diagram. Hence, $V(D^c)$ is an irreducible variety that has the same dimension as Y. We conclude that $Y = V(D^c)$. On the other hand, if D^a satisfies condition (A2) but fails condition (A1), then it fails it for the bracket with index $x_{\nu-1} + 1$ and the index ν. By the kernel bound, any linear space that intersects $L_{n_{x_{\nu-1}+1}}$ in a subspace away from the kernel of Q restricted to $Q^{r_{\nu-1}}_{d_{\nu-1}}$ has to be contained in $L^{\perp}_{n_{x_{\nu-1}+1}}$. The latter vector space is depicted in a symplectic diagram by changing ν to $\nu - 1$ and shifting $\}^{\nu-1}$ one unit to the right as in Step 2 of Algorithm 3.25. This argument applies as long as condition (A1) fails for the resulting sequence. We conclude that Y has to be contained in $V(D^c)$. Since Y and $V(D^c)$ are irreducible varieties of the same dimension, we conclude that $Y = V(D^c)$.

Now suppose that $x_{\nu-1} < s$ and $p(\}^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ in D. Then, by our dimension count, replacing the linear space $L_{n_{x_{\nu-1}+1}}$ by a linear space $F_{u_{x_{\nu-1}+1}}$ corresponding to a bracket of the form

$$\cdots a\ a{+}1 \ldots \nu{-}1\ \nu\ \nu{+}2 \ldots \nu{+}l{-}1\ \nu{+}l\ \nu{+}l] \cdots \to \cdots a]a{+}1 \ldots \nu{-}1\ \nu\ \nu{+}2 \ldots \nu{+}l{-}1\ \nu{+}l\ \nu{+}l \cdots$$

produces a locus Z that has the same dimension as Y. Replacing any other linear space results in a smaller dimensional locus. However, unless $F_{u_{x_{\nu-1}+1}} = \mathrm{Ker}(Q^{r_\nu}_{d_\nu}) \cap L_{n_{x_{\nu-1}+1}}$ not all linear spaces parameterized by Z can be in the flat limit. Observe that $W^{\perp}(t)$ intersects $L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a})$ in a subspace of dimension at least $\pi(a) + 1$ for every $W(t) \in V(D(t))$. By upper semi-continuity, the same has to hold of the flat limit at $t = 0$. Hence, unless $a = \nu$, we obtain a smaller dimensional variety. We conclude that $Y \subset V(D^b)$. If D^b is admissible, then both varieties are irreducible of the same dimension and we conclude that $Y = V(D^b)$. If D^b is not admissible, then by Proposition 3.39, D^b satisfies condition (A2) but fails condition (A1). Furthermore, it fails condition (A1) only for the bracket $\cdots a\ \nu] \cdots$. By the kernel bound, the linear spaces parameterized of dimensions $k - a, k - a + 1, \ldots, k - \nu + 2$ contained in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$, respectively, are contained in $(L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a}))^{\perp}$ in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$. Algorithm 3.26 replaces the linear spaces $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$ with $(L_{n_{x_{\nu-1}+1}} \cap \mathrm{Ker}(Q^{r_a}_{d_a}))^{\perp}$ in $Q^{r_{a+1}}_{d_{a+1}}, \ldots, Q^{r_{\nu-1}}_{d_{\nu-1}}$, respectively. Hence, Y is contained in $V(D^c)$. Finally, if during the process two braces occupy the same position, then the resulting locus Z has strictly smaller dimension by our dimension counts so does not lead to a locus Z containing Y. Since in all other cases Y and $V(D^c)$ are irreducible varieties of the same dimension, we conclude that $Y = V(D^c)$. This completes the proof that the support of the flat limit of the specialization is contained in the union of $V(D_i)$, where D_i are the admissible symplectic diagrams associated to D by Algorithm 3.29.

Finally, there remains to check that each of the irreducible components occur with multiplicity one. This is an easy local calculation. The point here is that taking the option D^a at each stage of the algorithm leads to a Schubert variety. Similarly, taking the option D^b at all allowed places in the algorithm leads to a Schubert variety. The classes of these two Schubert varieties occur in the class of $V(D)$ with multiplicity one. Therefore, by intersecting $V(D)$ with the dual of these Schubert varieties, we can tell the multiplicity of $V(D^a)$ and $V(D^b)$.

First, in each of the five cases we can assume that $\eta = 1$. Let U be the Zariski open set of our family of restriction varieties parameterizing linear spaces $W(t)$ such that $\dim(W(t) \cap Q_{d_\eta}^{r_\eta(t)}(t)) = k - \eta + 1$. Let Z be the family of symplectic restriction varieties obtained by applying the specialization to the admissible sequence $(L'_\bullet, Q'_\bullet)$ (represented by D') obtained from $(L_\bullet, Q_\bullet)$ by omitting the linear spaces $Q_{d_1}^{r_1}, \ldots, Q_{d_{\eta-1}}^{r_{\eta-1}}$. Then there exists a natural morphism $f : U \to Z$ sending $W(t)$ to $W(t) \cap Q_{d_\eta}^{r_\eta(t)}(t)$, which is smooth at the generic point of each of the irreducible components of the fiber of Z at $t = 0$. The fibers f over $W' \in Z$ is the linear spaces of dimension k that contain W' and satisfy the appropriate rank conditions with respect to the linear spaces $Q_{d_1}^{r_1}, \ldots, Q_{d_{\eta-1}}^{r_{\eta-1}}$. Notice that running Algorithm 3.29 on D' results in the same outcome as running in D and removing the braces with indices $i < \eta$. Hence, we can do the multiplicity calculation for the family Z. We may, therefore, assume that $\eta = 1$.

In all the cases, the argument is almost identical with very minor variations. We will give it in the hardest case, case (2)(i), and leave the minor modifications in the other cases to the reader. In case (2)(i), by a similar argument, we may further assume that $\kappa = 1$, $d_\kappa + r_\kappa = n - 2$, $x_\kappa = 0$ and $s \leq 1$. The most interesting case is when $s = 1$ and $2d_{k-s} \geq n$. Let y_1 be the minimal index l such that L_{n_1} is contained in $\mathrm{Ker}(Q_{d_l}^{r_l})$. We will check that the multiplicities are one by finding a cycle that intersects $V(D)$ in one point and exactly one of the limits in one point. If D^a is admissible, then consider the Schubert variety Σ defined with respect to a general isotropic flag with the following invariants

$$\lambda_i = n - d_i + 2 \text{ for } \kappa = 1 \leq i \leq l-1, \ \lambda_i = n - d_i + 1 \text{ for } l \leq i \leq k-1, \text{ and } \mu_k = n - n_1 + 1.$$

If D^a satisfies condition (A2) but not (A1), change the definition of λ_1 so that $\lambda_1 = n - d_1 + 2$. If D^a fails condition (A2), change the definition of Σ so that

$$\lambda_i = n - d_{i+1} + 1 \text{ for } 1 \leq i \leq l-2, \ \lambda_i = n - d_{i+1} \text{ for } l-1 \leq i \leq k-2, \text{ and } \mu_{k-1} = n - n_1, \mu_k = n - r_\kappa + 1.$$

By Kleiman's Transversality Theorem [K1], it is immediate that both $\Sigma \cap V(D)$ and $\Sigma \cap V(D^a)$ consist of a single reduced point, whereas $\Sigma \cap V(D^b)$ is empty. Since Σ requires the k-plane to be contained in a linear space of dimension $n - n_1 + 1$ and $V(D^b)$ requires the linear space to intersect a linear space of dimension less than n_1, these conditions cannot be simultaneously satisfied for general choices of linear spaces. Hence, $\Sigma \cap V(D^b)$ is empty. On the other hand, the intersection $L_{n_1} \cap F_{\mu_k}^{\perp}$ consists of a one-dimensional vector space W_1 and $Q_{d_i}^{r_i} \cap F_{\lambda_i}$ consist of one-dimensional linear spaces contained in $W_1^{\perp}$ when $l \leq i \leq k-1$ and two-dimensional linear spaces not contained in $W_1^{\perp}$ when $1 \leq i \leq l-1$. Since any linear space contained in $V(D) \cap \Sigma$ or $V(D^a) \cap \Sigma$ must intersect all these linear spaces in one-dimensional subspaces, we conclude that the k-dimensional linear space satisfying conditions imposed by $V(D)$ and Σ or $V(D^a)$ and Σ are uniquely determined. It follows that the multiplicity of $V(D^a)$ is one.

Similarly, if $p(]^1) - \pi(2) - 1 = y_1 - 2$, then D^b is admissible. Let Ω be the Schubert variety defined with respect to a general isotropic flag with the following invariants:

$$\lambda_i = n - d_i + 1 \text{ for } 1 \leq i \leq k-1, \ \mu_k = r_1.$$

By Kleiman's Transversality Theorem [K1], it is immediate that both $\Omega \cap V(D)$ and $\Omega \cap V(D^b)$ consist of a single reduced point, whereas $\Omega \cap V(D^a)$ is empty. The conditions imposed by Ω and $V(D^a)$ cannot be simultaneously satisfied, hence $\Omega \cap V(D^a)$ is empty. On the other hand, $F_{\lambda_i} \cap Q_{d_i}^{r_i}$ by construction are one-dimensional subspaces that need to be contained in any W contained in $\Omega \cap V(D)$ or $\Omega \cap V(D^b)$. These determine $(k-1)$-dimensional subspace W' of W. $L_{n_1}^b \cap F_{\mu_1}^{\perp}$ is also a one-dimensinonal subspace Λ that needs to be contained in W. Since $\Lambda \subset (W')^{\perp}$, this uniquely constructs $W \in \Omega \cap V(D^b)$. Similarly, $L_{n_1} \cap F_{\mu_1}^{\perp}$ is a y_1-dimensional linear space. However, the intersection of this linear space with $(W')^{\perp}$ is one-dimensional and must be contained in W. This uniquely constructs W in $V(D) \cap \Omega$. We leave the minor modifications necessary in the other cases to the reader (see [C2] for more details in the orthogonal case). This concludes the proof of the theorem. $\square$

6. Rigidity of Schubert classes

In this section, as an application of Algorithm 3.29, we discuss the rigidity of Schubert classes in $SG(k,n)$. Let G be an algebraic group and let P be a parabolic subgroup. Let $X = G/P$ be the corresponding homogeneous variety. A Schubert class c in the cohomology of X is called *rigid* if the only projective subvarieties of X representing c are Schubert varieties. A Schubert class c in the cohomology of X is called *multi rigid* if the only projective subvarieties of X representing kc, for any positive integer k, are unions of k Schubert varieties. For details about rigidity of Schubert classes we refer the reader to the papers [B], [Ho1], [RT], [C1] and [C6].

In many cases, symplectic restriction varieties provide explicit deformations of Schubert classes showing that the corresponding Schubert classes are not rigid. The following example is typical.

Example 6.1. The Grassmannian $SG(1,n)$ is isomorphic to $\mathbb{P}^{n-1}$. Hence, all the Schubert varieties $\mathbb{P}L_{n_j}$ are linear spaces. However, note that not all linear spaces are Schubert varieties. Points and codimension one linear spaces are always Schubert varieties. The restriction of Q to a codimension one linear space has a one-dimensional kernel W, hence it is of the form $W^{\perp}$. We conclude that points and codimension one linear spaces are rigid. Linear spaces $\mathbb{P}M$ with $1 < \dim(M) < n-1$ do not have to be isotropic, hence the corresponding Schubert classes are not rigid since they can be deformed to non-isotropic linear spaces.

The following theorem generalizes this example.

Theorem 6.2. *Let $\sigma_{\lambda_\bullet;\mu_\bullet}$ be a Schubert class in the cohomology of $SG(k,n)$.*

1. *If $s=0$ and $\mu_j > k-j+1$ for some j, then $\sigma_{\lambda_\bullet;\mu_\bullet} = \sigma_{\mu_\bullet}$ is not rigid.*
2. *If $s \geq 1$ and $\lambda_s > \max(\mu_{s+1}, \lambda_{s-1}+1)$, then $\sigma_{\lambda_\bullet;\mu_\bullet}$ is not rigid.*

Proof. In both cases, we find a symplectic restriction variety that has the same class as the Schubert variety but is not a Schubert variety. First, suppose that $s=0$. Consider a general Schubert variety $\Sigma_{a_\bullet}$ in $G(k,n)$ with class $\sigma_{a_\bullet}$, where $a_j = n - \mu_j$. Then the cohomology class of the restriction variety $V(D_{a_\bullet})$ is the Schubert class $\sigma_{\mu_\bullet}$. To prove this run Algorithm 3.29 on the diagram $D(a_\bullet)$. Since $n - \mu_j \geq m + j > 2j-1$, the diagram $D(a_\bullet)$ does not have any brackets. Furthermore, since the Schubert variety already satisfies condition (A2), any intermediate diagram satisfies (A2). Hence, there are no brackets in any of the intermediate diagrams (a position of the bracket cannot be larger than m). Therefore, the intermediate diagrams automatically satisfy condition (A1). We conclude that the algorithm only produces D^a and at each stage D^a is admissible. The formation of D^a does not change the position of the braces. Hence, when D^a becomes saturated in perfect order, $V(D^a)$ equals a Schubert variety with class $\sigma_{\mu_\bullet}$. If $\mu_j > k-j+1$ for some j, then $V(D_{a_\bullet})$ is not a Schubert variety. Let j be the largest index such that $\mu_j > k-j+1$. If $j=k$, then the span of the linear spaces parameterized by $V(D^a)$ is not isotropic. Hence, $V(D^a)$ cannot be a Schubert variety. If $j<k$, then the linear space $Q^{r_j}_{d_j}$ is distinguished and is not isotropic. Hence, $V(D^a)$ is not a Schubert variety.

Now assume that $s \geq 1$ and $\lambda_s > \max(\mu_{s+1}, \lambda_{s-1}+1)$. Consider the following partial flag:

$$F_{\lambda_1} \subset \cdots \subset F_{\lambda_{s-1}} \subset Q^{\lambda_s - 2}_{\lambda_s} \subset F^{\perp}_{\mu_{s+1}} \subset \cdots \subset F^{\perp}_{\mu_k},$$

where F_i are isotropic subspaces and $Q^{\lambda_s-2}_{\lambda_s}$ is a non-isotropic space contained in $F^{\perp}_{\lambda_{s-1}}$. By our assumption that $\lambda_s > \max(\mu_{s+1}, \lambda_{s-1}+1)$ such a sequence exists. Let Y be the Zariski closure of the locus of k-dimensional isotropic subspaces W that satisfy $\dim(W \cap F_{\lambda_i}) = i$ for $1 \leq i < s$, $\dim(W \cap F^{\perp}_{\mu_j}) = j$ for $s < j \leq k$ and $\dim(W \cap Q^{\lambda_s-2}_{\lambda_s}) = s$. Then the cohomology class of Y is $\sigma_{\lambda_\bullet;\mu_\bullet}$, but Y is not a Schubert variety. To calculate the class of Y, we run Algorithm 3.29 on the partial flag defining Y. If we omit the linear spaces $F^{\perp}_{\mu_{s+1}}, \dots, F^{\perp}_{\mu_k}$, we obtain an admissible sequence. The sequence is in order, so we are in case (2)(i) with $\kappa = k-s+1$. The Algorithm only produces D^a, which does not satisfy condition (A2). Step 1 of Algorithm 3.25, replaces $Q_{\lambda_s}^{\lambda_s-2}$ with F_{λ_s} and the result is a Schubert variety. We conclude that the class of Y is $\sigma_{\lambda_\bullet,\mu_\bullet}$. However, since $Q_{\lambda_s}^{\lambda_s-2}$ is not isotropic, Y is not a Schubert variety. This concludes the proof. □

Corollary 6.3. (1) *If the Schubert class $\sigma_{n-\mu_1,\dots,n-\mu_k}$ in the cohomology of $G(k,n)$ can be represented by a smooth subvariety of $G(k,n)$, then the Schubert class $\sigma_{;\mu_1,\dots,\mu_k}$ can also be represented by a smooth subvariety of $SG(k,n)$.*

(2) *If there exists an index $i<k$ such that $m-i-1>\mu_i>\mu_{i+1}+2$ or if there exists an index $1<i<k$ such that $m-i>\mu_{i-1}=\mu_i+1>\mu_{i+1}+2$, then $\sigma_{;\mu_1,\dots,\mu_k}$ cannot be represented by a smooth subvariety of $SG(k,n)$.*

(3) *If the Schubert class $\sigma_{\lambda_1,\dots,\lambda_k}$ in the cohomology of $G(k,m)$ can be represented by a smooth subvariety of $G(k,m)$, then the Schubert class $\sigma_{\lambda_1,\dots,\lambda_k;}$ in the cohomology of $SG(k,n)$ can be represented by a smooth subvariety of $SG(k,n)$.*

(4) *If there exists an index $i<k$ such that $i<\lambda_i<\lambda_{i+1}+2$ or an index $1<i<k-1$ such that $i-1<\lambda_{i-1}=\lambda_i-1<\lambda_{i+1}-2$, then $\sigma_{\lambda_1,\dots,\lambda_k;}$ cannot be represented by a smooth subvariety of $SG(k,n)$.*

Proof. By Theorem 6.2, the Schubert class $\sigma_{;\mu_1,\dots,\mu_k}$ is the class of the restriction variety $D(\sigma_{n-\mu_1,\dots,n-\mu_k})$. If $\sigma_{n-\mu_1,\dots,n-\mu_k}$ can be represented by a smooth subvariety Y of $G(k,n)$, then, by Kleiman's Transversality Theorem [K1], for a general translate of Y, $Y\cap SG(k,n)$ is a smooth subvariety of $SG(k,n)$ representing the Schubert class $\sigma_{;\mu_1,\dots,\mu_k}$. This proves (1).

A Schubert variety in $SG(k,n)$ with class $\sigma_{\lambda_1,\dots,\lambda_k;}$ parameterizes k-dimensional subspaces of a maximal isotropic space W, hence it is also a subvariety of $G(k,W)=G(k,m)$ with class $\sigma_{\lambda_1,\dots,\lambda_k}$. If the latter class can be represented by a smooth subvariety Y of $G(k,W)$, then Y also represents the class $\sigma_{\lambda_1,\dots,\lambda_k;}$ in $SG(k,n)$. This proves (3).

The Schubert variety Σ parameterizing k-dimensional isotropic subspaces contained in a fixed maximal isotropic space W is a smooth subvariety of $SG(k,n)$ isomorphic to $G(k,m)$ that has cohomology class $\sigma_{m-k+1,\dots,m;}$. If Y is a smooth subvariety representing $\sigma_{;\mu_1,\dots,\mu_k}$, then, by Kleiman's Transversality Theorem, the intersection of Σ with a general translate of Y is a smooth subvariety of $G(k,m)$ representing the class $\sigma_{m-\mu_1,\dots,m-\mu_k}$. Therefore, Theorem 1.6 of [C1] implies (2).

Under the inclusion $i: SG(k,n)\to G(k,n)$, a Schubert variety Σ in $SG(k,n)$ with class $\sigma_{\lambda_1,\dots,\lambda_k;}$ is a Schubert variety of $G(k,n)$ with class $\sigma_{\lambda_1,\dots,\lambda_k}$. If the former class can be represented by a smooth subvariety Y of $SG(k,n)$, then $i(Y)$ is a smooth subvariety that represents the latter class in $G(k,n)$. Hence, if the latter class cannot be represented by a smooth subvariety of $G(k,n)$, then Y cannot exist. Therefore, Theorem 1.6 of [C1] implies (4).

□

References

[BS] A. Berenstein and R. Sjamaar. Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion. *J. Amer. Math. Soc.*, **13** no. 2 (2000), 433–466.

[Bo] A. Borel. *Linear Algebraic Groups*, Springer-Verlag, New York, 1991.

[B] R. Bryant. *Rigidity and quasi-rigidity of extremal cycles in Hermitian symmetric spaces.* Princeton University Press, Ann. Math. Studies AM–153, 2005.

[CH] L. Caporaso and J. Harris. Counting plane curves of any genus. *Invent. Math.*, **131** no. 2 (1998), 345–392.

[C1] I. Coskun. Rigid and non-smoothable Schubert classes. *J. Differential Geom.*, **87** no. 3 (2011), 493–514.

[C2] I. Coskun. Restriction varieties and geometric branching rules. *Adv. Math.*, **228** no. 4 (2011), 2441–2502.

[C3] I. Coskun. A Littlewood-Richardson rule for two-step flag varieties. *Invent. Math.*, **176** no. 2 (2009), 325–395.

[C4] I. Coskun. Symplectic restriction varieties and geometric branching rules II. In preparation.

[C5] I. Coskun. Degenerations of surface scrolls and the Gromov-Witten invariants of Grassmannians. *J. Algebraic Geom.*, **15** (2006), 223–284.
[C6] I. Coskun. Rigidity of Schubert classes in orthogonal Grassmannian. Preprint.
[CV] I. Coskun and R. Vakil, Positivity in the cohomology of homogeneous spaces and generalized Schubert calculus. *Algebraic Geometry, Seattle 2005*, volume 1, 77–124.
[E] C. Ehresmann. Sur la topologie de certains espaces homogènes. *Ann. Math.* (2), **35** (1934), 396–443.
[EH1] D. Eisenbud and J. Harris, Divisors on general curves and cuspidal rational curves. *Invent. Math.*, **74** no. 3 (1983), 371–418.
[EH2] D. Eisenbud and J. Harris, A simpler proof of the Gieseker-Petri theorem on special divisors. *Invent. Math.*, **74** no. 2 (1983), 269–280.
[EH3] D. Eisenbud and J. Harris, Limit linear series: basic theory. *Invent. Math.*, **85** no. 2 (1986), 337–371.
[EH4] D. Eisenbud and J. Harris, The Kodaira dimension of the moduli space of curves of genus ≥ 23. *Invent. Math.*, **90** no. 2 (1987), 359–387.
[GH1] P. Griffiths and J. Harris, The dimension of the space of special linear series on a general curve. *Duke Math. J.*, **47** (1980), 233-272.
[GH2] P. Griffiths and J. Harris, On the Noether-Lefschetz theorem and some remarks on codimension-two cycles. *Math. Ann.*, **271** no. 1 (1985), 31–51.
[GH3] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley Interscience, 1978.
[GS] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, **67** no. 3 (1982), 491–513.
[H] J. Harris, On the Severi problem. *Invent. Math.*, **84** no. 3 (1986), 445–461.
[He] G. J. Heckman. Projections of orbits and asymptotic behavior of multiplicities for compact, connected Lie groups. *Invent. Math.*, **67** no. 2 (1982), 333–356.
[Ho1] J. Hong. Rigidity of smooth Schubert varieties in Hermitian symmetric spaces. *Trans. Amer. Math. Soc.*, **359** (2007), 2361–2381.
[K1] S. Kleiman. The transversality of a general translate. *Compos. Math.*, **28** no. 3 (1974), 287–297.
[Pr1] P. Pragacz. A generalization of the Macdonald-You formula, *J. Algebra*, **204** no. 2 (1998), 573–587.
[Pr2] P. Pragacz. Addendum to "A generalization of the Macdonald-You formula". *J. Algebra*, **226** no. 1 (2000), 639–648.
[P] K. Purbhoo. Vanishing and non-vanishing criteria in Schubert calculus. *Int. Math. Res. Not.*, 2006, Article ID 24590, 38 pp.
[RT] C. Robles and D. The. Rigid Schubert varieties in compact Hermitian symmetric spaces. To appear in *Selecta Math.*
[S] I.R. Shafarevich, *Basic Algebraic Geometry I.* Springer-Verlag, 1994.
[V] R. Vakil. A geometric Littlewood-Richardson rule, with an appendix joint with A. Knutson. *Ann. Math.* (2), **164** (2006), 371–421.

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