NOTES ON ALGEBRA

A LITTLE BIT OF HOMOLOGICAL ALGEBRA

Marc Culler - Spring 2005

1. tensor products over commutative rings

Definition 1.1. Suppose that R is a commutative ring (possibly a field) and that A, B and C are R-modules. A function $f : A \times B \to C$ is *bilinear* if

- $f(a_1 + a_2, b) = f(a_1, b) + f(a_1, b)$,
- $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$, and
- $f(ra, b) = f(a, rb) = rf(a.b)$.

1.2. The most basic examples of bilinear functions are the functions $R \times R \to R$ given by homogeneous quadratic polynomials of degree 2 (also known as a quadratic forms). For example, the function $f(x, y) = xy$ is a bilinear function from $R \times R \to R$. If F is a field, and $V = F^n$ then the dot product gives an example of a bilinear function $f : V \times V \to F$.

Theorem 1.3. *Let R be a commutative ring and let A and B be R-modules. There exists an R-module, denoted $A \otimes_R B$, and a bilinear function $\otimes : A \times B \to A \otimes_R B$, which has the following universal property with respect to bilinear functions:*

If C is any R-module and $f : A \times B \to C$ is any bilinear function then there exists a unique homomorphism $\phi : A \otimes_R B \to C$ which makes the following diagram commute

$$\begin{array}{ccc} A \times B & \xrightarrow{\otimes} & A \otimes_R B \\ & \searrow^{f} & \downarrow{\scriptstyle \phi} \\ & & C \end{array} .$$

Moreover, any R-module which has this universal property is isomorphic to $A \otimes_R B$.

Proof. Let F be a free R-module with a basis that corresponds one-to-one with the elements of $A \times B$. If $a \in A$ and $b \in B$ then we will denote the basis element that corresponds to (a, b) by $[a, b]$. Thus a typical element of F can be written as

$$r_1[a_1, b_1] + \cdots r_n[a_n, b_n].$$

Now let N be the submodule of F generated by all elements of the following three types:

Date: September 2, 2005.

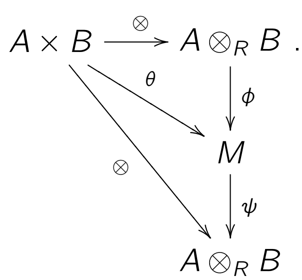
- $[a_1 + a_2, b] - [a_1, b] - [a_2, b]$,
- $[a, b_1 + b_2] - [a, b_1] - [a, b_2]$,
- $[ra, b] - [a, rb]$,

where $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $r \in R$.

We define $A \otimes_R B$ to be the quotient F/N and we denote the coset of $[a, b]$ by $a \otimes b$. Clearly $A \otimes_R B$ is generated by the elements $a \otimes b$, where a ranges over all elements of A and b ranges over all elements of B. The definition of N implies that the function $\otimes : A \times B \to A \otimes_R B$ defined by $\otimes(a, b) = a \otimes b$ is bilinear. For example, $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$ in $A \otimes_R B$, since the element $[a_1 + a_2, b] - [a_1, b] - [a_2, b]$ is contained in N.

Suppose we are given a bilinear function $f : A \times B \to C$. Since F is free we can define a homomorphism $\hat{\phi} : F \to C$ by specifying the images of the basis elements as $\hat{\phi}([a, b]) = f(a, b)$. The condition that f is bilinear implies that each generator of N is contained in the kernel of $\hat{\phi}$. Thus $\hat{\phi}$ determines a homomorphism $\phi : A \otimes_R B \to C$ which satisfies $\phi(a \otimes b) = f(a, b)$. There can only be one such homomorphism since the elements of the form $a \otimes b$ generate $A \otimes B$. This shows that $F/N = A \otimes_R B$ has the required universal property.

Suppose M were another R-module with this universal property, where the bilinear map from $A \times B \to M$ is denoted by θ. Then there would exist homomorphisms $\phi : A \otimes_R B \to M$ and $\psi : M \to A \otimes_R B$ making the following diagram commute:



The uniqueness part of the universal property guarantees that $\psi \circ \phi = \mathrm{id}_{A \otimes_R B}$ and $\phi \circ \psi = \mathrm{id}_M$. Thus M is isomorphic to $A \otimes_R B$. $\square$

1.4. One could take the view that the purpose of the tensor product is to simplify working bilinear maps. A bilinear map defined on $A \times B$ can be replaced by an ordinary homomorphism defined on $A \otimes_R B$. However, it may be hard to recognize this in one of the most common situations where tensor products are used, namely to accomplish "extension of scalars".

Example 1.5. To illustrate what is meant by extending scalars, we will show that $\mathbb{Q} \otimes Z^n \cong \mathbb{Q}^n$. This is an isomorphism of $\mathbb{Z}$-modules, i.e. of abelian groups. But we will also see that there is a natural way to define scalar multiplication of elements of $\mathbb{Q} \otimes Z^n$ by elements of $\mathbb{Q}$ so that it extends the scalar multiplication by elements of $\mathbb{Z}$. Thus tensoring with $\mathbb{Q}$ transforms a free $\mathbb{Z}$-module into a vector space over $\mathbb{Q}$.

We use the universal property to compute this tensor product. Define a function $\theta : \mathbb{Q} \times Z^n \to \mathbb{Q}^n$ by

$$\theta(r, (a_1, \ldots, a_n)) = (ra_1, \ldots, ra_n).$$

It is easy to check that θ is bilinear.

Let $e_1, \ldots, e_n$ be a basis of $\mathbb{Z}^n$. Then $\theta(1, e_1), \ldots, \theta(1, e_n)$ is a basis of $\mathbb{Q}^n$. Now suppose that C is a $\mathbb{Z}$-module and $f : \mathbb{Q} \times Z^n \to C$ is any bilinear function. Given any element $v = (s_1, \ldots, s_n)$ of $\mathbb{Q}^n$ there is an integer m such that $ms_1, \ldots, ms_n$ are all integers. In fact, the set of all such m is an ideal in $\mathbb{Z}$, which has the form (d) for some $d \in \mathbb{Z}$. We define $\phi(v) = f(1/m, (ms_1, \ldots, ms_n))$. Note that this does not depend on which element m of (d) we use. If m is an arbitrary element of the ideal (d), say with $m = kd$ then

$$f(1/m, (ms_1, \ldots, ms_n)) = f(k/m, ((m/k)s_1, \ldots, (m/k)s_n)) = f(1/d, (ds_1, \ldots, mds_n)).$$

Thus we have a well-defined homomorphism $\phi : \mathbb{Q}^n \to C$ such that $\phi \circ \theta = f$. There is only one such homomorphism since the condition $\phi(\theta(1, e_i)) = f(1, e_i)$ determines the images of the elements of a basis.

Exercise 1.1. Suppose that R is a subring of a ring S. Regard S as an R-module. Let A be another R module. Consider the tensor product $S \otimes_R A$. Show that $S \otimes_R A$ has the structure of an S-module in which

$$s(\sum_{i=1}^{n} s_i \otimes a_i) = \sum_{i=1}^{n} ss_i \otimes a_i$$

whenever $s_1, \ldots, s_n \in S$ and $a_1, \ldots, a_n \in A$. (That is, show that the formula above gives a well-defined scalar multiplication.)

Exercise 1.2. Let R be a commutative ring. Show that $R \otimes_R R \cong R$.

Exercise 1.3. If A, B and C are R-modules, define $A \otimes_R B \otimes_R C$ and show that

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R B \otimes_R C \cong A \otimes_R (B \otimes_R C).$$

Exercise 1.4. Let p and q be relatively prime integers. Let $A = \mathbb{Z}/p\mathbb{Z}$ and $B = \mathbb{Z}/q\mathbb{Z}$. Show that $A \otimes B \doteq A \otimes_{\mathbb{Z}} B = \{0\}$.

Exercise 1.5. Let F be a field and V a vector space of dimension 3 over F. Show that $V \otimes_F V$ is a vector space of dimension 9 over F. Find an element of $V \otimes_F V$ that is not contained in the image of the bilinear function $\otimes : V \times V \to V \otimes_F V$.

2. tensor products over non-commutative rings

Now we suppose that Λ is a ring, but is not necessarily commutative. When considering Λ-modules we must distinguish between left and right modules. (Although when we state a result about left modules we usually will not bother to give the corresponding statement about right modules.)

Recalling the basic example of a bilinear function in the commutative case, we consider the function $f : \Lambda \times \Lambda \to \Lambda$ given by $f(x, y) = xy$. In the non-commutative setting, it is not true in general that $f(\lambda x, y) = f(x, \lambda y)$. It is true, however, that $f(x\lambda, y) = f(x, \lambda y)$. It seems that the only way to view f as a bilinear function would be to treat the first factor of $\Lambda \times \Lambda$ as a right Λ-module, and the second factor as a left Λ-module.

This suggests that we cannot even get started in defining a tensor product of two left (or two right) Λ-modules. Instead, we should consider a right Λ-module and a left Λ-module. This is not the full extent of the difficulty, however. If we could define $A \otimes B$ in this setting, should the (unique) result be a left Λ-module or a right Λ-module? Our answer will be "neither." It will just be an abelian group, with no Λ-module structure.

Definition 2.1. Suppose that R is a ring, not necessarily commutative. Let A be a left Λ-module and B a right Λ-module. Let G be an abelian group. A function $f : A \times B \to G$ is Λ-*bilinear* if

- $f(a_1 + a_2, b) = f(a_1, b) + f(a_1, b)$,
- $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$, and
- $f(a\lambda, b) = f(a, \lambda b)$.

Note that it does not make sense here to "take scalars out" since there is no Λ-action on G.

Theorem 2.2. *Let Λ be a ring, let A be a left Λ-module and let B be a right Λ-module. There exists an abelian group, denoted $A \otimes_\Lambda B$, and a Λ-bilinear function $\otimes : A \times B \to A \otimes_\Lambda B$, which has the following universal property with respect to Λ-bilinear functions:*

If G is any abelian group and $f : A \times B \to G$ is any Λ-bilinear function then there exists a unique group homomorphism $\phi : A \otimes_\Lambda B \to G$ which makes the following diagram commute

$$\begin{array}{ccc} A \times B & \xrightarrow{\otimes} & A \otimes_\Lambda B \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle \phi} \\ & & G \end{array}.$$

Moreover, any abelian group which has this universal property is isomorphic to $A \otimes_\Lambda B$.

Exercise 2.1. Prove Theorem 2.2.

3. the tensor product functor

3.1. Let Λ be a ring, not necessarily commutative, and suppose that A is a right Λ-module. We can assign an abelian group to each left Λ-module X by the rule $X \mapsto A \otimes_\Lambda X$. It is important to realize that this defines a covariant functor from the category of left Λ-modules to the category of abelian groups. That is, given a homomorphism $f : X \to Y$ we obtain a homomorphism of abelian groups $f_\otimes : A \otimes X \to A \otimes Y$. (Also, $\mathrm{id}_\otimes$ is the identity homomorphism and $(f \circ g)_\otimes = f_\otimes \circ g_\otimes$.) The definition of $f_\otimes$ depends on the universal property, of course. The following diagram illustrates the situation:

$$\begin{array}{ccc} A \times X & \xrightarrow{\otimes} & A \otimes_\Lambda X \\ \big\downarrow{\scriptstyle f \times \mathrm{id}} & & \big\downarrow{\scriptstyle f_\otimes} \\ A \times Y & \xrightarrow{\otimes} & A \otimes_\Lambda Y \end{array} .$$

The map $f_\otimes$ is constructed by applying the universal property to the multilinear function $A \times X \to A \otimes_\Lambda X$ that we get by composing the left arrow and the bottom arrow.

Needless to say, if A is a left Λ-module then $X \mapsto X \otimes_\Lambda A$ is a functor from right Λ-modules to abelian groups.

Theorem 3.2. *Let A be a right Λ-module. Suppose that*

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow 0$$

is a short exact sequence of left Λ-modules. Then

$$A \otimes X \xrightarrow{i_\otimes} A \otimes Y \xrightarrow{p_\otimes} A \otimes Z \longrightarrow 0$$

is an exact sequence (but $i_\otimes$ need not be injective).

Proof. Since the homomorphism $f \times \mathrm{id} : A \times Y \to A \times Z$ is onto, and since the image of the bilinear function $\otimes : A \times Z \to A \otimes_\Lambda Z$ generates $A \otimes_\Lambda Z$, it follows that the image of the homomorphism $p_\otimes$ is a subgroup and generates $A \otimes_\Lambda Z$. Thus the homomorphism $p_\otimes$ must be surjective.

Since $p \circ i$ is the 0-map, it follows from naturality and the universal property of tensor products that $p_\otimes \circ i_\otimes$ is the 0-map. In other words we have $\mathrm{im}\, i_\otimes \subseteq \ker p_\otimes$, and therefore the homomorphism $p_\otimes$ determines a homomorphism $\hat{p} : A \otimes_\Lambda Y / \mathrm{im}\, i_\otimes \to A \otimes_\Lambda Z$. We must show that $\hat{p}$ is an isomorphism. We will do this by constructing an inverse. The construction will be based on the universal property, of course.

First we construct a function from $f : A \times Z \to A \otimes_\Lambda Y / \mathrm{im}\, i_\otimes$ by defining $f(a, z)$ to be the coset of $a \otimes y$ whenever $p(y) = z$. This is well defined since

$$p(y_1) = p(y_2) \;\Rightarrow\; y1 - y2 \in \mathrm{im}\, i \;\Rightarrow\; a \otimes (y1 - y2) \in \mathrm{im}\, i_\otimes .$$

We then use the universal property to construct a homomorphism $f_\otimes : A \otimes_\Lambda Z \to A \otimes_\Lambda Y$. Naturality implies that $\hat{p} \circ f$ is the identity. □

4. the Hom functors

4.1. If X and Y are left Λ-modules, we let $\mathrm{Hom}_\Lambda(X, Y)$ denote the left Λ-module whose elements are Λ-module homomorphisms from X to Y, with pointwise addition and Λ-multiplication. Thus $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda(f(x))$. It is easy to see that $X \mapsto \mathrm{Hom}(X, A)$ is a covariant functor. If $f : X \to Y$ then $f_* : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$ is just given by composition: $f_*(g) = f \circ g$ for $g \in \mathrm{Hom}(A, X)$. Similarly, $X \mapsto \mathrm{Hom}(A, X)$ is a contravariant functor. If $f : X \to Y$ then $f^* : \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X, A)$ is also given by composition: $f^*(g) = g \circ f$ for $g \in \mathrm{Hom}(Y, A)$. (Since we are not forced to mix left and right modules, we avoid the issue that forced the tensor product functor to take values in the category of abelian groups.)

Theorem 4.2. *Let A be a left Λ-module. Suppose that*

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow 0$$

is a short exact sequence of left Λ-modules. Then

$$0 \longrightarrow \mathrm{Hom}_\Lambda(A, X) \xrightarrow{i_*} \mathrm{Hom}_\Lambda(A, Y) \xrightarrow{p_*} \mathrm{Hom}_\Lambda(A, Z)$$

and

$$0 \longrightarrow \mathrm{Hom}_\Lambda(Z, A) \xrightarrow{p^*} \mathrm{Hom}_\Lambda(Y, A) \xrightarrow{i^*} \mathrm{Hom}_\Lambda(X, A)$$

are exact sequences (but the maps on the right may not be surjective).

Proof. Given $f \in \mathrm{Hom}_\Lambda(A, X)$ we have $i_*(f) = i \circ f$. Since i is injective, the composition $i \circ f$ is non-zero if $\mathrm{im}\, f \neq \{0\}$. This shows that i_* is injective. Similarly, if $f \in \mathrm{Hom}(Z, A)$ then, since p is surjective, the composition $f \circ p$ is non-zero if f is non-zero. this shows that p^* is injective.

Naturality implies that $p_* \circ i_*$ and $i^* \circ p^*$ are zero maps. Thus we have $\mathrm{im}\, i_* \subseteq \ker p_*$ and $\mathrm{im}\, p^* \subseteq \ker i^*$.

Suppose that $f \in Hom_\Lambda(A, Y)$ is in the kernel of p_*. We have

$$f \in \ker p_* \Leftrightarrow p \circ f = 0 \Leftrightarrow \mathrm{im}\, f \subseteq \ker p = \mathrm{im}\, i.$$

Since i is injective there is a homomorphism $j : \mathrm{im}\, i \to X$ such that $i \circ j$ is the identity on $\mathrm{im}\, i$. Thus $i_*(j \circ f) = f$. This shows that $\ker p_* \subseteq \mathrm{im}\, i_*$.

Suppose that $f \in \mathrm{Hom}_\Lambda(Y, A)$ is in the kernel of i^*. Construct $g \in \mathrm{Hom}_\Lambda(Z, A)$ by defining $g(z) = f(y)$ whenever $p(y) = z$. This is well-defined when $f \in \ker i^*$, since

$$f \in \ker i^* \Leftrightarrow i \circ f = 0 \Leftrightarrow \mathrm{im}\, i \subseteq \ker f.$$

This shows that $\ker i^* \subseteq \operatorname{im} p^*$. $\square$

5. projective and injective modules

Definition 5.1. A functor $\mathcal{F}$ from left Λ-modules to left Λ-modules is *exact* if

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\mathcal{F}(i)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(p)} \mathcal{F}(Z) \longrightarrow 0$$

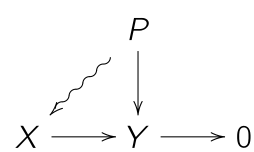
is exact whenever

$$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow 0$$

is exact.

It is natural to ask when the functors $X \mapsto \operatorname{Hom}_\Lambda(X, A)$ and $X \mapsto \operatorname{Hom}_\Lambda(A, X)$ are exact.

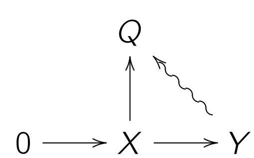
Definition 5.2. A left Λ-module P is *projective* if for every surjective homomorphism $X \to Y$ of left Λ-modules, and every homomorphism $P \to Y$, there exists a homomorphism $P \to X$ that makes the following diagram commute



Exercise 5.1. Show that $X \mapsto \operatorname{Hom}_\Lambda(A, X)$ is exact if and only if A is projective.

Exercise 5.2. Show that any free Λ-module is exact.

Definition 5.3. A left Λ-module Q is *injective* if for every injective homomorphism $X \to Y$ of left Λ-modules, and every homomorphism $X \to Q$, there exists a homomorphism $Y \to Q$ that makes the following diagram commute



Exercise 5.3. Show that $X \mapsto \operatorname{Hom}_\Lambda(X, A)$ is exact if and only if A is injective.

Exercise 5.4. Show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Theorem 5.4. *The following are equivalent for a left Λ-module P:*

- *P is projective;*
- *Every short exact sequence $0 \to X \to Y \to P \to 0$ splits;*
- *P is a direct summand of a free module;*

Exercise 5.5. Show that a direct sum of left Λ-modules is projective if and only if each summand is projective.

Theorem 5.5. *The following are equivalent for a left Λ-module Q:*

- *Q is injective;*
- *Every short exact sequence $0 \to Q \to Y \to Z \to 0$ splits;*
- *Given any $q \in Q$ and $\lambda \in \Lambda$ there exists $x \in Q$ such that $\lambda x = q$.*

Exercise 5.6. Show that a direct product of left Λ-modules is injective if and only if each factor is injective.

6. chain complexes

Definition 6.1. A *chain complex* C is a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

such that $d_i \circ d_{i+1} = 0$ for $i = 0, 1, \ldots$. For $n = 0, 1, \ldots$ The *homology* groups of C are defined to be the quotients $H_n(C) = Z_n(C)/N_n(C)$, where $Z_n(C) = \ker d_n$ and $B_n(C) = \operatorname{im} d_{n+1}$.

A *cochain complex* C is a sequence of abelian groups and homomorphisms

$$\cdots \xleftarrow{d^4} C^3 \xleftarrow{d^3} C^2 \xleftarrow{d^2} C^1 \xleftarrow{d^1} C^0 \xleftarrow{d^0} 0$$

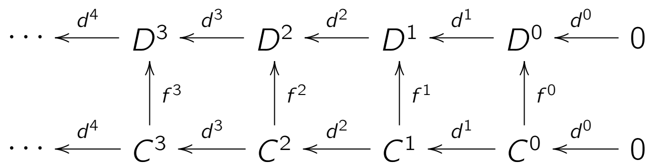
such that $d^{i+1} \circ d^i = 0$ for $i = 0, 1, \ldots$. For for $n = 0, 1, \ldots$ the *cohomology* groups of C are defined to be the quotient groups $H^n(C) = Z^n(C)/B^n(C)$, where $Z^n(C) = \ker d^{n+1}$ and $B^n(C) = \operatorname{im} d^n$.

Definition 6.2. A *chain map* from a chain complex C to a chain complex D is a sequence of homomorphisms $f_n : C_n \to D_n$ such that $d_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$ for $n = 0, 1, 2, \ldots$. In other words, the following diagram commutes:

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d_4} & C_3 & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & 0 \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\
\cdots & \xrightarrow{d_4} & D_3 & \xrightarrow{d_3} & D_2 & \xrightarrow{d_2} & D_1 & \xrightarrow{d_1} & D_0 & \xrightarrow{d_0} & 0
\end{array}$$

If C and D are chain complexes we will write $f : C \to D$ to indicate that $f = (f_n)$ is a chain map from C to$D$.

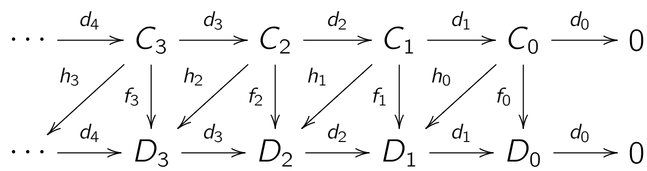
A *cochain map* from a cochain complex C to a cochain complex D is a sequence of homomorphisms $f^n : C^n \to D^n$ such that $d^{n+1} \circ f^n = f^{n+1} \circ d^{n+1}$ for $n = 0, 1, 2, \ldots$. In other words, the following diagram commutes:



If C and D are cochain complexes we will write $f : C \to D$ to indicate that $f = (f^n)$ is a cochain map from C to$D$.

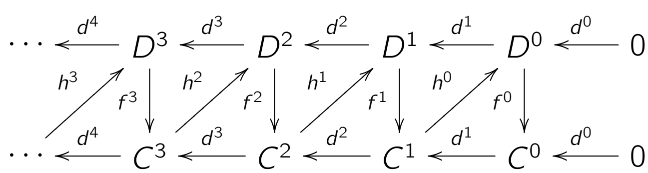
Exercise 6.1. Show that a homomorphism between two chain complexes (cochain complexes) determines a natural homomorphism between their homology (cohomology) groups. That is, H_n and H^n are functors from (co)chain complexes to abelian groups.

Definition 6.3. Suppose that C and D are chain complexes and that $f : C \to D$ is a homomorphism. A *null homotopy* for f is a sequence (h_n) of homomorphisms, for $n = 0, 1, 2, \ldots$, such that $h_n : C_n \to D_{n+1}$ satisfies $f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n$ for $n > 0$. The following diagram (which is not commutative) illustrates the situation.



Two chain maps $f : C \to D$ and $g : C \to D$ are *homotopic* if the homomorphism $f - g$ is null homotopic.

Similarly, a null homotopy for a cochain map is a sequence (h_n) of homomorphisms, for $n = 0, 1, 2, \ldots$, such that $h_n : C^{n+1} \to D^n$ satisfies $f^n = d^n \circ h^{n-1} + d^{n+1} \circ h^n$ for $n > 0$. The following diagram illustrating the situation:



Two cochain maps $f : C \to D$ and $g : C \to D$ are *homotopic* if the homomorphism $f - g$ is null homotopic.

Proposition 6.4. *Suppose that $f : C \to D$ and $g : C \to D$ are homotopic chain maps. Then the homomorphism $f_* : H_n(C) \to H_n(D)$ is equal to $g_* : H_n(C) \to H_n(D)$.*

Proof. It suffices to show that if $k = f - g$ is null-homotopic then k_* is the zero homorphism. Let h be a null-homotopy for k. We will show that $f_n(Z_n(C)) \subseteq B_n(D)$. If

$x \in Z_n(C)$ then $d_n(x) = 0$, so

$$f_n(x) = d_{n+1}(h_n(x)) + h_{n-1}(d_n(x)) = d_{n+1}(h_n(x)) \in B_n(D).$$

□

7. projective resolutions

Definition 7.1. Let A be a Λ-module. A *projective resolution* of Λ is an exact sequence of projective left Λ-modules

$$\cdots \xrightarrow{p_3} P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \longrightarrow 0$$

If we think of the modules P_i as abelian groups then a projective resolution is a chain complex. However, since the sequence is exact, the homology groups are all trivial.

7.2. A projective resolution of an arbitrary Λ-module A can be constructed inductively. Since A is a quotient of a projective module, there is a surjective homomorphism $p_0 : P_0 \to A$ for some projective module P_0. Let K_0 be the kernel of p_0, so we have a short exact sequence $0 \to K_0 \to P_0 \to A \to 0$. Since K_0 is itself a quotient of some projective module P_1, we have another short exact sequence $0 \to K_1 \to P_1 \to K_0 \to 0$. If we compose the homorphism from P_1 onto K_0 with the inclusion of K_0 into P_0, we obtain an exact sequence

$$0 \to K_1 \to P_1 \to P_0 \to A \to 0.$$

We may continue inductively, to construct a projective resolution of A.

Proposition 7.3. *Suppose that P and Q are projective resolutions of the left Λ-modules A and B. Let $f : A \to B$ be any homomorphism. Then f extends to a chain map $(f, f_0, f_1, \ldots)$ making the following diagram commute.*

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{p_3} & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & A & \longrightarrow 0 \\
& \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & \\
\cdots \xrightarrow{q_3} & Q_2 & \xrightarrow{q_2} & Q_1 & \xrightarrow{q_1} & Q_0 & \xrightarrow{q_0} & B & \longrightarrow 0
\end{array}$$

Proof. We construct the chain map by induction. Consider the composite homomorphism $f \circ p_0 : P_0 \to B$. Since the homomorphism $q_0 : Q_0 \to B$ is surjective, and P_0 is projective, there is a homomorphism $f_0 : P_0 \to Q_0$ such that $q_0 \circ f_0 = f \circ p_0$. This is the base case of the induction. For the induction step, suppose that we have constructed $f_i : P_i \to Q_i$ such that $q_i \circ f_i = f_{i-1} \circ p_i$. Then $q_i \circ f_i \circ p_{i+1} = f_{i-1} \circ p_i \circ p_{i+1} = 0$. It follows that the image of $f_i \circ p_{i+1}$ is contained in $\ker q_i = \operatorname{im} q_{i+1}$. By the projective property of P_{i+1} there exists $f_{i+1} : P_{i+1} \to Q_{i+1}$ such that $q_i \circ f_{i+1} = f_i \circ p_{i+1}$. This completes the proof □

Definition 7.4. A chain map satisfying the conclusion of Proposition 7.3 is a *chain extension* of f.

Obviously any two chain extensions of f induce the same (trivial) homomorphism on the homology groups of a projective resolution, since all of the homology groups are 0 to begin with. However, the following proposition says a little bit more.

Proposition 7.5. *Suppose that P and Q are projective resolutions of left Λ-modules A and B and that $f : A \to B$ is a homomorphism. Then any two chain extensions of f are chain homotopic.*

Proof. It suffices to show that any chain extension of the zero homomorphism is null-homotopic. Suppose that (f_i) is a chain extension of the zero homomorphism from A to B. We will inductively construct homomorphisms $h_i : P_i \to Q_{i+1}$ so that $q_{i+1} \circ h_i + h_{i-1} \circ p_i = f_i$, for $i = 1, 2, \ldots$. We have that $q_0 \circ f_0 = 0$, so $\operatorname{im} f_0 \subseteq \ker q_0 = \operatorname{im} q_1$. Therefore the projective property of P_0 implies that there is a homomorphism $h_0 : P_0 \to Q_1$ such that $q_1 \circ h_0 = f_0$. It follows that $q_1 \circ h_0 \circ p_1 = f_0 \circ p_1 = q_1 \circ f_1$. Thus $q_1 \circ (f_1 - h_0 \circ p_1) = 0$, so the image of $f_1 - h_0 \circ p_1$ is contained in $\ker q_0 = \operatorname{im} q_1$. The projective property of P_1 then guarantees the existence of $h_1 : P_1 \to Q_1$ such that $q_2 \circ h_1 = f_1 - h_0 \circ p_1$.

For $i > 0$, suppose we are given $h_i : P_i \to Q_{i+1}$ such that $q_{i+1} \circ h_i = f_i - h_{i-1} \circ p_i$. Consider $f_{i+1} - h_i \circ p_{i+1}$. Composing with q_{i+1} we have

$$q_{i+1} \circ f_{i+1} - q_{i+1} \circ h_i \circ p_{i+1} = f_i \circ p_{i+1} - f_i \circ p_{i+1} = 0.$$

Thus the image of $f_{i+1} - h_i \circ p_{i+1}$ is contained in $\ker q_{i+1} = \operatorname{im} q_{i+2}$. Now the projective property of P_{i+1} guarantees the existence of a homomorphism $h_{i+1} : P_{i+1} \to Q_{i+2}$ such that $q_{i+2} \circ h_{i+1} = f_{i+1} - h_i \circ p_{i+1}$. This completes the induction step. □

Corollary 7.6. *Suppose that P and Q are projective resolutions of a left Λ-module A and that $f : A \to A$ is a homomorphism. If M is a left Λ-module then any two chain extensions of f induce the same homomorphism between the cohomology groups $H^*(\operatorname{Hom}(Q, M))$ and $H^*(\operatorname{Hom}(P, M))$. Similarly, if N is a right Λ-module then any two chain extensions of $f : A \to A$ induce the same homomorphisms between the homology groups $H_n(N \otimes_\Lambda P)$ and $H_n(N \otimes_\Lambda Q)$.*

Corollary 7.7. *Suppose that P and Q are projective resolutions of a left Λ-module A. If M is a left Λ-module then $H^n(\operatorname{Hom}(Q, M)) \cong H^n(\operatorname{Hom}(P, M))$ for all $n \geq 0$. If N is a left Λ-module then $H_*(N \otimes_\Lambda P) \cong H_*(N \otimes_\Lambda Q)$ for all $n \geq 0$.*

8. group cohomology

Definition 8.1. Let R be a commutative ring and G a group. The *group ring RG* is the direct product of copies of R, indexed by elements of G, with a multiplication operation to be defined below. We can represent an element of RG as a formal sum $a_1 g_1 + \cdots + a_n g_n$ where $a_i \in R$ and $g_i \in G$ for $i = 1, \ldots, n$. Alternatively, we may write an element of RG as a sum $\sum_{g\in G} a_g g$, where $a_g = 0$ for all but a finite number of elements of G. In terms of this notation the multiplication is given by the rule

$$(\sum_{g\in G} a_g\, g)(\sum_{g\in G} b_g\, g) = \sum_{g\in G} c_g\, g$$

where

$$c_g = \sum_{xy=g} a_x b_y.$$

8.2. We will immediately specialize to the case $R = \mathbb{Z}$. To define a $\mathbb{Z}G$-module structure on an abelian group A it suffices to specify the scalar multiplication by elements of G. In other words, any group action $G \to \mathrm{Aut}(A)$ (denoted by $\cdot$) corresponds to a $\mathbb{Z}G$-module structure on A, where $ga = g \cdot a$, and

$$(n_1 g_1 + \cdots + n_k g_k)a = n_1(g_1 \cdot a) + \cdots n_k(g_k \cdot a).$$

A $\mathbb{Z}G$-module is *trivial* if it corresponds to the trivial homomorphism from G to $\mathrm{Aut}(A)$ which sends each element of G to the identity automorphism. That is, A is a trivial $\mathbb{Z}G$-module if and only if $ga = g \cdot a = a$ for all $g \in G$ and $a \in A$.

Definition 8.3. Suppose that G is a group. Regard $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module and suppose that P is a projective resolution of $\mathbb{Z}$, so we have an exact sequence of projective left $\mathbb{Z}G$-modules

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to Z \to 0.$$

If A is a right $\mathbb{Z}G$-module then we can construct a chain complex by tensoring with A:

$$\cdots \to A \otimes_{\mathbb{Z}G} P_3 \to A \otimes_{\mathbb{Z}G} P_2 \to A \otimes_{\mathbb{Z}G} P_1 \to A \otimes_{\mathbb{Z}G} P_0 \to 0.$$

The homology groups of this chain complex are denoted by $H_n(G; A)$, and are called homology groups of G with coefficients in A. If A is a left $\mathbb{Z}G$-module then we can form a cochain complex by applying the functor $X \mapsto \mathrm{Hom}(X, A)$:

$$\cdots \leftarrow \mathrm{Hom}(P_3, A) \leftarrow \mathrm{Hom}(P_2, A) \leftarrow \mathrm{Hom}(P_1, A) \leftarrow \mathrm{Hom}(P_0, A) \leftarrow 0.$$

The cohomology groups of this cochain complex are denoted by $H^n(G; A)$ and are called the cohomology groups of G with coefficients in A.

Note that Corollary 7.7 shows that the homology and cohomology groups of G do not depend on the choice of projective resolution. In particular, we could use a free resolution if we want.

8.4. In order to get some idea of what sort of thing the cohomology of G might be telling us about G, we will specialize to the case where A is a trivial $\mathbb{Z}G$-module, and we will concentrate on the groups $H^1(G; A)$ and $H^2(G; A)$. As observed above, we may use a free resolution to compute these groups. We will start by describing a very concrete free resolution, which is called the *homogeneous bar resolution*.

We introduce some special symbols (with bars in them) that will be used to denote the basis elements of our (left) free modules. The first module B_0 will have rank 1, making it isomorphic to $\mathbb{Z}G$, and its basis element will be denoted $[\ ]$. There is a surjective homomorphism $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ (called the *augmentation map*) which is determined by the property that it sends each element of G to 1. Thus $\epsilon(a_1 g_1 + \cdots a_n g_n) = a_1 + \cdots a_n$. The augmentation map is the homomorphism $B_0 \to \mathbb{Z}$ that is used at the end of the bar resolution. The basis elements for B_n, $n > 0$, are in one-to-one correspondence with n-tuples of non-identity elements of G, and will be denoted by $[g_1|\cdots|g_n]$. Thus the elements of B_n are ZG-linear combinations of these symbols. We will make the convention that the symbol $[g_1|\cdots|g_n]$ represents the element 0 if $g_i = 1$ for some $i \in \{1, \ldots, n\}$.

For $n > 0$ we define $d_n : B_n \to B_{n-1}$ by the following formula

$$d_n([g1|\cdots|g_n]) = g_1[g_2|\cdots|g_n] - [g_1 g_2|\cdots|g_n] + \cdots (-1)^{n-1}[g_1|\cdots|g_{n-1}g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

(Of course, it suffices to specify the homomorphism d_n on the basis elements.) We will let $B(G)$ denote the sequence of free $\mathbb{Z}G$-modules and maps

$$\cdots \xrightarrow{d_3} B_2 \xrightarrow{d_2} B_1 \xrightarrow{d_1} B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where B_n, d_n and $d_0 = \epsilon$ are as defined above.

Proposition 8.5. *The sequence $B(G)$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.*

Proof. By construction the modules B_n are free left $\mathbb{Z}G$-modules and the maps d_n are $\mathbb{Z}G$-module homomorphisms. It remains to show that $\ker d_n = \operatorname{im} d_{n+1}$. For this purpose we are allowed to forget about the module structure and treat the B_n as abelian groups and the d_n as homomorphisms of abelian groups. (The kernel and image are the same sets.)

As an abelian group B_n is free with a basis consisting of all elements $g[g_1|\cdots g_n]$. We may therefore construct a homomorphism of abelian groups $h_n : B_n \to B_{n+1}$ for each $n \geq 0$ by defining its values on this basis by the formula

$$h_n(g[g_1|\cdots g_n]) = [g|g_1|\cdots g_n].$$

We claim that $d_{n+1} \circ h_n + h_{n-1} \circ d_n$ is the identity for all $n \geq 0$. We have

$$d_{n+1} \circ h_n(g[g_1|\cdots|g_n]) = d_{n+1}([g|g_1|\cdots|g_n])$$
$$= g[g_1|\cdots|g_n] - [gg1|\cdots|g_n] + \cdots + (-1)^n[g|g_1|\cdots|g_{n-1}g_n] + (-1)^{n+1}[g|g_1|\cdots|g_n]$$

while

$$h_{n-1} \circ d_n(g[g_1|\cdots|g_n]) = h_{n-1}(g d_n([g_1|\cdots|g_n]))$$
$$= h_{n-1}(g(g_1[g_2|\cdots|g_n] - [g_1g_2|\cdots|g_n] + \cdots + (-1)^{n-1}[g_1|\cdots|g_{n-1}g_n] + (-1)^n[g_1|\cdots|g_{n-1}]))$$
$$= [gg_1|g_2|\cdots|g_n] - [g|g_1g_2|\cdots|g_n] + \cdots + (-1)^{n-1}[g|g_1|\cdots|g_{n-1}g_n] + (-1)^n[g|g_1|\cdots|g_{n-1}].$$

Careful inspection reveals that when these two expressions are added all of the terms cancel except for $g[g_1|\cdots|g_n]$. (And it is easy to check the case $n = 0$ directly. This proves the claim.

Next we show by induction that $d_{n-1} \circ d_n = 0$. For $n = 1$ we have $d_0(d_1([g])) = \epsilon(g[\,] - [\,]) = 0$. For $n > 1$ we have

$$d_n \circ d_{n+1} \circ h_n = d_n \circ (\mathrm{id} - h_{n-1} \circ d_n) = d_n - d_n \circ h_{n-1} \circ d_n = d_n - (\mathrm{id} - h_{n-2} \circ d_{n-1}) \circ d_n = d_n - d_n + 0 = 0$$

Since h_n is surjective, this shows that $d_n \circ d_{n+1} = 0$, and hence that $B(G)$ is a chain complex. But we have also shown that (h_n) is a null-homotopy of the identity map from $B(G)$ to itself. In other words, the homology groups of $B(G)$ are all zero, and $B(G)$ is hence an exact sequence. □

8.6. We can use the bar resolution to determine the cohomology group $H^1(G; A)$ where A is a trivial $\mathbb{Z}G$-module. We consider the cochain complex

$$\cdots \xleftarrow{\delta^3} \mathrm{Hom}(B_3, A) \xleftarrow{\delta^2} \mathrm{Hom}(B_2, A) \xleftarrow{\delta^1} \mathrm{Hom}(B_1, A) \xleftarrow{\delta^0} \mathrm{Hom}(B_0, A) \longleftarrow 0$$

where $\delta^i(f) = f \circ d_{i+1}$.

An element ϕ of $\mathrm{Hom}(B_1, A)$ is a 1-cocycle if and only if $\phi(d_2([g_1|g_2])) = 0$ for all elements $g_1, g_2 \in G$. This means

$$0 = \phi(g_1[g_2] - [g_1g_2] + [g_1]) = g_1\phi([g_2]) - \phi([g_1g_2]) + \phi([g_1]) = \phi([g_2]) - \phi([g_1g_2]) + \phi([g_1])$$

where the last step uses that A is a trivial $\mathbb{Z}G$-module. Thus ϕ is a 1-cocycle if and only if $\phi([g_1g_2]) = \phi([g_1]) + \phi([g_2])$. In other words the 1-cocyles can be naturally identified with the group of homomorphisms from G to A. On the other hand, if $\psi \in \mathrm{Hom}(B_0, A)$ then $d^0(\psi) = \psi \circ d_1 \in \mathrm{Hom}(B_1, A)$, and

$$\psi \circ d_1([g]) = \psi(g[\,] - [\,]) = g\psi([\,]) - \psi([\,]) = \psi([\,]) - \psi([\,]) = 0.$$

Thus the subgroup of 1-coboundaries is trivial, and $H^1(G; A) \cong \mathrm{Hom}(G, A)$ (as abelian groups). Incidentally, this also shows that $H^0(G; A) \cong \mathrm{Hom}(B_0, A) \cong A$, since an element of $\mathrm{Hom}(B_0, A)$ is determined by the image of $[\,]$, which can be arbitrary.

Exercise 8.1. Describe $H^1(G; A)$ if A is a non-trivial $\mathbb{Z}G$-module.

Problem 8.2. Suppose that F is a field and G is a finite group of automorphisms of F. Show that the abelian group $F^\times$ can be given the structure of a FG-module in which $\sigma \cdot x = \sigma(x)$ for $\sigma \in G$. Show that $H^1(G; F^\times) = \{0\}$.

8.7. Next we will describe $H^2(G; A)$, where A is a trivial $\mathbb{Z}G$-module.