MATH 215 – The Real Numbers (RN)

We will assume the existence of a set $\mathbb{R}$, whose elements are called real numbers, along with a well-defined binary operation $+$ on $\mathbb{R}$ (called addition), a second well-defined binary operation $\cdot$ on $\mathbb{R}$ (called multiplication), and a relation $<$ on $\mathbb{R}$ (called less than), and that the following fourteen statements involving $\mathbb{R}$, $+$, $\cdot$, and $<$ are true:

A1. For all a, b, c in $\mathbb{R}$, $(a+b)+c = a+(b+c)$.
A2. There exists a unique real number 0 in $\mathbb{R}$ such that $a+0 = 0+a = a$ for every real number a.
A3. For every a in $\mathbb{R}$, there exists a unique real number $-a$ in $\mathbb{R}$ such that $a+(-a) = (-a)+a = 0$.
A4. For all a, b in $\mathbb{R}$, $a+b = b+a$.
M1. For all a, b, c in $\mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
M2. There exists a unique real number 1 in $\mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a$ for all a in $\mathbb{R}$.
M3. For all non-zero a in $\mathbb{R}$, there exists a unique real number a^{-1} in $\mathbb{R}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
M4. For all a, b in $\mathbb{R}$, $a \cdot b = b \cdot a$.
D1. For all a, b, c in $\mathbb{R}$, $a \cdot (b+c) = a \cdot b + a \cdot c$.
NT1. $1 \neq 0$.
O1. For all a in $\mathbb{R}$, exactly one of the following statements is true: $0 < a$, $a = 0$, $0 < -a$.
O2. For all a, b in $\mathbb{R}$, if $0 < a$ and $0 < b$, then $0 < a+b$.
O3. For all a, b in $\mathbb{R}$, if $0 < a$ and $0 < b$, then $0 < a \cdot b$.
C1. A completeness axiom. (to be introduced in a later course)

Remark 1 *Our assumption that the operations addition and multiplication are* **well-defined** *means that the following statements involving equality and operations of addition and multiplication, respectively, are true, even though we haven't stated them as axioms:*
E1. For all a, b, c, d in $\mathbb{R}$, if $a = b$ and $c = d$, then $a + c = b + d$.
E2. For all a, b, c, d in $\mathbb{R}$, if $a = b$ and $c = d$, then $a \cdot c = b \cdot d$.

Notation 2 *We will use the common notation ab to denote $a \cdot b$.*

Notation 3 *We will also use the notation $a > b$ (greater than) to denote $b < a$ (less than).*

Proposition 4 *For every a in $\mathbb{R}$, $a \cdot 0 = 0$.*

Proposition 5 *Let a, b be real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.*

Proposition 6 0 *has no multiplicative inverse. In other words, there is no real number a such that $a \cdot 0 = 1$.*

Proposition 7 *For all a, b, c in $\mathbb{R}$, if $a + b = a + c$, then $b = c$.*

Proposition 8 *For all a, b, c in $\mathbb{R}$, if $a \neq 0$ and $ab = ac$, then $b = c$.*

Proposition 9 *For every a in $\mathbb{R}$, $-(-a) = a$.*

Proposition 10 *For all real numbers a and b, $(-a)b = -(ab)$.*

Proposition 11 *For all real numbers a and b, $(-a)(-b) = ab$.*

Proposition 12 $(-1)(-1) = (1)(1) = 1$.

Proposition 13 $0 < 1$.

Proposition 14 *For all real numbers a and b, if $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.*