Part I: A model theory primer – structures, formulas, and compactness.

Defn 1: A language $\mathcal{L}$ is given by:

1) A set of function symbols, $\mathcal{F}$, and $n_f \in \mathbb{Z}_{>0}$ for each $f \in \mathcal{F}$.
2) A set of relation symbols, $\mathcal{R}$, and $n_r \in \mathbb{Z}_{>0}$ for each $r \in \mathcal{R}$.
3) A set of constant symbols, $\mathcal{C}$.

n_f, n_r give the arity of the corresponding functions and relations.

e.g.

- $\mathcal{F}, \mathcal{R}, \mathcal{C}$ might be empty.
- Common and important example: $\mathcal{L}_{Rings} = \{\underbrace{+, -, \cdot}_{\text{binary functions}}, \underbrace{0, 1}_{\text{constants}}\}$
- $\mathcal{L}_{or} = \{<\}$ ← Binary relation.
- $\mathcal{L}_{Graphs} = \{E\}$ ← Binary Relation.

Defn 2: A $\mathcal{L}$-structure $\mathcal{M}$ is given by:

1) A nonempty set, which we denote by $\underline{\mathcal{M}}$. ← This is called the universe or domain of $\mathcal{M}$.
2) A function $f^{\mathcal{M}}: \underline{\mathcal{M}}^{n_f} \longrightarrow \underline{\mathcal{M}}$ for each $f \in \mathcal{F}$.
3) A set $R^{\mathcal{M}} \subseteq \underline{\mathcal{M}}^{n_R}$ for each $R \in \mathcal{R}$.
4) An element $c^{\mathcal{M}} \in \underline{\mathcal{M}}$ for all $c \in \mathcal{C}$.

← These are called interpretations of the symbols.

e.g. Give the most natural examples of structures from previous example.

Defn 3: Suppose $\mathcal{M}, \mathcal{N}$ are $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $f: \mathcal{M} \longrightarrow \mathcal{N}$ is an injective map $f: \underline{\mathcal{M}} \longrightarrow \underline{\mathcal{N}}$ that preserves symbols:

1) $f(f^{\mathcal{M}}(a_1, \dots a_{n_f})) = f^{\mathcal{N}}(f(a_1) \dots f(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \dots a_n \in \underline{\mathcal{M}}$
2) $(a_1, \dots a_{n_R}) \in R^{\mathcal{M}} \iff (f(a_1) \dots f(a_{n_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \dots a_{n_R} \in \underline{\mathcal{M}}$.
3) $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all $c \in \mathcal{C}$.

$\mathcal{L}$-isomorphism – bijective $\mathcal{L}$-embedding.

If $\underline{\mathcal{M}} \subseteq \underline{\mathcal{N}}$ and the inclusion map is an $\mathcal{L}$-embedding, we say $\mathcal{M}$ is a substructure of $\mathcal{N}$ and we write $\mathcal{M} \leq \mathcal{N}$.

e.g. Give the most natural examples from previous examples.

$(\mathbb{Z}, +, 0) \leq (\mathbb{R}, +, 0)$

$x \longmapsto e^x \quad (\mathbb{Z}, +, 0) \longrightarrow (\mathbb{R}, \cdot, 1)$.

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Defn: The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ s.t.:

1) $c \in \mathcal{T}$ for each $c \in \mathcal{C}$
2) each variable $v_i \in \mathcal{T}$ for $i = 1, 2, \dots$
3) If $t_1 \dots t_{n_f} \in \mathcal{T}$, $f \in \mathcal{F}$, then $f(t_1 \dots t_{n_f}) \in \mathcal{T}$.

Explain how any term in which variables $v_1, \dots, v_n$ gives a map $t^{\mathcal{M}} : \underline{\mathcal{M}}^n \longrightarrow \underline{\mathcal{M}}$.

Defn:[5] ϕ is an atomic $\mathcal{L}$-formula if ϕ is either

1) $t_1 = t_2$ for terms t_1, t_2.
2) $R(t_1, \dots, t_{n_R})$ for $R \in \mathcal{R}$ and $t_1, \dots t_{n_R}$ are $\mathcal{L}$-terms.

The set of $\mathcal{L}$-formulas is the smallest set W containing the atomic $\mathcal{L}$-formulas s.t.

1) $\phi \in W \Rightarrow \neg\phi \in W$
2) $\phi, \psi \in W \Rightarrow \phi \wedge \psi \in W$ and $\phi \vee \psi \in W$.
3) $\phi \in W \Rightarrow \exists v_i \phi \in W$ and $\forall v_i \phi \in W$.

Free versus bound variables

Sentence = No free variables

Defn:[6] Let ϕ have free variables among $(v_1, \dots v_m)$ and let $\bar{a} = (a_1 \dots a_m) \in \underline{\mathcal{M}}^m$.
We define $\mathcal{M} \models \phi(\bar{a})$ inductively:

1) If $\phi = $ "$t_1 = t_2$", then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.
2) If $\phi = $ "$R(t_1, \dots, t_{n_R})$" then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) \dots t_{n_R}^{\mathcal{M}}(\bar{a}) \in R^{\mathcal{M}}$.
3) If $\phi = $ "$\neg\psi$" then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$.
4) If $\phi = $ "$(\psi \wedge \theta)$" then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$.
5) If ϕ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\theta(\bar{a})$.
6) If ϕ is $\exists v_j\, \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if there is some $b \in \underline{\mathcal{M}}$ s.t. $\mathcal{M} \models \psi(\bar{a}, b)$.
7) If ϕ is $\forall v_j\, \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if for every $b \in \underline{\mathcal{M}}$, $\mathcal{M} \models \psi(\bar{a}, b)$.

Prop:[7] Suppose $\mathcal{M} \subseteq \mathcal{N}$, $\bar{a} \in \underline{\mathcal{M}}$ and $\phi(\bar{v})$ is quantifier free. Then, $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\bar{a})$.

Defn:[8] An $\mathcal{L}$-theory, T, is a set of $\mathcal{L}$-sentences.

$\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all $\phi \in T$.

A class K of $\mathcal{L}$-structures is an elementary class if $K = \{\mathcal{M} : \mathcal{M} \models T\}$ for some theory T.

Defn:[9] (Logical consequence) $T \models \phi$ if $\mathcal{M} \models \phi$ for any $\mathcal{M} \models T$.

e.g.: Let T = Theory of ordered abelian groups

$$\_\_\_\_\_ T \models \forall x\,(x \neq 0 \longrightarrow x + x \neq 0).$$

Model theory, as currently practised, is in large part the Theory of Definable sets. As one understands the structures in a given class, attention inevitably turns toward understanding the definable sets, both because many other questions essentially depend on this understanding and because definable sets are usually interesting in their own right.

Defn: Let $\mathcal{M}$ be an $\mathcal{L}$-str.

$X \subseteq \mathcal{M}^n$ is definable if there is an $\mathcal{L}$-formula $\phi(v_1, \dots v_n, w_1, \dots, w_m)$ and $\bar{b} \in \mathcal{M}^m$ s.t.

$$X = \{\bar{a} \in \mathcal{M}^n \mid \mathcal{M} \models \phi(\bar{a}, \bar{b})\}.$$

Let $A \subseteq \mathcal{M}$. We say X is A-definable if the $\bar{b}$ above can be taken in A.

e.g.: We will work with $\mathcal{L}_{Rings} = \{+, -, \cdot, 0, 1\}$ and consider a ring R with the natural interpretation.

Let $p(x) \in R[x]$. $\{x \in R \mid p(x) = 0\}$ is definable.

e.g.:

Now, let $R = \mathbb{R}$, and consider

$$\phi(x, y) := \text{“} \exists z\,(z \neq 0 \wedge y = x + z^2) \text{”}$$

Then $\mathcal{M} \models \phi(a, b) \iff a < b$.

So, even though $<$ is not in $\mathcal{L}_{Rings}$, if we are interested in the real numbers, and her definable subsets, then we must inevitably consider the ordering.

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Prop:[11] Suppose $\mathcal{M}$ is a substr. of $\mathcal{N}$, $\bar{a} \in \mathcal{M}$. Then if $\phi(\bar{v})$ is q.f., then $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a})$.

Pf: By induction on complexity of formula.

Defn:[12] $\mathcal{M} \equiv \mathcal{N} \iff \mathcal{M} \models \phi$ iff $\mathcal{N} \models \phi$ for all $\mathcal{L}$-sentences ϕ.

Thm:[13] Suppose $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$.

Pf: 1) Show j commutes with term maps.

2) Induction on complexity of formulas.

Defn:[14] Theories, logical consequence, and an example. ← Torsion free from ordered.

Defn:[15] Definable, A-definable, and an example. ← Varieties ordering in $\mathbb{R}$

e.g.: $\mathcal{M} = (\mathbb{C}(x), +, -, \cdot, 0, 1)$

Consider $\{v \mid \exists x \exists y,\ y^2 = v \wedge x^3 + 1 = v\}$

For any $a \in \mathbb{C}$, we can find y, x s.t. $y^2 = a$ and $x^2 + 1 = a$.

Now let $h \in \mathbb{C}(x) \setminus \mathbb{C}$.

If there $f, g \in \mathbb{C}(x)$ s.t. $g^2 = h$ and $f^3 + 1 = h$.

Then $x \longmapsto (f(x), g(x))$ is a non-constant rational map

$\mathbb{C} \longrightarrow E = \{y^2 = x^3 + 1\}$ and no such map exists, by Riemann-Roch, for instance.

One can generalize this to show:

Given
$$\begin{array}{c} \mathbb{C}(x)^{alg} \\ | \\ F \\ | \\ \mathbb{C}(x) \end{array}$$
← Finite extension.

Then $\mathbb{C}$ is defn. in F.

Prop:[16] Let $\mathcal{M}$ be an $\mathcal{L}$-str. If $X \subseteq \mathcal{M}^n$ is A-defn., then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A ptwise fixes X setwise.

Pf: $\psi(\bar{v}, \bar{a})$ an $\mathcal{L}$-form defin. X with $A \in A$. Let $\sigma \in Aut(\mathcal{M}/A)$. Let $\bar{b} \in \mathcal{M}^n$.

By Thm', $\mathcal{M} \models \psi(\bar{b}, \bar{a}) \Leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{b}), \sigma(\bar{a})) \Leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{b}), \bar{a})$ ∎.

Defn:[17] An $\mathcal{L}_0$-structure $\mathcal{N}$ is defn. interpreted in $\mathcal{M}$ if $\exists\ X \subseteq \mathcal{M}^n$ and defn. functions and sets on X s.t. when $\mathcal{L}_0$ is interpreted on X with these func. & sets,

$$\mathcal{N} \cong X.$$

Defn:[18] $\mathcal{N}$ is interpretable if we allow a defn. eq. relation $\sim$ and the defn. func. & sets are $\sim$-eqivariant.

If $\mathcal{M}, \mathcal{N}$ defn. interpret each other, then

$$\begin{array}{ccccc} \mathcal{M} & \xleftarrow[\cong]{f} & X & & \\ & & \cap\!| & & \\ & & \mathcal{N}^n & \xleftarrow[\cong]{g} & Y \\ & & & & \cap\!| \\ & & & & \mathcal{M}^m \end{array}$$

So, we have some R, a copy of $\mathcal{M}$ interpreted defn. in $\mathcal{M}^m$.

Do we have a defn. isom between R and $\mathcal{M}$?

What about vice versa?

If so, we call $\mathcal{M}, \mathcal{N}$ biinterpretable.

Open Problem: Is every infinite f.g. field bi-interpretable with $\mathbb{N}$?

e.g.: $\mathcal{L} = \mathcal{L}_{Rings}$, $\mathcal{M} = (K, +, -, \cdot, 0, 1)$. Then any linear algebraic group is def interpretable in $\mathcal{M}$. ← This is easy!

Sometimes the opposite assertion is true. ← Not so easy!

Let
$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a, b \in K,\ a \neq 0 \right\}$$

Let
$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}, \text{ with } \tau \neq 0, 1.$$

Let
$$A = \{ g \in G \mid g\alpha = \alpha g \} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in K \right\}$$
$$B = \{ g \in G \mid g\beta = \beta g \} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \neq 0 \right\}$$

A, B are definable. $B \curvearrowright A$ by conjugation:

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y/x \\ 0 & 1 \end{pmatrix}.$$

Define $i : A \setminus \{1\} \longrightarrow B$ via $i(a) = b \Leftrightarrow b^{-1} a b = \alpha$.

$$i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Define $*$ on A by

$$a * b = \begin{cases} i(b)\, a\, (i(b))^{-1} & \text{if } b \neq I \\ I & \text{if } b = I \end{cases}$$

Now $(K, +, \cdot, 0, 1) \cong (A, \cdot, *, 1, \alpha)$.

Notes: Suppose G is an algebraic group over an alg. cl. field.

One can show the field is interpretable in any simple algebraic group (see Poizat, Stable Groups, 4.17).

It is an interesting question to characterize which groups interpret the field operations. See Baldwin, Some notes on stable groups, 1989.

e.g.: $\mathcal{M} = (\mathbb{Q}_p, +, -, \cdot, 0, 1)$. Suppose $p \neq 2$.

Let $\varphi(x) := $ "$\exists y,\ y^2 = px^2 + 1$". We claim φ defines $\mathbb{Z}_p$.

Say $y^2 = pa^2 + 1$. $v(pa^2) = 2v(a) + 1$. If $v(a) < 0$, $v(pa^2 + 1) =$ odd negative integer, so $y^2 = pa^2 + 1$ is impossible. So if $\mathcal{M} \models \varphi(a)$, then $v(a) \geq 0$, so $a \in \mathbb{Z}_p$.

If $a \in \mathbb{Z}_p$, $F(X) = X^2 - (pa^2 + 1)$. Let $\bar{F}$ be reduction mod p. $\bar{F}(x) = x^2 - 1$ and $\bar{F}'(x) = 2x$, so $\bar{F}(1) = 0$ and $\bar{F}'(1) \neq 0$, so by Hensel $\exists b \in \mathbb{Z}_p$ s.t. $F(b) = 0$. So, $\mathcal{M} \models \varphi(a)$.

e.g.: $\mathbb{P}^n(K) = \{(a_0, \dots, a_n) \mid \text{not all } a_i = 0\} / \sim$

with $(a_0 \dots a_n) \sim (b_0 \dots b_n)$ if $\bar{a} = \lambda \bar{b}$ for some $\lambda \in K$.

Now we build towards the first theorem of model theory.

19

Defn: An $\mathcal{L}$-theory T is satisfiable if there is $\mathcal{M} \models T$.

If T is infinite, this seems hard to check.

20 **Thm:** (Compactness) T is satisfiable iff every finite subset of T is satisfiable

Our proof is not the original, which went through the notion of formal proof and Gödel's Completeness Theorem. These ideas are no longer central in model theory, and so we will avoid them.

21 **Defn:** An $\mathcal{L}$-theory T has the witness property if when $\phi(v)$ is an $\mathcal{L}$-formula with free variable v, there is some $c \in \mathcal{C}$ with the property that $T \models (\exists v \phi(v) \longrightarrow \phi(c))$.

22 **Defn:** An $\mathcal{L}$-theory is maximal if for all ϕ, either $\phi \in T$ or $\neg\phi \in T$.

23 **Lem:** T maximal, fin. satisfiable. Then if $\Delta \subseteq T$ finite and $\Delta \models \psi$, then $\psi \in T$.

Pf: If $\psi \notin T$, then $\neg\psi \in T$ and so $\Delta \cup \{\neg\psi\}$ is finite, in T, and not sat.

24 **Lem:** Suppose T is maximal and fin. sat with the witness property. Then if $|\mathcal{C}| \leq \kappa$, there is $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \kappa$.

Pf: Let $c, d \in \mathcal{C}$. Set $c \sim d$ if $T \models$ "$c = d$"

Claim: $\sim$ is an eq. relation.

Set $\mathcal{M} = \mathcal{C}/\sim$

$c^{\mathcal{M}} = c/\sim$.

Let quickly explain how to interpret $\mathcal{L}$ on $\mathcal{M}$.

Claim: Let R be an n-ary relation symbol. Let $c_1, \dots, c_n, d_1, \dots, d_n \in \mathcal{C}$, $c_i \sim d_i$ for $i = 1, \dots, n$. Then $R(\bar{c}) \in T$ iff $R(\bar{d}) \in T$.

This will be verified as an exercise.

So, $R^{\mathcal{M}}$ is well-defined.

Let f be an n-ary function, for $c_1, \dots, c_n \in \mathcal{C}$. But $\emptyset \models \exists v\, f(c_1, \dots, c_n) = v$

So, there is $c_{n+1} \in \mathcal{C}$ s.t. "$f(c_1, \dots, c_n) = c_{n+1}$" $\in T$. But then for $i = 1, \dots, n+1$, we have $c_i \sim d_i$, then "$f(d_1, \dots, d_n) = d_{n+1}$" $\in T$.

Since f is a function symbol, if $e_i \sim c_i$ for $i = 1, \dots, n$ and $f(e_1, \dots, e_n) = e_{n+1} \in T$, then $e_{n+1} \sim c_{n+1}$. Thus, we get a well-defined function

$$f^{\mathcal{M}}: \mathcal{M}^n \longrightarrow \mathcal{M} \quad \text{by} \quad f^{\mathcal{M}}(c_1^* \dots c_n^*) = d^* \quad \text{iff "} f(c_1 \dots c_n) = d \text{"} \in T$$

So, $\mathcal{M}$ is an $\mathcal{L}$-structure. But we nts that $\mathcal{M} \models T$.

Claim: For any term t with free variables $v_1, \ldots, v_n$, if $c_1 \ldots c_n, d \in C$ then "$t(c_1, \ldots, c_n) = d$" $\in T$ iff $t^{\mathcal{M}}(c_1^* \ldots c_n^*) = d^*$.

$\Rightarrow$ If t is c, then "$c = d$" $\in T$ iff $\overset{c^{\mathcal{M}}}{\overset{\parallel}{c^*}} = d^*$.

Suppose true for $t_1, \ldots t_m$ and $t = f(t_1 \ldots, t_m)$, then using witness prop & Lem a, there are $d, d_1, \ldots, d_n \in C$ s.t. "$t_i(c_1, \ldots, c_n) = d_i$" $\in T$ for $i \leq m$ and "$f(d_1, \ldots, d_m) = d$" $\in T$. By induction $t_i^{\mathcal{M}}(c_1^* \ldots c_m^*) = d_i^*$ and $f^{\mathcal{M}}(d_1^* \ldots d_m^*) = d^*$. Thus $t^{\mathcal{M}}(c_1^* \ldots c_n^*) = d^*$.

$\Leftarrow$ Suppose, $t^{\mathcal{M}}(c_1^* \ldots c_n^*) = d^*$. By witness prop, Lem a, there is $e \in C$ s.t.

"$t(c_1 \ldots c_n) = e$" $\in T$. By $\Rightarrow$, we have $t^{\mathcal{M}}(c_1^* \ldots, c_n^*) = e^*$. Thus $e^* = d^*$ and "$e = d$" $\in T$. By Lem a, "$t(c_1 \ldots c_n) = d$" $\in T$.

Claim 4: For all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$, $c_1 \ldots c_n \in C$, $\mathcal{M} \models \phi(\bar{c}^*)$ iff $\phi(\bar{c}) \in T$.

If ϕ is $t_1 = t_2$, then by Lem 2.16 & witness prop, we get d_1, d_2 s.t. $t_1(\bar{c}) = d_1$ and $t_2(\bar{c}) = d_2$ are in T. By Claim 3, $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$ for $i = 1, 2$. Then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{c}^*) &\iff d_1^* = d_2^* \\ &\iff \text{"}d_1 = d_2\text{"} \in T \\ &\iff \text{"}t_1(\bar{c}) = t_2(\bar{c})\text{"} \in T \quad \text{by lem. a.} \end{aligned}$$

Now, if $\phi = R(t_1 \ldots t_m)$. T has witness prop, by Lem. a., $\exists d_1, \ldots, d_m \in C$ s.t. "$t_i(\bar{c}) = d_i$" $\in T$ and, Claim 4, $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$ for $i = 1, \ldots, m$. Thus,

$$\begin{aligned} \mathcal{M} \models \phi(c^*) &\iff \bar{d}^* \in R^{\mathcal{M}} \\ &\iff R(\bar{d}) \in T \\ &\iff \phi(\bar{c}) \in T \text{ by Lem. a.} \end{aligned}$$

Now suppose the claim is true for ϕ. Show as an exercise it holds for $\neg\phi$.

Now suppose the claim is true for ϕ and ψ. Show as an exercise it holds for $\phi \wedge \psi$

Let $\phi = \exists v\, \psi(v)$. Suppose the claim is true for ψ. If $\mathcal{M} \models \psi(d^*, \bar{c}^*)$ then, by induction, $\psi(d, \bar{c}) \in T$ and $\exists v\, \psi(v, \bar{c}) \in T$ by Lem. a.

If $\exists v\, \psi(v, \bar{c}) \in T$, then by witness prop & Lem a, $\psi(d, \bar{c}) \in T$ for some c. By induction $\mathcal{M} \models \psi(d^*, \bar{c}^*)$ and

so $\mathcal{M} \models \exists v\, \psi(v, \bar{c}^*)$. ∎

Exercises:

Lem[25]: Let T be fin. sat. $\mathcal{L}$-theory. There is $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ fin. sat. s.t. any $\mathcal{L}^*$- theory extending T^* has the witness property. We can pick $\mathcal{L}^*$ s.t.

$|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0.$

26
Lem: Suppose T is fin. sat $\mathcal{L}$-theory and ϕ an $\mathcal{L}$ sentence. Then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is f.s.

27
Cor: If T is fin sat. $\mathcal{L}$-theory, then there is a maximal fin sat $\mathcal{L}$-theory $T' \supseteq T$.

28
Thm: If T is a finsat $\mathcal{L}$-theory and κ is infinite card with $\kappa \geq |\mathcal{L}|$ then there is a model of T with card at most κ.

ex: A <u>filter</u> on I is a collection D of elements of $\mathcal{P}(I)$ s.t.

1) $I \in D$
2) $\emptyset \notin D$
3) $A, B \in D \implies A \cap B \in D$
4) $A \in D$ and $A \subseteq B \implies B \in D$

An <u>ultra filter</u> is a filter so that for all $X \subseteq I$ either $X \in D$ or $I \setminus X \in D$.

Let $\mathcal{M}_i$ be an $\mathcal{L}$-str. for all $i \in I$. For an ultrafilter D, we define

$\mathcal{M} := \prod_{i\in I} \mathcal{M}_i / D$ via the equiv. relation $\sim$ on $\prod_{i \in I} \mathcal{M}_i$ defined:

$$\prod_{i\in I} \mathcal{M}_i = \{ f : I \to \bigcup \mathcal{M}_i \mid f(i) \in \mathcal{M}_i \}$$

$f \sim g$ if $\{ i \in I \mid f(i) = g(i) \} \in D$.

For $c \in \mathcal{C}$, let $c^{\mathcal{M}} = \sim$-class of $f_c(i) = c^{\mathcal{M}_i}$ for all $i \in I$.
For $g_1 \cdots g_{n_f} \in \prod \mathcal{M}_i$, $f \in \mathcal{F}$, define $f^{\mathcal{M}}(g_1/\sim, \dots, g_{n_f}/\sim) = g/\sim$ where $g(i) = f^{\mathcal{M}_i}(g_1(i), \dots, g_{n_f}(i))$

Let $R \in \mathcal{R}$. $R^M := \{ (g_1/\sim, \dots, g_{n_R}/\sim) \mid \{ i \in I \mid (g_1(i), \dots, g_{n_R}(i)) \in R^{\mathcal{M}_i} \} \in D \}$

Show the above interpretations give a well-defn. $\mathcal{L}$-str.

Let $\phi(\bar{v})$ be a formula. Then show

$$\mathcal{M} \models \phi(g_1/\sim, \dots, g_n/\sim) \iff \{ i \in I \mid \mathcal{M}_i \models \phi(g_1(i), \dots, g_n(i)) \} \in D$$

ex: Let $\mathbb{F}_p$ denote the field with p elements. Let $\mathcal{L} = \mathcal{L}_{Rings}$.
Let D be a nonprincipal (every cofinite set is in the filter) on the primes.

$$\text{Let } K = \prod_{p \in \operatorname{Spec}\mathbb{Z}} \mathbb{F}_p / D$$

a) Is K a field?
b) What is char K?
c) Is $K \equiv \mathbb{R}$?
d) Does K contain any irrational algebraic number?
e) Is $K \equiv \mathbb{C}$?

f) Show K has a unique alg. extension of each degree.
g) Is there a solution to $x^2+1=0$ in K?
h) Let $a, b \in K \setminus \{0\}$. Show there are infinitely many $(x,y) \in K$ s.t. $y^2 = x^3 + x$.

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Defn: Let $\kappa \in \text{Card}$. Let T be a theory with models of size κ.
T is κ-categorical if T has only one model of size κ up to $\cong$.

eg 1) Torsion free DAG is κ-categorical for all uncountable κ.

2) ACF_p, $p=0$ or $p \in \operatorname{Spec}\mathbb{Z}$ is κ-categorical for all uncountable κ.

3) eq-relation with two infinite classes is $\aleph_0$-categorical.

30

Thm: Let T be κ-cat for some infinite κ. Let T have no finite models
Then T is complete.

Pf: Use compactness.

31

Defn: $\mathcal{M}, \mathcal{N}$ are $\mathcal{L}$-structures, then an $\mathcal{L}$-embedding $j: \mathcal{M} \to \mathcal{N}$ is elementary if

$$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(j(\bar{a}))$$

for all $\mathcal{L}$-formulas ϕ and all $\bar{a} \in \mathcal{M}$.

32

Defn: Let $\mathcal{L}_{\mathcal{M}}$ be the language $\mathcal{L}$ together with constants for each element in $\mathcal{M}$.

$$\operatorname{Diag}(\mathcal{M}) = \{\phi \in \mathcal{L}_{\mathcal{M}} \mid \phi \text{ is atomic or neg. atomic } \& \ \mathcal{M} \models \phi\}$$

$$\operatorname{Diag}_{el}(\mathcal{M}) = \{\phi \in \mathcal{L}_{\mathcal{M}} \mid \mathcal{M} \models \phi\}$$

33

Prop: Suppose $\mathcal{N} \models \operatorname{Diag}(\mathcal{M})$. Then $\exists f: \mathcal{M} \to \mathcal{N}$ an $\mathcal{L}$-embedding.
If $\mathcal{N} \models \operatorname{Diag}_{el}(\mathcal{M})$, then $\mathcal{M} \preccurlyeq \mathcal{N}$.

Thm:[34] $\mathcal{M}$ an infinite $\mathcal{L}$-str. and $\kappa \geq |\mathcal{M}| + |\mathcal{L}|$. Then there is an infinite $\mathcal{L}$-str. of card κ and $j: \mathcal{M} \to \mathcal{N}$ elementary

Pf: Note $\text{diag}_{el}(\mathcal{M})$ is not too large.

Prop:[35] (Tarski-Vaught) Suppose $\mathcal{M} \leq \mathcal{N}$. Then $\mathcal{M} \preceq \mathcal{N}$ iff for any $\phi(v, \bar{w})$ and $\bar{a} \in \mathcal{M}$, if there is $b \in \mathcal{N}$ s.t. $\mathcal{N} \models \phi(b, \bar{a})$ then there is $c \in \mathcal{M}$ s.t. $\mathcal{N} \models \phi(c, \bar{a})$.

Pf: $\Rightarrow$ is trivial.

$\Leftarrow$ Induction on formulas. If $\phi(\bar{v})$ is q.f. then $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{a})$. Negations and conjunctions are easily handled.

Let us assume the claim holds for $\psi(v, \bar{w})$. Let $\bar{a} \in \mathcal{M}$. If $\mathcal{M} \models \exists v \psi(v, \bar{a})$, there is $b \in \mathcal{M}$ s.t. $\mathcal{M} \models \psi(b, \bar{a})$. Then $\mathcal{N} \models \psi(b, \bar{a})$, so $\mathcal{N} \models \exists v \psi(v, \bar{a})$.

Now, if $\mathcal{N} \models \exists v \psi(v, \bar{a})$. Then there is $c \in \mathcal{M}$ s.t. $\mathcal{N} \models \psi(c, \bar{a})$. By induction there is $c \in \mathcal{M}$, s.t. $\mathcal{N} \models \psi(c, \bar{a})$. By induction $\mathcal{M} \models \psi(c, \bar{a})$ so, $\mathcal{M} \models \exists v \psi(v, \bar{a})$. ∎

Defn:[36] T has built in Skolem functions if for all $\mathcal{L}$-forms $\phi(v, w_1, \ldots, w_n)$, there's a function symbol f s.t. $T \models \forall \bar{w} (\exists v \phi(v, \bar{w}) \to \phi(f(\bar{w}), \bar{w}))$.

Lem:[37] Let T be an $\mathcal{L}$-theory. There are $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ an $\mathcal{L}^*$-th s.t. T^* has built-in Skolem functions and if $\mathcal{M} \models T$, then we can expand $\mathcal{M}$ to $\mathcal{M}^* \models T^*$. $\mathcal{L}^*$ can be chosen s.t. $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.

Pf: Let $\mathcal{L}_0 = \mathcal{L}$. Given $\mathcal{L}_i$, let $\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{f_\phi : \phi(v, w_1, \ldots, w_n) \text{ an } \mathcal{L}_i\text{-form}\}$ f_ϕ n-ary function. For $\phi(v, \bar{w})$ an $\mathcal{L}_i$-form, we let

$$\Psi_\phi := \forall \bar{w} (\exists v \phi(v, \bar{w}) \to \phi(f_\phi(\bar{w}), \bar{w}))$$

Let $T_{i+1} = T_i \cup \{\Psi_\phi \mid \phi \in \mathcal{L}_{i+1}\}$

Expand $\mathcal{M}_i$ to $\mathcal{M}_{i+1}$ in the natural manner. Then $\mathcal{M}_i \models T_{i+1}$ and

let $\mathcal{L}^* = \bigcup \mathcal{L}_i$
$T^* = \bigcup T_i$

and expand $\mathcal{M}$ to $\mathcal{M}^*$ unioning the expansions.

Finish by calculating $|\mathcal{L}^*|$ and showing that $\mathcal{M}^*$ has built in skolems. ■

Thm: [32] ($\downarrow$ Lowenheim Skolem) Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $X \subseteq \mathcal{M}$. Then there is $\mathcal{N} \preccurlyeq \mathcal{M}$ s.t. $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

Pf: By previous lemma, we can assume $Th(\mathcal{M})$ has built-in skolem functions.

Let $X_0 = X$. Given X_i, let $X_{i+1} = X_i \cup \left\{ f^{\mathcal{M}}(\bar{a}) : \begin{array}{l} f \text{ an } n\text{-ary function} \\ \bar{a} \in X_i^n \text{ and } n = 1, 2 \ldots \end{array} \right\}$

Let $\mathcal{N} = \bigcup X_i$. Then $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

If f is n-ary function symbol, $\bar{a} \in \mathcal{N}$, then $\bar{a} \in X_i$ for some i, and $f^{\mathcal{M}}(\bar{a}) \in X_{i+1} \subseteq \mathcal{N}$. Thus $f^{\mathcal{M}} \restriction_{\mathcal{N}} : \mathcal{N}^n \to \mathcal{N}$.

Thus, we interpret f as $f^{\mathcal{N}} = f^{\mathcal{M}} \restriction_{\mathcal{N}}$.

If R a relation $R^{\mathcal{N}} = R^{\mathcal{M}} \cap \mathcal{N}$.

If c a constant, there is a function s.t. $f(x) = c^{\mathcal{M}}$ for any $x \in \mathcal{M}$. so, $c^{\mathcal{M}} \in \mathcal{N}$.

So $\mathcal{N}$ is an $\mathcal{L}$-str.

If $\phi(v, \bar{w}) \in \mathcal{L}$, $b \in \mathcal{M}$, $\mathcal{M} \models \phi(b, \bar{a})$, then $\mathcal{M} \models \phi(f(\bar{a}), \bar{a})$ for some f. Thus $f^{\mathcal{M}}(\bar{a}) \in \mathcal{N}$, so by T-V,

$\mathcal{M} \preccurlyeq \mathcal{N}$. ■

Defn: [39] An $\mathcal{L}$-theory T has a <u>universal axiomatization</u> if there is a set of universal $\mathcal{L}$-sentences Γ s.t. $\mathcal{M} \models \Gamma$ iff $\mathcal{M} \models T$ for all $\mathcal{L}$-str. $\mathcal{M}$.

Thm: [40]

An $\mathcal{L}$-th. T has a univ. axiom. iff whenever $\mathcal{M} \models T$ and $\mathcal{N} \subseteq \mathcal{M}$, then $\mathcal{N} \models T$.

Pf: Say $\mathcal{N} \subseteq \mathcal{M}$. If $\phi(\bar{v})$ is q.f., $\bar{a} \in \mathcal{N}$, then if $\bar{a} \in \mathcal{N}$,

$\mathcal{N} \models \phi(\bar{a}) \Leftrightarrow \mathcal{M} \models \phi(\bar{a})$.

($\Rightarrow$)

So $\mathcal{M} \models \forall \bar{v}\, \phi(\bar{v}) \Longrightarrow \mathcal{N} \models \forall \bar{v}\, \phi(\bar{v})$.

($\Leftarrow$) Say T is preserved by $\subseteq$.

$\Gamma := \{\phi : \phi \text{ universal}, T \models \phi\}$

Suppose $\mathcal{N} \models \Gamma$.

Claim: $T \cup \text{diag}(\mathcal{N})$ is satisfiable.

If not, $\Delta \subseteq \text{Diag}(\mathcal{N})$ finite with $T \cup \Delta$ not sat.

Let $\Delta = \{\psi_1, \dots, \psi_n\}$. Let $\bar{c}$ be new constant symbols from $\mathcal{N}$ used in ψ_i.

$\psi_i = \phi_i(\bar{c})$ for ϕ_i a q.f. $\mathcal{L}$-form. $\bar{c} \notin T$, so we only need to check satisfiability of:

$$T \cup \{\exists \bar{v} \bigwedge \phi_i(\bar{v})\}$$

If not, $T \models \forall \bar{v} \bigvee \neg \phi_i(\bar{v})$ $\leftarrow$ Universal

So, $\mathcal{N} \models \forall \bar{v} \bigvee \neg \phi_i(\bar{v})$ $\times\!\!\!\times$.

So, now $\exists\, \mathcal{M}, \supseteq \mathcal{N}$ with $\mathcal{M}, \models T$. T is preserved under $\subseteq$, so $\mathcal{N} \models T$ and thus $\Gamma \models T$.

41 Defn: Elementary chains.

42 Prop: Unions preserve elementary chains.

ex: If $\mathcal{M}_i$ are $\mathcal{L}$-str. and

$$\begin{tikzcd} \mathcal{M}_1 & & \mathcal{M}_2 \\ & \mathcal{M}_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

are elementary. Show $\exists\, \mathcal{N}$ and elementary $f_i : \mathcal{M}_i \to \mathcal{N}$ s.t.

$$\begin{tikzcd} & \mathcal{N} & \\ \mathcal{M}_1 \arrow[ur, "f_1"] & \circlearrowright & \mathcal{M}_2 \arrow[ul, "f_2"'] \\ & \mathcal{M}_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

43 Defn: A theory is Model-complete if $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}, \mathcal{N} \models T$.

44

Prop: If T has QE, then T is model-comp.

Pf: (easy) exercise.

Games, $\equiv$, and $\cong$.

There are two players, $\forall$nna and $\exists$lsa.

Let $\gamma \in \mathrm{Ord}$.

$EF_\gamma(\mathcal{M}, \mathcal{N})$ has γ-sequences of moves.

At step α, $\forall$ picks $a \in \mathcal{M}$ or $b \in \mathcal{N}$.

$\exists$ picks an element of the other model.

A play of $EF_\gamma(\mathcal{M}, \mathcal{N})$ is a γ-sequence of $(a_\alpha, b_\alpha)_{\alpha<\gamma}$ with $a_\alpha \in \mathcal{M}$, $b_\alpha \in \mathcal{M}$.

$\exists$ wins if $a_\alpha \longmapsto b_\alpha$ has the property:

For any atomic $\varphi(\bar{v})$, $\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(\bar{b})$.

See HW #3, pg 2 for a primer on infinitary logic.

$\mathcal{M} \sim_\gamma \mathcal{N}$ if $\exists$ has a winning strategy for $EF_\gamma(\mathcal{M}, \mathcal{N})$.

eg.: $(\mathbb{Q}, <) \sim_\omega (\mathbb{R}, <)$

$(\mathbb{Q}, <) \nsim_{\omega_1} (\mathbb{R}, <)$

Agree on $\mathcal{L}_{\infty\omega}$-sentences

Thm[45]: If $\mathcal{L}$ is countable, then $\mathcal{M} \sim_\omega \mathcal{N}$ iff $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$

Pf: $\Rightarrow$ By induction on Φ.

- If Φ is atomic then $\mathcal{M} \sim_\omega \mathcal{N} \Rightarrow \mathcal{M} \sim_0 \mathcal{N} \Rightarrow \mathcal{M} \models \Phi \Leftrightarrow \mathcal{N} \models \Phi$ by defn.
- $\neg$ are easy by induction.
- $\Phi = \bigvee \Psi$, then for all $\psi \in \Psi$, $\mathcal{M} \models \psi \Leftrightarrow \mathcal{N} \models \psi$.
- $\Phi = \exists x \Psi$

Say $\mathcal{M} \models \Phi$. Then for some $a \in \mathcal{M}$, $\mathcal{M} \models \Psi(a)$.

$\exists$lsa has a winning strategy, so let a be $\forall$nna's 0^{th} move. Let $b \in \mathcal{N}$ be $\exists$lsa's play responding to a.

So (a, b) is a winning position for $\exists$lsa

$\Leftrightarrow$ $\exists$lsa wins $EF_\omega(\mathcal{M}_a, \mathcal{N}_b)$.

By induction $\mathcal{N}_b \models \Psi(b)$. So $\mathcal{N} \models \exists x \Psi(x)$.

The argument works identically with $\mathcal{M}, \mathcal{N}$ swapped. So $\mathcal{N} \models \Phi \Leftrightarrow \mathcal{M} \models \Phi$.

($\Leftarrow$) Suppose $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$. If ∀nna plays $a \in A$, then ∃lsa picks $b \in B$ such that $tp(a) = tp(b)$ in $\mathcal{L}_{\infty\omega}$. Why can we do this? Set

$$\Phi := \{ \varphi(x) \in \mathcal{L}_{\infty\omega} \mid \mathcal{M} \models \varphi(a) \}$$

Then $\mathcal{M} \models \exists x \bigwedge \Phi(x)$. So $\mathcal{N} \models \exists x \bigwedge \Phi(x)$.

Let $\mathcal{N} \models \bigwedge \Phi(b)$.

So, $\mathcal{M}_a \equiv_{\infty\omega} \mathcal{N}_b$. Now iterate this construction. ∃lsa wins. ∎

46

Defn: An <u>unnested</u> <u>atomic</u> <u>formula</u> is one of the form:

1) $x = c$, $c \in \mathcal{C}$
2) $f(\bar{x}) = y$ for $f \in \mathcal{F}$
3) $R(\bar{x})$ for $R \in \mathcal{R}$
4) $x = y$

An unnested formula is one built up in the usual way, but from unnested atomic formulae (rather than atomic formulae).

The <u>unnested</u> <u>E.-F.</u> <u>game</u> <u>of</u> <u>length</u> <u>γ</u> is denoted by

$$EF_\gamma[\mathcal{M}, \mathcal{N}]$$

Notice the square brackets.

It is played exactly as $EF_\gamma(\mathcal{M}, \mathcal{N})$. All that differs is the rule for when ∃lsa wins:

∃lsa wins $EF_\gamma(\mathcal{M}, \mathcal{N})$ precisely when for each <u>unnested</u> atomic $\varphi(x_\beta)_{\beta<\gamma}$, $\mathcal{M} \models \varphi(\bar{a}) \Leftrightarrow \mathcal{N} \models \varphi(\bar{b})$

We write $\mathcal{M} \approx_\gamma \mathcal{N}$ if ∃lsa has a winning strategy to $EF_\gamma[\mathcal{M}, \mathcal{N}]$.

Usually, the games we think about have $\gamma \leq \omega$.
Some set theorists think about longer games.

Rem: Comparing $\mathcal{M} \sim_\alpha \mathcal{N}$ and $\mathcal{M} \approx_\alpha \mathcal{N}$

1) Clearly, $\mathcal{M} \sim_\alpha \mathcal{N} \Rightarrow \mathcal{M} \approx_\alpha \mathcal{N}$.

2) The converse fails: let $\mathcal{L} = \{0, 1\}$.

Let $\mathcal{M}$ be a $\mathcal{L}$-str. with $0^{\mathcal{M}} = 1^{\mathcal{M}}$.
Let $\mathcal{N}$ be a $\mathcal{L}$-str. with $0^{\mathcal{N}} \neq 1^{\mathcal{N}}$.

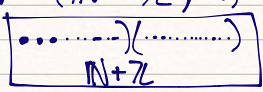
Then $\mathcal{M} \approx_0 \mathcal{N}$ but $\mathcal{M} \not\sim_0 \mathcal{N}$.

3) Note $\mathcal{M} \not\approx_1 \mathcal{N}$.

Ex: Prove $\mathcal{M} \approx_\omega \mathcal{N} \Leftrightarrow \mathcal{M} \sim_\omega \mathcal{N}$ or find a counterexample.

Ex: Prove if $\mathcal{M} \approx_k \mathcal{N}$ for all $k \in \mathbb{N}$ then $\mathcal{M} \approx_\omega \mathcal{N}$ or find a counterexample

$\mathcal{M} = (\mathbb{N}, <)$, $\mathcal{N} = (\mathbb{N} + \mathbb{Z}, <)$



Have $\forall$nna pick all the elements of $\mathbb{Z}$. $\exists$lsa runs out before ω, but she could hold out past any finite k.

47 **Defn:** For a formula, $qr(\varphi)$ = # nested quantifiers:

$qr(\text{atomic}) = 0$
$qr(\varphi \wedge \psi) = qr(\varphi \vee \psi) = \max(qr(\varphi), qr(\psi))$
$qr(\neg\varphi) = qr(\varphi)$
$qr(\exists\varphi) = qr(\forall\varphi) = qr(\varphi) + 1$

48 **Lem:** For finite $\mathcal{L}$, $k, n \in \mathbb{N}$, there is a finite set $\Theta_{n,k}$ of unnested formulae of $qr \leq k$ in n free var. $x_0 \dots x_{n-1}$ s.t. :

a) Distinct elements $\varphi_0, \varphi_1 \in \Theta_{n,k}$, $\models \forall \bar{x} (\varphi_0 \to \neg \varphi_1)$

b) φ an $\mathcal{L}$-form of q.r. $\leq k$ and free variables $x_0 \dots x_{n-1}$ then

$\exists\, \Phi \subseteq \Theta_{n,k}$ s.t. $\models \forall \bar{x} (\varphi \leftrightarrow \bigvee \Phi)$.

c) Given $\mathcal{M}, \mathcal{N}$, $\mathcal{L}$-str. then for any n-tuples $\bar{a} \in \mathcal{M}$ and $\bar{b} \in \mathcal{N}$,

$\mathcal{M}_{\bar{a}} \approx_k \mathcal{N}_{\bar{b}}$ iff for each $\theta \in \Theta_{n,k}$,

$\mathcal{M} \models \theta(\bar{a}) \Leftrightarrow \mathcal{N} \models \theta(\bar{b})$.

Notation: $\varphi^1 := \varphi$
$\varphi^{-1} := \neg\varphi$

Pf: We build $\Theta_{n,k}$ by induction on k.

$\Phi = \{ \rho \mid \rho \text{ unnested atomic } \mathcal{L}\text{-form in } x_0, \dots, x_{n-1} \}$ ← Finite set

$\Theta_{n,0} = \{ \bigwedge_{\rho \in \Phi} \rho^{s(\rho)} \mid s : \Phi \longrightarrow \{-1, 1\} \}$

$\Theta_{n,k+1} := \{ \bigwedge_{\rho \in Y} \exists x_n \, \rho(\bar{x}, x_n) \wedge \bigwedge_{\rho \in Z} \forall x_n \neg \rho(\bar{x}, x_n) \,/\, Y, Z \text{ partition } \Theta_{n+1,k} \}$

Now we need to check a), b), c).

a) is pretty clear.

b) For $\Theta_{n,k}$ we took all possible conjunctions of a set of formulas and their negations, so this clearly partitions the universe of any model.
Any $qr \leq k$ formula in n-var. is equivalent to a boolean combination of $\Theta_{n,k}$.

c) Induction on k.

- $(k=0)$ $(\mathcal{M}, \bar{a}) \approx_0 (\mathcal{N}, \bar{b})$ means

$$\mathcal{M} \models \psi(\bar{a}) \Leftrightarrow \mathcal{N} \models \psi(\bar{b})$$

but $\Theta_{n,0}$ = atoms in boolean algebra generated by unnested atomic form.

- $(k+1)$ Suppose $(\mathcal{M}, \bar{a}) \approx_{k+1} (\mathcal{N}, \bar{b})$.

Let $\rho \in \Theta_{n,k+1}$.

$$\rho = \bigwedge_{\eta \in Y} \exists x_n \eta \wedge \bigwedge_{\tau \in Z} \forall x_n \neg \tau \quad \text{for some } Y, Z \text{ partitioning } \Theta_{n+1,k}.$$

Say $\mathcal{M} \models \rho(\bar{a})$.

Then for $\eta \in Y$, $\mathcal{M} \models \exists x_n \eta(\bar{a}, x_n)$, so let $c \in \mathcal{M}$ be s.t.
$\mathcal{M} \models \eta(\bar{a}, c)$.

There is by hypothesis, $d \in \mathcal{N}$ s.t. $\mathcal{N} \models \eta(\bar{b}, d)$ so
$\mathcal{N} \models \exists x_n \eta(\bar{b}, x_n)$.

Similarly for $\tau \in Z$.

So $\mathcal{N} \models \rho(\bar{b})$ and the reverse argument by symmetry.

Now, assume $(\mathcal{M}, \bar{a})$ and $(\mathcal{N}, \bar{b})$ agree on $\Theta_{n,k+1}$.

Suppose Anna plays $c \in \mathcal{M}$.

$\Theta_{n+1,k}$ partitions $\mathcal{M}^{n+1}$, there is exactly one $\rho \in \Theta_{n+1,k}$ s.t.

$\mathcal{M} \models \rho(\bar{a}, c)$.

But $\Theta_{n,k+1}$ partitions $\mathcal{M}^n$, so there is exactly one formula $\psi \in \Theta_{n,k+1}$ with

$\mathcal{M} \models \psi(\bar{a})$.

So, $\psi(\bar{x}) \longrightarrow \exists x_n \rho(\bar{x}, x_n)$.

But $(\mathcal{M}, \bar{a}), (\mathcal{N}, \bar{b})$ agree on $\Theta_{n,k+1}$. So, $\mathcal{N} \models \psi(\bar{b})$.

But then $\mathcal{N} \models \exists x_n \rho(\bar{x}, x_n)$.

Let d be a witness. Then by induction

$$(\mathcal{M}, \bar{a}, c) \approx_k (\mathcal{N}, \bar{b}, d).$$

Now finish with induction and the symetry of the argument. ∎

Thm:[49] Let $\mathcal{L}$ be finite, $\mathcal{M}, \mathcal{N}$ $\mathcal{L}$-str.

$$\mathcal{M} \equiv \mathcal{N} \iff \text{For all } k < \omega, \mathcal{M} \approx_k \mathcal{N}.$$

Pf: $\mathcal{M} \equiv \mathcal{N} \Rightarrow \mathcal{M}, \mathcal{N}$ agree on all unnested form, so all $\Theta_{n,k}$

exercise: Every formula is logically equiv to an unnested form. ∎

50 **Thm:** DLO has QE.

Pf: If ϕ is a sentence and DLO $\models \phi$ then DLO $\models$ "$\phi \leftrightarrow x_1 = x_1$"

If ϕ is a sentence and DLO $\models \neg\phi$ then DLO $\models$ "$\phi \leftrightarrow x_1 \neq x_1$"

By completeness of DLO, one of these options holds.

If ϕ has free variables $x_1, \dots, x_n$ with $n \geq 1$. We will find $\psi(\bar{x})$ q.f. s.t. $\mathbb{Q} \models \forall \bar{x} (\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

For $\sigma : \{(i,j) \mid 1 \leq i \leq j \leq n\} \longrightarrow 3$, let $\chi_\sigma(x_1, \dots, x_n)$ be the formula

$$\left(\bigwedge_{\sigma(i,j)=0} x_i = x_j\right) \wedge \left(\bigwedge_{\sigma(i,j)=1} x_i < x_j\right) \wedge \left(\bigwedge_{\sigma(i,j)=2} x_i > x_j\right).$$

Let Λ_ϕ be the set of above sign conditions, σ, s.t. there is $\bar{a} \in \mathbb{Q}$ s.t. $\mathbb{Q} \models \chi_\sigma(\bar{a}) \wedge \phi(\bar{a})$.

Case 1: $\Lambda_\phi = \emptyset$. Then $\mathbb{Q} \models \forall x \neg \phi(\bar{x})$, so $\mathbb{Q} \models \phi(\bar{x}) \leftrightarrow x_1 \neq x_2$.

Case 2: $\Lambda_\phi \neq \emptyset$. Let $\psi_\phi(x) = \bigvee_{\sigma \in \Lambda_\phi} \chi_\sigma(\bar{x})$.

Then $\mathbb{Q} \models \phi(\bar{x}) \longrightarrow \psi_\phi(\bar{x})$.

Suppose $\bar{b} \in \mathbb{Q}$ and $\mathbb{Q} \models \psi_\phi(\bar{b})$. Let $\sigma \in \Lambda_\phi$ s.t. $\mathbb{Q} \models \chi_\sigma(\bar{b})$. Then there is $\bar{a} \in \mathbb{Q}$ s.t. $\mathbb{Q} \models \phi(\bar{a}) \wedge \chi_\sigma(\bar{a})$. But now there is some aut. of $\mathbb{Q}$ s.t. $f(\bar{a}) = \bar{b}$. So, $\mathbb{Q} \models \phi(\bar{b})$ and thus $\mathbb{Q} \models \phi(\bar{x}) \longleftrightarrow \psi_\phi(\bar{b})$. ∎

51 **Thm:** Suppose $\mathcal{L}$ has at least one constant, c.

T an $\mathcal{L}$-theory, $\phi(\bar{v})$ an $\mathcal{L}$-formula. TFAE

a) There is a q.f. $\mathcal{L}$-form $\psi(\bar{v})$ s.t. $T \models \forall \bar{v} (\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

b) If $\mathcal{M}, \mathcal{N} \models T$, A an $\mathcal{L}$-str., $A \subseteq \mathcal{M}, \mathcal{N}$. Then $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{N} \models \phi(\bar{a})$.

Pf: a $\Rightarrow$ b) q.f. forms preserved under substructure...

b $\Rightarrow$ a) Say $T \models \forall \bar{v}\, \phi(\bar{v})$. Then $T \models \forall \bar{v} (\phi(\bar{v}) \leftrightarrow c = c)$

Say $T \models \forall \bar{v} \neg \phi(\bar{v})$. Then $T \models \forall \bar{v} (\phi(\bar{v}) \leftrightarrow c \neq c)$.

So, wlog, neither of the above situations occur.

Let $\Gamma(\bar{v}) = \{\psi(\bar{v}) \mid \psi \text{ is qf and } T \models \forall \bar{v} (\phi(\bar{v}) \longrightarrow \psi(\bar{v}))\}$.

Let $d_1, \dots, d_m$ be new constants. We show that

$$T \cup \Gamma(\bar{d}) \models \phi(\bar{d}).$$

By compactness, this implies there is $\psi_1 \in \Gamma$ s.t.

$T \cup \psi_1(\bar{d}) \models \phi(\bar{d})$.

But then:

$$T \models \forall \bar{v} (\psi_1(\bar{v}) \to \phi(\bar{v}))$$

The reverse implication also holds (since $\psi \in \Gamma$).

So, we'll be done if we show:

Claim: $T \cup \Gamma(\bar{d}) \models \phi(\bar{d})$.

Suppose not. Then let $\mathcal{M} \models T \cup \Gamma(\bar{d}) \cup \{\neg \phi(\bar{d})\}$

Let $\mathcal{A} \leq \mathcal{M}$, $\mathcal{A} = \langle \bar{d} \rangle$.

$\Sigma := T \cup \mathrm{Diag}(\mathcal{A}) \cup \phi(\bar{d})$.

If Σ is not sat., then there are $\psi_1 \dots \psi_n \in \mathrm{Diag}(\mathcal{A})$ s.t.

$$T \models \forall \bar{v} (\bigwedge \psi_i(\bar{v}) \to \neg \phi(\bar{v}))$$

But then $T \models \forall \bar{v} (\phi(\bar{v}) \to \bigvee \neg\psi_i(\bar{v}))$

So, $\bigvee \neg \psi_i(\bar{v}) \in \Gamma$.

$\mathcal{A} \models \bigvee \neg \psi_i(\bar{d})$. ※

So Σ is sat. Let $\mathcal{N} \models \Sigma$, so $\mathcal{A} \leq \mathcal{N}$. But $\mathcal{M} \models \neg\phi(\bar{d})$ and $\mathcal{N} \models \phi(\bar{d})$.

This contradicts b). ■

52

Lem: Let T be an $\mathcal{L}$-theory. Suppose any qf $\theta(\bar{v}, w)$ has qf $\psi(\bar{v})$ s.t. $T \models \exists w\, \theta(\bar{v}, w) \leftrightarrow \psi(\bar{v})$. Then T has QE.

Pf: Let $\phi(\bar{v})$ be any $\mathcal{L}$-form. Goal: Find qf $\psi(\bar{v})$ equiv. to $\phi(\bar{v})$ modulo T.

Induction on complexity of $\phi(\bar{v})$.
All cases easy except when:

$$\phi(\bar{v}) = \exists w\, \theta(\bar{v}, w).$$

Let $\psi_0(\bar{v}, w)$ be qf equiv. to $\theta(\bar{v}, w)$.

Then $\phi(\bar{v}) \leftrightarrow \exists v\, \theta(\bar{v}, w) \leftrightarrow \exists w\, \psi_0(\bar{v}, w) \leftrightarrow \psi(\bar{v})$

Use the hypothesis.

■

Cor 53: Let T be an $\mathcal{L}$-theory.

Suppose that for all q.f. $\mathcal{L}$-form $\phi(\bar{v}, w)$, if $\mathcal{M}, \mathcal{N} \models T$, $A \subseteq \mathcal{M}, \mathcal{N}$, and all $\bar{a} \in A$, $\exists\, b \in \mathcal{M}$ s.t. $\mathcal{M} \models \phi(\bar{a}, b)$, then there is $c \in \mathcal{N}$ s.t. $\mathcal{N} \models \phi(\bar{a}, c)$.

Then T has QE.

Defn 54: We say T has alg. prime models if for any $A \models T_\forall$, there is $\mathcal{M} \models T$ and an embedding $i : A \to \mathcal{M}$ s.t. for any $\mathcal{N} \models T$ s.t. $j : A \hookrightarrow \mathcal{N}$.
There is

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "h"] & & \mathcal{N} \\ & A \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

Defn 55: Say $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \subseteq \mathcal{N}$. We write $\mathcal{M} \prec_s \mathcal{N}$ if for any q.f. formula $\phi(\bar{v}, w)$ and any $\bar{a} \in \mathcal{M}$, if $\mathcal{N} \models \exists w\, \phi(\bar{a}, w)$ then so does $\mathcal{M}$.

Cor 56: Suppose T is an $\mathcal{L}$-theory s.t.

1) T has alg. prime models
2) $\mathcal{M} \prec_s \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ are models of T.

Then T has QE.

Pf: Let $A \subseteq \mathcal{M}, \mathcal{N}$ with $\mathcal{M}, \mathcal{N} \models T$. Then $A \models T_\forall$. So, let $\mathcal{M}_0$ be prime over A.

Then we have:

$$\begin{tikzcd} \mathcal{M} & & \mathcal{N} \\ & \mathcal{M}_0 \arrow[ul, dash] \arrow[ur, dash] & \\ & A \arrow[u, dash] & \end{tikzcd}$$

So, let $\rho(v, \bar{w})$ be q.f. and let $\bar{a} \in A$. Then by 2),

$$\mathcal{M} \models \exists v \rho(v, \bar{a}) \Leftrightarrow \mathcal{M}_0 \models \exists v \rho(v, \bar{a}) \Leftrightarrow \mathcal{N} \models \exists v \rho(v, \bar{a})$$

So, by Cor 53 we are done ■.

Now we move on to several applications.

Application #1: DAG and ODAG

Let DAG be the theory of torsion-free divisible abelian groups.

Lem[57]: Let G, H be torsion free divisible Ab. groups, $G \subseteq H$, $\psi(\bar{v}, w)$ q.f., $\bar{a} \in G$ $b \in H$, $H \models \phi(\bar{a}, b)$. Then $\exists c \in G$, $G \models \phi(\bar{a}, c)$.

Pf: (Note: this gives i) of Cor[56].

Put ψ is disjunctive normal form:

$$\psi(\bar{v}, w) \longleftrightarrow \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m_i} \theta_{ij}(\bar{v}, w).$$

$H \models \psi(\bar{a}, b)$, so $H \models \bigwedge \theta_{ij}(\bar{a}, b)$ for some fixed i.

So wlog $\psi = \bigwedge \theta_j(\bar{v}, w)$ ← Atomic or $\neg$ Atomic.

If θ_j is atomic, $\theta_j(\bar{v}, w) \longleftrightarrow \sum_{i=1}^{n} n_i v_i + nw = 0$.

This specifies that w is a specific element in div. hull of $\bar{v}$. So, b must be in G in the 1st place.

Now suppose all θ_j are negated atomics. Then each formula only "bans" fin. many elements. Pick something else. ∎

Lem[58]: If G is a torsion-free Ab. group, then there is a DAG, $\overset{G \subseteq}{\mathrm{Hull}(G)}$ s.t. if $G_1 \models$ DAG and with $G \subseteq G_1$ then:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Hull}(G) \\ & \searrow & \vdots\ \exists! \leftarrow \boxed{\text{Unique embedding.}} \\ & & G_1 \end{array}$$

← **Ex:** Explain why Lem[58] shows

$\mathrm{DAG}_\forall$ = "Torsion-free Abelian Grp"

Pf: Exercise

Thm[59]: DAG has QE.

Pf: Apply Cor[56] with the last two lemmas giving i) & 2). ∎

Defn: A theory T is strongly minimal if whenever $\mathcal{M} \models T$, the sets defn. in a single variable in $\mathcal{M}$ are all finite or cofinite

Ex: DAG is str. min.

Lem[60]: Let G be an ordered div. ab. group. Let H be the div. hull of G. Show H is an ordered gr with an ord. extending that on G.
Show if $H' \geq G$ and $H' \models$ ODAG, then

$$\begin{array}{ccc} G & \longrightarrow & H \\ & \searrow & \swarrow \\ & H' & \end{array}.$$

Pf: exercise. ∎

Ex:
Explain why Lem[60] $\Rightarrow$ ODAG has alg. prime models and $\text{ODAG}_\forall$ = "ordered group"

Ex: Formulate and prove the analog of Lem[57] for ODAG to show ODAG has QE.

Defn: An ordered structure $(\mathcal{M}, <, \dots)$ is o-minimal if every defn. subset in one variable is a boolean combination of fin. many intervals.

Ex: Show that ODAG is o-min.

Ex: Suppose that $(G, <, \dots)$ is o-min. Show $G \models$ ODAG.

#2
Presberger Arithmetic:

Let $\mathcal{L} = \{+, -, <, 0, 1\} \cup \{P_n : n \in \mathbb{N}\}$

We make $\mathbb{Z}$ an $\mathcal{L}$-str. by interpreting:

$$P_n(x) \iff \exists y (\underbrace{y + y \dots + y}_{n} = x)$$

We will show $\text{Th}(\mathbb{Z})$ has QE in $\mathcal{L}$. This is called Presburger arithmetic.

At this point in lecture one should explain that getting QE is itself not the goal in model theory. Goal = understand defn. sets.

One can always obtain QE in a big enough language. This is only useful if we understand the language.

Let PR be the $\mathcal{L}$-theory:

1) Ordered Abelian Gr.
2) $0 < 1$
3) $\forall x \, (x \leq 0 \text{ or } x \geq 1)$
4) $\forall x (P_n(x) \leftrightarrow \exists y (\sum_{i=1}^{n} y = x))$ for $n \in \mathbb{N}$
5) $\forall x \bigvee_{i=0}^{n-1} \left(P_n(x+i) \wedge \bigwedge_{i \neq j} \neg P_n(x+i) \right)$ for $n \in \mathbb{N}$.

What is $\text{PR}_\forall$?

Let T be given by:

1), 2), 3), 5) above and:

6) P_n closed under $+,-$.

7) $\forall x,y\ (\sum_{i=1}^{n} y = x) \to P_n(x)$.

8) For $m|n$, $\forall x (P_m(x) \longrightarrow P_n(x))$

9) $\forall x\ (P_{kn}(\sum_{i=1}^{k} x) \to P_n(x))$

$T \subseteq Pr_\forall$ is clear.

Lem[61]: Let $G \models T$. There is $H \geq G$ with $H \models Pr$ and if $H' \geq G$ with $H' \models Pr$, then there is embedding $h: H \longrightarrow H'$ with $h\restriction_G = id$.

Pf: H will be a subgr. of Hull(G).

$H := \{\frac{x}{n} \mid x \in G, n = 1 \text{ or } P_n(x)\}$

Set $P_n^H = nH$.

$H \models$ "$0<1$". Suppose $\frac{x}{m} \in H$ and $0 < \frac{x}{m} < 1$. Then $0 < x < m$.

But $P_m^H \cap (0,m) = \varnothing$, by 5) in G.

We need to verify 5) in H.

Let $\frac{x}{m} \in H$. Then $P_m(x)$.

By 5, $\exists!\ i$, $0 \leq i < mn$ s.t. $x+i \in P_{mn}$. By 8), $x+i \in P_m$.

P_m a subgrp, so $i \in P_m$. So $i = l \cdot m$ for $0 \leq l < n$.

In H, $\exists y$ s.t. $\sum_{i=1}^{mn} y = x + lm$.

So, $\sum_{i=1}^{n} y = \frac{x}{m} + l$. There is only one choice for i, so only one choice for l.

So, $H \models Pr$. So, $T = Pr_\forall$.

Now, if $H' \models Pr$, let $x \in G$ s.t. $G \models P_m(x)$.

There is $y \in H'$ s.t. $my = x$. Thus there is an embedding $H \hookrightarrow H'$ fixing G. ∎

Lem[62]: Let $G \subseteq H$, $G, H \models Pr$.

Then $G \preceq_s H$.

Pf: Say $\bar{a} \in G$, $\phi(\bar{v}, w)$ q.f.

Let $b \in H$ s.t. $H \models \phi(b, \bar{a})$.

Claim: WMA

$\phi(v,\bar{a}) = $ " $\bigwedge m_i v = g_i$

$\wedge \bigwedge P_{n_i}(s_i v + h_i)$

$\wedge \bigwedge c_i < v < d_i$ "

We're just putting the atomics in a very particular form.

Note: $\neg P_n(x) \Leftrightarrow \bigvee_{i=1}^{n-1} P_n(x+i)$

Put ϕ in norm. form and

Now replace $\neg P_n$ by the above disjunction.

Now, we need only consider a single disjunct, sat. by b.

Replace $mv > g_i$ by $v > h$ with $h \in G$ and $mh \leq g_i < m(h+1)$.

The claim follows.

If some conjunct is $m_i v = g$, then $b = \frac{g}{m_i}$.

So WMA this does not occur.

There are c and d in G s.t. for all i,

$c_i \leq c < b < d \leq d_i$. If there are such c, d with $d-c$ finite, we are done so, assume $d-c$ is infinite.

Then for all i, we have j_i s.t.

- $0 \leq j_i < n_i$
- $P_{n_i}(h_i - j_i)$

So, $P_{n_i}(s_i v + j_i) \Longleftrightarrow P_{n_i}(s_i v + h_i)$.

So, b solves:

$$s_1 v + j_1 \equiv 0 \bmod n_1$$
$$\vdots$$
$$s_m v + j_m \equiv 0 \bmod n_m$$

Let $N = \prod n_i$. Let $\ell \in \omega$ st. $P_N(b - \ell)$. Then ℓ is a solution to this system.

$d-c$ is infinite, so there is $g \in G$ with $c < g < d$, and $P_N(g - \ell)$.

So $G \models \phi(\bar{a}, g)$. ■

Ex: Show Pr is complete.

#3 Algebraically Closed Fields

Lem:[63] $ACF_\forall$ = "Int. domains"

Pf: exercise.

Thm:[64] ACF has QE.

Pf: ACF has prime models.

Let $F \subseteq K$ and $K, F \models ACF$. Then Let $\phi(x, \bar{y})$ be q.f. and $\bar{a} \in F$.

Let $K \models \phi(b, \bar{a})$.

WMA: ϕ is conj. of atomic & neg. atomic.

Exercise: Finish the proof. ∎

Cor:[65] ACF is model complete and ACF_p is complete.

Review of some alg. geom notation:

$V(S)$ for $S \subseteq K[x_1 \dots x_n]$
$I(X)$ for $X \subseteq K^n = \mathbb{A}^n(K)$.

Fact:[66]
1) $I(X)$ radical.
2) X closed $\Leftrightarrow X = V(I(X))$
3) X, Y closed $X \subseteq Y \iff I(Y) \subseteq I(X)$
4) X, Y closed $\Rightarrow$ $X \cup Y = V(I(X) \cap I(Y))$ and
$X \cap Y = V(I(X) + I(Y))$.

Thm:[67] (Hilb. Basis) K a field $\Rightarrow K[x_1 \dots x_n]$ is noetherian (No ∞ ascending chains of ideals).

Cor:[68] No ∞-desc. seq. of closed sets.

Zar. Closed sets in $\mathbb{A}^n(K)$ $\rightsquigarrow$ Preserved by fin. unions
$\rightarrow$ Preserved by any intersections.

Constructible = Boolean Combin. of Zar. Cl. sets

Cor:[69] Let $K \models ACF$.
1) Defn. = Constructible
2) The image of a constructible set is constructible.

Cor:[70] ACF is str. minimal.

Fact:[71] If $I \subseteq K[\bar{x}]$ is radical, then $\exists P_1 \dots P_m \supseteq I$ s.t. $I = P_1 \cap \dots \cap P_m$,
$I \neq \bigcup_{j \in J} P_j$ for any $J \subsetneq \{1, \dots, m\}$,

and the collection $\{P_1 \dots P_m\}$ is unique.

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Thm:[72] (Nullstellensatz) Let K be an alg. closed field. Suppose that I, J are radical ideals in $K[x_1, \dots x_n]$ and $I \subsetneq J$. Then $V(I) \supsetneq V(J)$.

$$\left\{ \begin{array}{c}\text{Radical}\\ \text{ideals}\end{array} \; I \subseteq K[\bar{x}] \right\} \underset{I(X) \longleftarrow X}{\overset{I \longrightarrow V(I)}{\rightleftarrows}} \left\{ \begin{array}{c}\text{Zariski}\\ \text{Closed}\end{array} \; X \subseteq \mathbb{A}^n(K) \right\}$$

Pf: Let $p \in J \setminus I$. By **Fact**[71], there is $P \supseteq I$, $P \in \operatorname{Spec}(K[x_1 \dots x_n])$ such that $p \notin P$.

Claim: There is $x \in V(P) \subseteq V(I)$ such that $p(x) \neq 0$. Then $V(I) \neq V(J)$.

P prime $\Rightarrow K[\bar{x}]/P$ a domain.

So, let $F = \left(\operatorname{Frac}(K[\bar{x}]/P)\right)^{alg}$.

Let $q_1 \dots q_m$ generate J. Let $a_i := X_i/P$.

$q_i \in P$, $p \notin P$ so:

$$F \models \bigwedge_{i=1}^{m} q_i(\bar{a}) = 0 \wedge p(\bar{a}) \neq 0.$$

So, $$F \models \exists \bar{w} \bigwedge_{i=1}^{m} q_i(\bar{w}) = 0 \wedge p(\bar{w}) \neq 0$$

By model completeness of ACF,

$$K \models \exists \bar{w} \bigwedge_{i=1}^{m} q_i(\bar{w}) = 0 \wedge p(\bar{w}) \neq 0.$$

So, we have $\bar{b} \in K^n$ s.t. $q_1(\bar{b}) \dots = q_m(\bar{b}) = 0$ and $p(\bar{b}) \neq 0$. So $\bar{b} \in V(P) \setminus V(J)$. ■

Cor:[73] If $J \subseteq K[\bar{x}]$ is radical, then $J = I(V(J))$.

Defn: $f : X \to K$ is <u>quasi</u> rational if

1) K has char 0, and f is rational.

2) K has char p and for rational $f(\bar{x}) \in K(\bar{x})$, $f(\bar{x}) = g(\bar{x})^{\frac{1}{p^n}}$

Prop:[74] If $X \subseteq \mathbb{A}^n(K)$ is constr., $f : X \to K$ defn., then there are constructable $X_1, \dots, X_n$ and quasirational $\rho_1 \dots \rho_m$ s.t. $X = \bigcup X_i$ and $f\restriction_{X_i} = \rho\restriction_{X_i}$.

Pf: Let $\Gamma(v_1, \dots v_n) = \{ f(\bar{v}) \neq \rho(\bar{v}) : \rho \text{ quasirational function} \} \cup \{ \bar{v} \in X \} \cup ACF \cup \operatorname{Diag}(K)$.

We will show Γ is not consistent. Assume Γ is consistent.

Let $L \models ACF \cup \operatorname{Diag}(K)$ with $b_1, \dots b_n \in L$ s.t. for all $\gamma(\bar{v}) \in \Gamma$, $L \models \gamma(\bar{b})$.

Let $K_0 = K(\bar{b})$.

K_1 = closure of B, K under quasirational functions $= \bigcup K_0^{\frac{1}{p^n}}$

Also called perfect closure.

$K \prec L$, so $f^L : X(L) \to L$.

$L \models \Gamma(\bar{b}) \Rightarrow f(\bar{b}) \notin K_1$. K_1 perfect, so, for any $a \notin K_1$ there is $\sigma \in Aut(L/K_1)$ s.t. $\sigma(a) \neq a$. But for any $\sigma \in Aut(L/K_1)$, $\sigma(f(\bar{b})) = f(\bar{b})$, since $\bar{b} \in K_1$ and f is K-defn with $K \leq K_1$. #

So Γ is unsatisfiable.

Now finish by compactness. ■

Imaginaries and Galois Theory:

Generally, we are interested in quotient structures, e.g. G/H for G, H defn. groups even though G/H is not definable, only interpretable. Sometimes G/H is defn. $\cong$ to a defn. group H_1, in which case we can prove various nice results.

We are interested in knowing general situations in which we can turn interpretable into defn.

e.g.: Group Quotients
Projective Space.

Let $K \models ACF$.

Let $\bar{x}, \bar{y} \in K^{n \cdot m}$. $\bar{x} \sim \bar{y}$ iff $\bar{x} = (\bar{x}_1 \ldots \bar{x}_m)$ is a permutation of $\bar{y} = (\bar{y}_1 \ldots \bar{y}_m)$.

Lem:[75] There is a defn. function $f : K^{nm} \to K^{\ell}$ for $\ell \in \omega$ s.t. $\bar{c} E \bar{d}$ iff $f(\bar{c}) = f(\bar{d})$.

Pf: Let $\bar{c} = (\bar{c}_1 \ldots \bar{c}_m)$ with $\bar{c}_i = (c_{i,1} \ldots c_{i,n})$

$$q_i(\bar{c}) := Y - \sum_{j=1}^{n} c_{i,j} X_j \in K[X_1 \ldots X_n, Y] \text{ and } p(\bar{c}) = \prod_{i=1}^{m} q_i(\bar{c}).$$

Let

$$f(\bar{c}) = (\text{coefficients of } p(\bar{c}))$$

Then $p(\bar{c}) = p(\bar{d})$ iff $\bar{c}$ permutation of $\bar{d}$. ■

Lem:[76] Let E be an eq. rel. on K^n. Let $\psi(\bar{x}, \bar{y}; \bar{d})$ define E.
Let $\bar{a} \in K^n$.
There is some $\bar{c}$ alg. over $\bar{a}/E, \bar{d}$ s.t. $\bar{c} E \bar{a}$.

Pf: Let $0 \leq m \leq n$ be maximal s.t. $\exists\, c_1 \ldots c_m$ alg. / $\bar{a}/E, \bar{d}$ st.

$$K \models \exists v_{m+1} \ldots v_n\, \psi(c_1 \ldots c_m, v_{m+1} \ldots v_n, \bar{a}; \bar{d}).$$

Lets say $m < n$. Then let

$$X = \{ x \in K \mid K \models \exists w_{m+2} \ldots \exists w_n\, \psi(c_1 \ldots c_m, x, w_{m+2} \ldots w_n, \bar{a}; \bar{d}) \}$$

X finite implies there is $c_{m+1} \in X$ alg. over $\bar{a}/_E, \bar{d}, c_1, \dots c_m$.

So, c_{m+1} alg. $/ \bar{a}/_E, \bar{d}$.

If X is infinite, $K \setminus X$ is finite, so choosing c_{m+1} is easy.

This contradicts m maximal. ■

Thm:[77] Say $K \models ACF$, $A \subseteq K$, and E an A-defn. eq. relation on K^n. Then for some ℓ there is an A-defn. function $f: K^n \to K^\ell$ s.t. $\bar{x} E \bar{y}$ iff $f(\bar{x}) = f(\bar{y})$.

Pf: Assume E defn. $/\emptyset$.

For any formula $\phi(\bar{x}, \bar{y})$, $k > 0$, $\Theta_{\phi,k}(\bar{y})$ be the conjunction of:

1) $\forall \bar{x} (\phi(\bar{x}, \bar{y}) \to \bar{x} E \bar{y})$
2) $\forall \bar{x} \forall \bar{z} (\bar{y} E \bar{z} \to (\phi(\bar{x}, \bar{y}) \leftrightarrow \phi(\bar{x}, \bar{z})))$
3) $|\{\bar{x} : \phi(\bar{x}, \bar{y})\}| = k$

By Lem[76], for any $\bar{a} \in K^n$, there is ϕ and k s.t. $\Theta_{\phi,k}(\bar{a})$ holds.

By 2), if $\Theta_{\phi,k}(\bar{a})$ and $\bar{b} E \bar{a}$ implies $\Theta_{\phi,k}(\bar{b})$.

Let $X = \{\bar{a} : \Theta_{\phi,k}(\bar{a})\}$. Then if $\bar{a} \in X$, let $Y_{\bar{a}} = \{\bar{b} : \phi(\bar{b}, \bar{a})\}$.

For $\bar{a}, \bar{b} \in X$, $\bar{a} E \bar{b}$ iff $Y_{\bar{a}} = Y_{\bar{b}}$.

By Lem[75], there is a $\emptyset$-defn. $f: X \to K^\ell$ for some ℓ s.t. $Y_{\bar{a}} = Y_{\bar{b}}$ iff $f(\bar{a}) = f(\bar{b})$

Now, by compactness, $\exists\ \phi_1 \dots \phi_m$ and $k_1 \dots k_m$ s.t. some $\Theta_{\phi_i,k_i}(\bar{y})$ holds for each element of K^n. Set $X_i = \{\bar{y} \mid \Theta_{\phi_i,k_i}(\bar{x})\}$.

There is $f_i : X_i \to K^{\ell_i}$ s.t. $\bar{a} E \bar{b}$ iff $f_i(\bar{a}) = f_i(\bar{b})$. Extend f_i to K^n by letting $f_i(\bar{b}) = \bar{0}$ for $\bar{b} \notin X_i$.

Now define $f: K^n \to K^{\sum \ell_i}$ by

$$\bar{x} \mapsto (f_1(\bar{x}) \dots f_m(\bar{x}))$$

Then f eliminates E ■.

For the next part, assume that structures are defn. closed: $A = dcl(A)$.

When $b \in acl(A)$, we call $\phi(y, \bar{a})$, $\bar{a} \in A$, **minimal for b** if:

1) $\mathcal{M} \models \phi(b, \bar{a})$
2) There is no ψ and $\bar{a}_1 \in A$ s.t. $\mathcal{M} \models \psi(b, \bar{a}_1)$.

$$|\{c \in \mathcal{M} \mid \mathcal{M} \models \psi(c, \bar{a}_1)\}| < |\{c \in \mathcal{M} \mid \mathcal{M} \models \phi(c, \bar{a})\}|$$

Define: $\deg(b/A) = |\{c \in \mathcal{M} \mid \mathcal{M} \models \varphi(c, \bar{a})\}|$ for φ minimal for b/A.

$\deg(B/A) = \min_{b \text{ gen. } B}\{\deg(b/A)\}$ for B f.gen. over A.

Defn: B/A is normal if $\mathrm{Aut}(\mathcal{M}/A)$ fixes B, set-wise.

ex: Let B/A be f.gen. of fin. degree. Show:

$$|\mathrm{Aut}(B/A)| = \deg(B/A) \iff B/A \text{ is normal.}$$

Defn: Let $C = \mathrm{dcl}(C)$ be a fin. ext. of $A = \mathrm{dcl}(A)$.

$G := \mathrm{Aut}(C/A)$. For $H \leq G$,

$$\mathrm{Fix}(H) := \{c \in C \mid \forall \sigma \in H,\ \sigma(c) = c\}$$

For $A \subseteq B \subseteq C$,

$$G(B) = \{\sigma \in G \mid \sigma(b) = b \text{ for all } b \in B\}$$

ex: Show $\mathrm{Fix}(H)$ is defn. closed.

Lem:[78] If $A < B < C$, all defn. closed, C/A is normal, $G := \mathrm{Aut}(C/A)$

then $\boxed{H = \mathrm{Aut}(C/B) \trianglelefteq G} \iff \boxed{B/A \text{ is normal.}}$

$\Leftarrow$

Pf: Assume B/A normal.

We claim:

$$0 \to \mathrm{Aut}(C/B) \to \mathrm{Aut}(C/A) \to \mathrm{Aut}(B/A) \to 0 \quad \text{is exact.}$$

$\mathrm{Aut}(C/B) \leq \mathrm{Aut}(C/A)$, but B is normal and thus $\mathrm{Aut}(C/A)$ invariant, setwise.
so restriction

$$\sigma \in \mathrm{Aut}(C/A) \longrightarrow \sigma\restriction_B \in \mathrm{Aut}(B/A)$$

is a group homomorphism and the kernel of the map is clearly $\mathrm{Aut}(C/B)$.

($\Rightarrow$). Say B is not normal. Then $\exists\, b \in B$ with $\sigma \in G$ s.t. $\sigma(b) \notin B$.
But it is an exercise to see the orbit $\mathcal{O}(b/A)$ is defn. $/A$.

So $\exists^{\geq 2}$ elements $c, d \in \mathcal{O}(b/A) \setminus B$.

Now let $h \in G$ s.t. $h(c) = d$. Let $g \in G$ s.t. $g(b) = c$.

So, $h(g(b)) = d$, $d \neq c$, so $g^{-1}(d) \neq g^{-1}(c) = b$. So $g^{-1}hg(b) \neq b$.
Thus $g^{-1}hg \notin \mathrm{Aut}(C/B)$ $\times\!\!\!\times$ ■.

Defn: T codes finite sets if

for any $n \in \mathbb{N}$, finite $F \subseteq \mathcal{M}^n$, there is $\bar{b} \in \mathcal{M}^m$ s.t. for all $\sigma \in \mathrm{Aut}(\mathcal{M})$,

$$\sigma(\bar{b}) = \bar{b} \iff \sigma(F) = F.$$

Thm:[79] Say T codes fin. sets. C/A normal, $G = \mathrm{Aut}(C/A)$. Then

$$\text{Subgroups of } G \xleftrightarrow{\text{bijection}} \text{int. defn. cl. structures } A \subset B \subset C$$

$$H \longmapsto \mathrm{Fix}(H) \qquad \mathrm{Aut}(C/B) \longleftarrow B$$

Pf: Say $c \notin \mathrm{dcl}(B)$. Then $\exists\, \sigma \in \mathrm{Aut}(\mathcal{M}/B)$ s.t. $\sigma(c) \neq c$.
So $c \notin \mathrm{Fix}(\mathrm{Aut}(C/B))$. So, $\mathrm{Fix}(\mathrm{Aut}(C/B)) = B = \mathrm{dcl}(B)$.

Now we nts $\mathrm{Aut}(C/\mathrm{Fix}(H)) = H$. We only need to find $\bar{b} \in B$ s.t. $\sigma(\bar{b}) \neq \bar{b}$ for $\sigma \notin H$.

So, let c gen. C/A. $C = \mathrm{dcl}(Ac)$.

$F := \{h(c) \mid h \in H\}$, and let b code F.

Note: H acts transitively and faithfully on F, so

$b \in \mathrm{Fix}(H)$.

But if $g \in G$ with $g \notin H$, then $g(c) \notin F$, since c gen. $\mathrm{Fix}(H)$.
So, F is not g-invariant, and so $g(b) \neq b$. ∎

Defn: An imaginary is an E-class of a tuple where E is a defn. eq. relation

T has EI if every imaginary e has tuple $a \in \mathcal{M}$ s.t. $\mathrm{dcl}^{eq}(e) = \mathrm{dcl}^{eq}(a)$.

T has WEI if $\forall$ imaginaries e, there is $a \in \mathcal{M}$ s.t. $e \in \mathrm{dcl}^{eq}(a)$ and $a \in \mathrm{acl}^{eq}(e)$.

Fact: TFAE
1) EI
2) R a defn relation. $\exists\, a \in \mathcal{M}$ s.t. $\forall \sigma \in \mathrm{Aut}(\mathcal{M})$, $\sigma(a) = a \iff \sigma(R(M)) = R(M)$.
3) For every $\varphi(x)$ with param. $\exists\, \psi(x,y)$ w/o param. s.t.

$\varphi(x) \longleftrightarrow \psi(x,a)$ and $a \in \mathcal{M}$ with this prop. is unique.

Fact: TFAE
1) WEI
2) R a defn relation. $\exists$ fin. set A of tuples from $\mathcal{M}$ s.t. $\forall \sigma \in \mathrm{Aut}(\mathcal{M})$, $\sigma(A) = A \iff \sigma(R(M)) = R(M)$

3) For every $\varphi(x)$ with param. $\exists\ \psi(x,y)$ w/o param. s.t.

$\varphi(x) \leftrightarrow \psi(x,a)$ and only fin. many $a \in \mathfrak{M}$ have this property.

4) For any defn. relation $\exists$ smallest alg closed substr. over which it is defined

<u>Types</u>: Let $A \subseteq \mathcal{M}$ in language $\mathcal{L}$.

Let $p(\bar{x})$ be a set of $\mathcal{L}$-form. with param. in A.

If $p \cup Th_{\mathcal{L}_A}(\mathcal{M})$ is satisfiable, we call $p(\bar{x})$ a type. (over A).

If $p(\bar{x})$ has the prop. that for all $\varphi(\bar{x}) \in \underline{\mathcal{L}_A\text{-form}}$

either $\varphi(\bar{x}) \in p(\bar{x})$ <u>or</u> $\neg\varphi(\bar{x}) \in p(\bar{x})$.

then we call $p(\bar{x})$ <u>complete</u>.

If $\bar{x} = (x_1 \dots x_n)$ then $p(\bar{x})$ is an <u>n-type</u>.

$S_n^{\mathcal{M}}(A) :=$ all $\overset{\text{complete}}{\text{n-types}}$ over A.

We make $S_n^{\mathcal{M}}(A)$ into a top. space by taking a basis of open sets:

$$[\varphi(\bar{x})] := \{p(\bar{x}) \in S_n^{\mathcal{M}}(A) \mid \varphi(\bar{x}) \in p(\bar{x})\}$$

clopen $\leftarrow [\neg\varphi(\bar{x})]$ also open.

Let $\mathcal{N} \succ \mathcal{M}$ and $\bar{a} \in \mathcal{N}$, $\bar{a} = (a_1, \dots, a_n)$.

$tp(\bar{a}/A) := \{\varphi(\bar{x}) \in \mathcal{L}_A \mid \mathcal{N} \models \varphi(\bar{a})\}$

Then its easy to see $tp(\bar{a}/A)$ is a complete n-type$/A$.

<u>Prop</u>: Let $\mathcal{M}, \mathcal{L}$ be as above, $A \subseteq \mathcal{M}$, and p an $\overset{\text{complete}}{\text{n-type}}/A$. Then there is $\mathcal{N} \succ \mathcal{M}$ and $\bar{a} \in \mathcal{N}$ s.t. $p = tp(\bar{a}/A)$.

<u>Pf</u>: Let $\Gamma = p \cup Diag_{el}(\mathcal{M})$. $\leftarrow$ <u>Claim</u>: Γ is satisfiable.

Let $\Delta \subseteq \Gamma$ be finite.

$$\Delta = \{\phi(v_1, \dots, v_n, a_1, \dots, a_m) \wedge \psi(a_1, \dots a_m, \underline{b_1 \dots b_\ell})\} \qquad b_1 \dots b_\ell \in \mathcal{M} \setminus A.$$

$\phi(\bar{v}, \bar{a}) \in p$ and $\mathcal{M} \models \psi(\bar{a}, \bar{b})$.

$\mathcal{N}_0 \models \underline{p \cup Th_A(\mathcal{M})}$

$\exists \bar{w}\, \psi(\bar{a}, \bar{w}) \in Th_A(\mathcal{M})$, so $\mathcal{N}_0 \models \phi(\bar{v}, \bar{a}) \wedge \exists \bar{w}\, \psi(\bar{a}, \bar{w})$.

So, $\mathcal{N}_0 \models \Delta$. Compactness $\Rightarrow \Gamma$ is sat.

<u>Finish</u>: Diagram argument.

Thm:[80] Let $\bar{a}, \bar{b} \in \mathcal{M}$ s.t. $tp(\bar{a}/A) = tp(\bar{b}/A)$.

There is some $\mathcal{N} \succ \mathcal{M}$ and $\sigma \in \underline{Aut(\mathcal{N}/A)}$ s.t. $\sigma(\bar{a}) = \bar{b}$.

Defn: Let $f: \underset{\mathcal{M}}{\overset{\cap}{B}} \longrightarrow \mathcal{N}$. We say f is partial elementary

$$\mathcal{M} \models \phi(\bar{b}) \Leftrightarrow \mathcal{N} \models \phi(f(\bar{b})).$$

Lem:[81] Let $f: B \longrightarrow \mathcal{N}$ be part. ele. If $b \in \mathcal{M}$, there is $\mathcal{N}_1 \succ \mathcal{N}$ and

$g: B \cup \{b\} \to \mathcal{N}_1$ which is part. el. and $g\restriction_B = f$.

Pf: $\Gamma = \{\phi(v, f(a_1) \ldots f(a_n)) \mid \mathcal{M} \models \phi(b, a_1, \ldots a_n),\ a_i \in B\} \cup \mathrm{Diag}_{el}(\mathcal{N})$.

If $\mathcal{M} \models \phi(b, a_1, \ldots a_n)$, then $\mathcal{N} \models \exists v\, \phi(v, f(a_1) \ldots f(a_n))$

$\Downarrow$ $\quad\nearrow$ f part. el.

$\mathcal{M} \models \exists v\, \phi(v, a_1, \ldots a_n)$

So Γ is sat. $\blacksquare$

Cor:[82] $f: \underset{\mathcal{M}}{\overset{\cap}{B}} \longrightarrow \mathcal{N}$ partial el. We can extend f to $g: \mathcal{M} \longrightarrow \mathcal{N}_1$ with $\mathcal{N}_1 \succ \mathcal{N}$
partial elem.

So g is an el. embedding.

Thm:[83] Let $\bar{a}, \bar{b} \in \mathcal{M}$ s.t. $tp(\bar{a}/A) = tp(\bar{b}/A)$.

There is some $\mathcal{N} \succ \mathcal{M}$ and $\sigma \in \underline{Aut(\mathcal{N}/A)}$ s.t. $\sigma(\bar{a}) = \bar{b}$.

Pf: Let $f: A \cup \{a\} \to A \cup \{b\}$
$f(a) = b$ and $f\restriction_A = id_A$.
} This is part. elem.

Let $\mathcal{N}_0 \succeq \mathcal{M}$ and elem. emb. extends f.
extending f by prev. cor.

We will build

$$\mathcal{M} = \mathcal{M}_0 \prec \mathcal{N}_0 \prec \mathcal{M}_1 \prec \mathcal{N}_1 \prec \mathcal{M}_2 \ldots$$

$f_i: \mathcal{M}_i \to \mathcal{N}_i$ s.t. $f_0 \subseteq f_1 \subseteq \ldots$

Assuming we have this, $\mathcal{N} = \bigcup \mathcal{M}_i = \bigcup \mathcal{N}_i \succ \mathcal{M}$

$\sigma := \bigcup f_i$

Assume we have $f_i : \mathcal{M}_i \to \mathcal{N}_i$

f_i^{-1} : $\underset{\text{of } \mathcal{N}_i}{\text{subset}} \longrightarrow \mathcal{M}_i$

Let $\mathcal{M}_{i+1}$ be as in cor and extend f_i^{-1} to

$g_i : \mathcal{N}_i \longrightarrow \mathcal{M}_{i+1}$.

Repeat this argument with g_i^{-1} to build $\mathcal{N}_{i+1}$ and $f_{i+1} : \mathcal{M}_{i+1} \to \mathcal{N}_{i+1}$

with desired properties. ∎

Defn: Let κ be an inf. cardinal. We say $\mathcal{M} \models T$ is κ-homogeneous if when $A \subseteq \mathcal{M}$ with $|A| < \kappa$, $f : A \to \mathcal{M}$ part. el. and $a \in \mathcal{M}$ then there is $f^* : A \cup \{a\} \to \mathcal{M}$ part. el. and $f^* \restriction_A = f$.

We say $\mathcal{M}$ is homogeneous if $\mathcal{M}$ is $|\mathcal{M}|$-homogeneous.

Defn: $\mathcal{M}$ is κ-saturated if for all $A \subseteq \mathcal{M}$, $|A| < \kappa$, any partial type over A is realized in $\mathcal{M}$. $\mathcal{M}$ is saturated if $\mathcal{M}$ is $|\mathcal{M}|$-sat.

Fact: Saturated models always exist under GCH or large cardinals (inaccessible). But it is also possible, eg. DLO, RCF that no sat models exist. This is related to stability.

Working in a saturated model has advantages, e.g., galois theoretic style arguments

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Lem: Let $\mathcal{M}$ be κ-sat & let $|A| < \kappa$. Let X be defn. X is A-defn iff $\sigma(X) = X$ for all $\sigma \in Aut(\mathcal{M}/A)$.

Pf: $\Rightarrow$ If $X = \phi(\mathcal{M}, b)$ for $b \in A$, then $\sigma(b) = b$, so

$$\mathcal{M} \models \phi(a,b) \Leftrightarrow \mathcal{M} \models \phi(\sigma(a), \sigma(b)) = \phi(\sigma(a), b).$$

So $\sigma(X) = X$.

$\Leftarrow$ Let $X = \phi(X, b)$ for $b \in \mathcal{M}$. Let $p(y) = tp(b/A)$.

We claim:

$p(y) \models \forall x (\phi(x,y) \leftrightarrow \phi(x,b))$.

Let $b' \models p(y)$. Then there is $\sigma \in Aut(\mathcal{M}/A)$ st. $\sigma(b) = b'$.

$\sigma(X) = \phi(\mathcal{M}, b') = X = \phi(\mathcal{M}, b)$.

By compactness, there is some $\psi(y) \in p(y)$ st. $\psi(y) \models \forall x (\phi(x,y) \leftrightarrow \phi(x,b))$

Let $\theta(x) = \exists y (\psi(y) \wedge \phi(x,y))$.

Claim: $X = \theta(\mathcal{M})$. If $a \in X$, then $\mathcal{M} \models \phi(a,b)$ so $X \subseteq \theta(\mathcal{M})$.

If $\mathcal{M} \models \theta(a)$, then let b' be s.t. $\mathcal{M} \models \psi(b') \wedge \phi(a,b')$.

but by choice of ψ, we see $\mathcal{M} \models \phi(a,b)$. ∎

Ex:
The topology on $S_{x,y}(A)$ is not the product topology.

Lem: [85] Define $S_{x,y}(A) \longrightarrow \bigcup_{q \in S_y(A)} S_x(Ab_q)$ where $b_q \models q(y) \in S_y(A)$.

Given $p(x,y)$, let $(a,b) \models p$ and $b = b_q$ for $q = p_x(y)$. The map

$p \longmapsto tp(a/Ab_q)$ is injective.

Defn: $\phi(v_1,\dots,v_n)$ an $\mathcal{L}$-form s.t. $T \cup \{\phi(\bar{v})\}$ is satisfiable.
Let $p(v)$ be an n-type. We say $\phi(v)$ isolates $p(v)$ if

$T \models \forall \bar{v}\, \phi(\bar{v}) \longrightarrow \psi(\bar{v})$.

Prop: [86] If $\phi(v)$ iso. $p(v)$ then in any $\mathcal{M} \models T$, $p(v)$ is realized.

Pf: (easy) ∎

Thm: [87] (Omitting types) $\mathcal{L}$ countable, T an $\mathcal{L}$-th., p a (maybe incomplete) type which is not isolated over $\emptyset$. Then there is countable $\mathcal{M} \models T$ omitting T.

Pf: (Modified Henkin)

$C = \{c_0 \dots\}$ new const. $\mathcal{L}^* = \mathcal{L} \cup C$.

We will build $T = T_0 \subseteq \dots$ s.t.
1) T_n consistent in $\mathcal{L}_n = \mathcal{L} \cup \{c_i \mid i < n\}$
2) $T_{i+1} \setminus T_i$ is finite
3) For $\ell, k < i$, there is some $\theta_{\ell,k} \in p_k$ s.t. $\neg\theta_{\ell,k}(c_\ell) \in T_i$
4) c_ℓ is a "henkin constant for f_ℓ:

$$\exists x\, f_\ell(x) \longrightarrow f_\ell(c_\ell) \in T_{\ell+1}.$$

Now, given T_i, we build T_{i+1}:

$$T_{i,0} := T_i \cup \{\exists x\, f_i(x) \longrightarrow f_i(c_i)\}$$

List all pairs ℓ, k with $\ell, k < i+1$. We want to build $T_{0,*}$ by adding $\neg\theta_{\ell,k}(c_\ell)$ to T_0.

If 3) cannot be accomplished, then $T_{i,*} \vdash \theta_{\ell,k}(c_i)$ for some $\ell, k < i$.

But $T_{i,*} = T \cup \{\eta_1(\bar{c}) \dots \eta_t(\bar{c})\}$ so,

$$T \vdash \bigwedge \eta_s(\bar{c}) \longrightarrow \theta_{\ell,k}(c_\ell) \quad \text{so,}$$

$T \vdash \forall x_0 \dots x_i \bigwedge \eta_s(\bar{x}) \longrightarrow \theta_{e,k}(x_e)$.

$T \vdash \forall x_e \left(\exists x_0 \dots x_{e-1}, x_{e+1} \dots x_i\right)\left(\bigwedge \eta_s(\bar{x}) \longrightarrow \theta_{e,k}(x_e)\right)$

Then p_k is isolated by $\exists x_0 \dots x_{e-1}, x_{e+1} \dots x_i)\left(\bigwedge \eta_s(\bar{x}) \longrightarrow \theta_{e,k}(x_e)\right)$.

So $T_{i,*+1} = T_{i,*} \cup \{\neg \theta_{e,k}(c_e)\}$. $T_{i+1} = \bigcup_j T_{i,j}$ ■.

Defn: $\mathcal{M}$ is <u>prime</u> if for any $\mathcal{N} \equiv \mathcal{M}$ there is el. embedding $\mathcal{M} \longrightarrow \mathcal{N}$.

Defn: $\mathcal{M}$ is <u>atomic</u> if every n-type realized in $\mathcal{M}$ is principal.

	Atomic	Prime
$\mathbb{Q}^{alg}$	yes	yes
Uncountable Set	Yes	No

Reading the next two propositions, you can see that set theory shenannigans are really the difference between prime and atomic.

Prop: 88 $\mathcal{M}$ prime in a countable language $\Rightarrow$ $\mathcal{M}$ atomic.

Prop: 89 $\mathcal{M}$ countable atomic in countable language $\Rightarrow$ $\mathcal{M}$ prime.

Prop: 90 $S_n(\emptyset)$ countable for all n $\Rightarrow$ T has an atomic model.

Thm: 91 Let $\mathcal{M}, \mathcal{N} \models T$ be countable, atomic. Then $\mathcal{M} \cong \mathcal{N}$.

Thm: 92 (Ryll-Nardzewski) $\mathcal{L}$-countable. T complete with infinite models. TFAE: is

1) T is $\aleph_0$-cat.
2) All models of T are atomic.
3) All countable models atomic.
4) $\mathrm{Aut}(\mathcal{M}) \curvearrowright \mathcal{M}$ oligomorphically for all $\mathcal{M} \models T$ countable.
5) $\mathrm{Aut}(\mathcal{M}) \curvearrowright \mathcal{M}$ oligomorphically for some $\mathcal{M} \models T$
6) $|S_n(\emptyset)| < \infty$ for all $n \in \mathbb{N}$.

Thm: 93 Let $\kappa \geq |\mathcal{L}|, \aleph_0$. Let $\mathcal{M}$ be an $\mathcal{L}$-str. TFAE

1) $\mathcal{M}$ is κ-sat.
2) $\mathcal{M}$ is κ-hom. and κ^+ universal.

Say $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$. We say $\mathcal{M}$ is prime over A if whenever $N \models T$, $f: A \to N$ part. elem there is $f^*: \mathcal{M} \to N$ extending f.

e.g. DLO, ACF, RCF

Defn: Let T be complete. $\mathcal{L}$, countable, κ infinite. T is κ-stable if whenever $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$ and $|A| = \kappa$, then $|S_n^{\mathcal{M}}(A)| = \kappa$.

Note: people always say ω-stable.

94 Thm: ω-stable $\Rightarrow$ κ-stable for all κ infinite cardinals.

Pf: Let $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, $|A| = \kappa$ and $|S_n^{\mathcal{M}}(A)| > \kappa$.

There are κ-many formulas in $\mathcal{L}_A$.
So by pidgeon hole, there is $\phi_\emptyset \in \mathcal{L}_A$ s.t. $|[\phi_\emptyset]| > \kappa$.

Claim: there is $\psi \in \mathcal{L}_A$ s.t. $|[\phi_\emptyset \wedge \psi]|$ and $|[\phi_\emptyset \wedge \neg\psi]| > \kappa$.

If not, we can write $[\phi_\emptyset]$ as a union of sets of size κ.

So, we build a binary tree of formulas, inductively. ∎

95 Prop: Let T be complete, $\mathcal{L}$ countable. If T is ω-stable, then for any $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, the isolated types are dense in $S_n^{\mathcal{M}}(A)$.

Pf: Find isolated type in $[\phi]$ $\leftarrow$ if not build a binary tree. ∎

96 Thm: Suppose T is ω-stable, $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$. There is $\mathcal{M}_0 \preceq \mathcal{M}$ s.t. $\mathcal{M}_0$ is prime over A. More, each type realized over A is isolated.

Pf: $A_0 = A$

$A_\alpha = \bigcup_{\beta < \alpha} A_\beta$ when α is a limit.

$$A_{\alpha+1} = \begin{cases} A_\alpha & \text{if } tp(a/A_\alpha) \text{ is not isolated for all } a \in \mathcal{M} \setminus A_\alpha \\ A_\alpha \cup \{a_\alpha\} & \text{for } tp(a_\alpha/A_\alpha) \text{ isolated and } a_\alpha \in \mathcal{M} \setminus A_\alpha \end{cases}$$

Let $\mathcal{M}_0 = \lim_{\alpha \to \infty} A_\alpha$

Claim: $\mathcal{M}_0 \preceq \mathcal{M}$.
Tarski-Vaught & isolated types dense & realized in $\mathcal{M}$.

Claim: $\mathcal{M}_0$ prime$/A$.
Build up a chain of part. ele. maps $f_\alpha: A_\alpha \to N$ for any $N \supseteq A$.

To finish the atomic bit.

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Lem:[11] Say $A \subseteq B \subseteq \mathcal{M} \models T$ and every $\bar{b} \in B$ is isolated in $S_m^{\mathcal{M}}(A)$.
Let $\bar{a} \in \mathcal{M}^n$ realize a type isolated in $S_n^{\mathcal{M}}(B)$.

Then $tp(\bar{a}/A)$ is isolated in $S_n^{\mathcal{M}}(A)$.

Pf: $\theta(\bar{w})$ iso. $\bar{b}/A$
$\psi(\bar{b}, \bar{v})$ iso $\bar{a}/B$ $\Rightarrow$ $\theta(\bar{w}) \wedge \psi(\bar{w}, \bar{v})$ iso $(\bar{b}, \bar{a})$. $/A$

$$\underset{\boxed{\text{exercise}}}{\Rightarrow} \quad \exists \bar{w}\ \theta(\bar{w}) \wedge \psi(\bar{w}, \bar{v}) \text{ iso. } \bar{a}/A \quad \blacksquare$$

Thm:[98] (Keisler Shelah) T complete theory in a countable language.
Then let
$$f_T(\kappa) = \sup\{|S_1(\mathcal{M})| : \mathcal{M} \models T, |\mathcal{M}| = \kappa\}.$$

$f_T(\kappa) = \kappa,\ \kappa + 2^{\aleph_0},\ ded(\kappa),\ ded(\kappa)^{\aleph_0}$, or 2^κ

ex: Show each of these possibilities can happen for some T.

Note: $Ded(\kappa) = Sup\{|I| \mid I$ is a linear with a dense set of size $\kappa\}$

Using PCF theory, one can show $2^\kappa \leq \underbrace{ded(ded(ded(ded}_{\text{optimal?}}(\kappa))))$

Lem:[99] $\kappa < ded(\kappa)$

Pf: Let μ be minimal such that $2^\mu > \kappa$. Consider $2^{<\mu}$ ordered lexicographically. There are 2^μ many cuts. $\blacksquare$

ex: Show $ded(\aleph_0) = 2^{\aleph_0}$.

Show $ded(\kappa) \leq 2^\kappa$
Show under GCH that $ded(\kappa) = 2^\kappa$.

Defn: $\phi(x,y)$ has the k-order property if there are a_i, b_j s.t. $\phi(a_i, b_j) \Leftrightarrow i < j$ for all $i, j \in [k]$.

$\phi(x,y)$ is stable if ϕ does not have the k-order property for some $k \in \omega$.

Prop:[100] T unstable $\Rightarrow f_T(\kappa) \geq ded(\kappa)$ for all $\kappa \geq |T|$.

Thm:[101] (Ramsey) $\aleph_0 \to (\aleph_0)^n_k$ for all $n, k \in \mathbb{N}$.

Lem:[102] Let $\phi(x,y)$, $\psi(x,z)$ be stable.

1) $\phi^*(y,x) = \phi(x,y)$ is stable.
2) $\neg\phi(x,y)$ is stable.
3) $\theta(x, yz) = \phi(x,y) \wedge \psi(x,z)$ is stable.
4) $y = uv \Rightarrow \phi(x, uc)$ is stable for any c.
5) T stable $\Rightarrow$ $\mathcal{L}^{eq}$-forms stable.

Thm:[103] (Erdős-Makkai) Let B be infinite, and $F \subseteq \mathcal{P}(B)$ a collection with $|B| < |F|$. Then there are $(b_i : i \in \omega)$, $(S_i : i \in \omega)$ st one of:

1) $b_i \in S_j \Leftrightarrow j < i$ for all $i, j \in \omega$.
2) $b_i \in S_j \Leftrightarrow i < j$ for all $i, j \in \omega$.

Defn: A complete ϕ-type over A is a maximal consistent collection of instances of $\phi(x,a)$ and $\neg\phi(x,a)$ for $a \in A$. $S_\phi(A)$ = space of ϕ-types $/A$.

Prop:[104] Suppose $|S_\phi(A)| > |A|$ for some infinite A. Then ϕ is unstable.

Defn: Δ a set of forulas, $\theta(x)$ a partial type over $\mathcal{M}$.

- $R_\Delta(\theta(x)) \geq 0$ iff $\theta(x)$ is consistent.
- $R_\Delta(\theta(x)) \geq n+1$ if for some $\phi(x,y) \in \Delta$ and $a \in \mathcal{M}_y$ we have

$$R_\Delta(\theta(x) \wedge \phi^i(x,a)) \geq n \quad \text{for } i = 0 \text{ or } 1.$$

Prop: [105] ϕ stable $\Rightarrow R_\phi(x=x)$ is finite.

Defn: 1) $\phi(x,y) \in \mathcal{L}$. $p(x) \in S_\phi(A)$ is defn $/B$ if there is $\mathcal{L}_B$-form. $\psi(y)$ s.t. for all $\phi(x,a) \in p(x) \Leftrightarrow \psi(a)$.

2) $p \in S_x(A)$ is defn $/B$ if $p\restriction_\phi$ is defn $/B$ for all $\phi \in \mathcal{L}$.

3) p is defn if it is defn. $/\mathrm{dom}(p)$.

4) Types in T are uniformly defn if for all $\phi(x,y)$, there is $\psi(y,z)$ s.t. every type can be defn by an instance of ψ, ie

$\forall\, p \in S_\phi(A)$ there is some $b \in A$ st $\phi(x,a) \in p \Leftrightarrow\ \models \psi(a,b)$ for all $a \in A$.

Prop[106]**:** Let $\phi(x,y)$ be stable. Then all ϕ-types are uniformly defn.

Pf: Say $p \in S_\phi(A)$, Call $p_i \subseteq p$ one element minimal if $R_\phi(q) = R_\phi(p_i)$ for all $p_i \subseteq q \subseteq p$ with $|\mathrm{dom}(q) \setminus \mathrm{dom}(p_i)| = 1$.

For any $p \in S_\phi(A)$, we claim there is a one-element min $p_i \subseteq p$ with $|p_i| \leq R_\phi(x=x)$. To see this, let $p_0 = \emptyset$. Given p_i, let $p_{i+1} \subseteq p$ be any one-element extension of p_i, if one exists.

Next, we claim for any $p \in S_\phi(A)$, if $p_i \subseteq p$ is one-element minimal, then p is defined by

$R_\phi(p_i(x) \wedge \phi(x,a)) = R_\phi(p_i)$

For $a \in A$, $\phi(x,a) \notin p \Rightarrow \neg\phi(x,a) \in p \Rightarrow R_\phi(p_i \cup \{\neg\phi(x,a)\}) = R_\phi(p_i) \Rightarrow R_\phi(p_i \cup \{\phi(x,a)\}) \neq R_\phi(p_i)$. $\blacksquare$

Thm[107]**:** TFAE
1) $\phi(x,y)$ stable
2) $R_\phi(x=x) < \omega$
3) All ϕ-types are uniformly defn.
4) All ϕ-types are defn.
5) $|S_\phi(M)| \leq \kappa$ for all $\kappa \geq |\mathcal{L}|$, $M \models T$, $|M| = \kappa$
6) $\exists\, \kappa$ s.t. $|S_\phi(M)| < \mathrm{ded}(\kappa)$ for all $M \models T$ with $|M| = \kappa$.

Thm[108]**:** T complete. TFAE.

1) T stable
2) No sequence $(a_i : i \in \omega)$ from M and $\phi(z_1, z_2) \in \mathcal{L}(M)$ s.t. $\models \phi(c_i, c_j) \Leftrightarrow i < j$.
3) $f_T(\kappa) \leq \kappa^{|T|}$ for all infinite κ.
4) $\exists\, \kappa$ s.t. $f_T(\kappa) \leq \kappa$.
5) $\exists\, \kappa$ s.t. $f_T(\kappa) < \mathrm{ded}(\kappa)$
6) All formulas of the form $\phi(x,y)$ with x a singleton, are stable.

Defn: Linear order I. Given $(a_i : i \in I)$ with $a_i \in \mathcal{M}_x$ is indiscernible over A iff $a_{i_0} \dots a_{i_n} \equiv_A a_{j_0} \dots a_{j_n}$, whenever i_k, j_k are increasing.

Prop:[109] Let $\bar{a} = (a_i : i \in \omega)$ be a sequence. Then define

$$EM(\bar{a}/A) := \left\{ \phi(x_0 \dots x_n) \in \mathcal{L}(A) \;\middle|\; \forall i_0 < i_1 < \dots < i_n,\ \models \phi(a_{i_0} \dots a_{i_n}),\ n < \omega \right\}$$

For any small linear order I, we can find $(b_i : i \in I)$ s.t.

$$EM(\bar{b}/A) = EM(\bar{a}/A) \quad \text{with } \bar{b} \text{ an } A\text{-indisc. seq.}$$

Cor:[110] If $(a_i : i \in I)$ is A-indisc, and $J \supseteq I$ is a lin. order, then there is A-ind seq. $\bar{b} = (b_j : j \in I)$

s.t. $a_i = b_j$ if $i = j$.

Prop:[111] Let κ, λ be small with $(a_i : i \in \lambda)$ a sequence with $|a_i| < \kappa$ and B be given. If $\lambda \geq \beth_{(2^{\kappa + |B| + |T|})^+}$ + there is $(a_i')_{i \in \omega}$ st. $\forall n \in \omega$, there are $i_0 \dots i_n \in \kappa$ s.t.

$$a_0' \dots a_n' \equiv_B a_{i_0} \dots a_{i_n}.$$

Defn: $(a_i : i \in I)$ is <u>totally indiscernible</u> over A if $a_0 \ldots a_{i_n} \equiv_A a_{j_0} \ldots a_{j_n}$ for any $i_0 \neq \ldots \neq i_n$, $j_0 \neq \ldots \neq j_n$ in I.

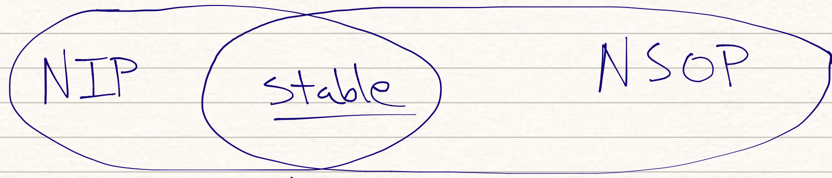
Thm[112]**:** T stable $\Leftrightarrow$ every ind. seq. is totally ind.

Prop[113]**:** For any stable $\phi(x,y)$ there is $k_\phi \in \omega$ depending on ϕ s.t. for any indis. seq. $I \subseteq \mathcal{M}^{|x|}$, and any $b \in \mathcal{M}^{|y|}$, either

$$|\phi(I,b)| \text{ or } |\neg\phi(I,b)| \leq k_\phi .$$

T stable:

Defn: $Av(I/A) := \{\phi(x,a) : \phi \text{ holds cofinitely on } I\}$.



strict order property

Defn: 1) $\phi(x,y) \in \mathcal{L}$ has SOP if there are $b_i \in \mathcal{M}$ for $i \in \omega$ s.t.

$$\phi(\mathcal{M}, b_i) \subsetneq \phi(\mathcal{M}, b_j) \text{ for all } i < j \in \omega.$$

2) T has SOP if some formula does.

3) T is NSOP if it does not have SOP

eg: DLO

ex: T has NSOP iff all formulas $\phi(x,y)$ with x a singleton have NSOP

ex: T has NSOP $\Leftrightarrow$ there is a defn. part. order with infinite chains.

Defn: