HW1, Math 506, Fall 2016, Due 8-31

August 22, 2016

1. (1 point) Using the signature $\{+, -, \cdot, 0, 1\}$ give an example of a field, F (regarded as a structure of this signature in *the* natural manner) which is not finitely generated as a structure in this signature, but has the property that there is an element b such that F is the smallest subfield containing b.

2. (2 points) Regard $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}$ as structures in the language of groups $\{+, 0\}$ (in *the* natural manner). Given an example of a sentence which $\mathbb{Q}$ satisfies, but $\mathbb{Z}$ does not. Given an example of a sentence which $\mathbb{Z}$ satisfies but $\mathbb{Z} \times \mathbb{Z}$ does not satisfy. Extra: Given an example which shows that $\mathbb{Z}^n \not\equiv \mathbb{Z}^m$ when $m \neq n$.

3. (2 points) Given a sentence ϕ in some signature τ, we define

   $$FinMod(\phi) := \{n \in \mathbb{N} \mid \text{there is a } \tau\text{-structure } \mathcal{A} \text{ with } \mathcal{A} \models \phi \text{ and } |dom(\mathcal{A})| = n\}.$$

   Find ϕ such that:

   1. $FinMod(\phi)$ is equal to the set of natural numbers of the form $2^m 3^n$.
   2. $FinMod(\phi)$ is equal to the set of natural numbers which are a power of a prime.
   3. $FinMod(\phi)$ is equal to the set of composite numbers.
   4. $FinMod(\phi)$ is equal to the set of prime numbers.

4. (1 point) Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $f : \mathcal{M}^n \to \mathcal{M}^m$ is definable if the graph of f is a definable set in M^{m+n}. Show that the composition of two definable functions is definable. Show that the image of a definable set under a definable function is definable. Show that a definable injection has a definable inverse.

5. (1 point) Let $\mathcal{M}$ be a τ-structure. We say b is definable over $A \subseteq M$ if there is some formula $\phi(v, \bar{w})$ and $\bar{a} \in A$ such that $\mathcal{M} \models \phi(b, \bar{a})$ and $\mathcal{M} \models \forall y \phi(y, \bar{a}) \to y = b$. Show that if σ is an automorphism of $\mathcal{M}$ which fixes A pointwise and b is definable over A, then $\sigma(b) = b$. Let $dcl(A) = \{b \in \mathcal{M} \mid b \text{ is } A\text{-definable}\}$. Show that $dcl(dcl(A)) = dcl(A)$.

6. (1 point) Let $\mathcal{M}$ be a τ-structure. We say b is algebraic over $A \subseteq M$ if there is some formula $\phi(v, \bar{w})$ and $\bar{a} \in A$ such that $\mathcal{M} \models \phi(b, \bar{a})$ and $\{\bar{y} \in M \mid \mathcal{M} \models \phi(y, \bar{a})\}$ is finite. Let b be algebraic over A; show that there are $b_1, \ldots, b_n$ such that for any $\sigma \in Aut(\mathcal{M}/A)$, $\sigma(b) = b_i$ for some i. Show that $acl(acl(A)) = acl(A)$. Show that if $b \in acl(A)$, then there is some finite A_0 such that $b \in acl(A_0)$. Show that if $A \subseteq B$ then $acl(A) \subseteq acl(B)$.

7. (1 point) Let $\kappa < \lambda < \theta$ be infinite cardinals. Give an example of a structure $\mathcal{A}$ in some signature τ such that $|dom(\mathcal{A})| = \theta$ and $\mathcal{A}$ has a substructure of cardinality κ, but not one of cardinality λ.

8. (1 point) An ordered group is a group G which is linearly ordered by $<$ with the property that
$$\forall x \forall y \forall z (x < y \to x + z < y + z).$$
Show that the additive group of the real numbers is an ordered group. Show that no nontrivial finite group can be ordered. Show that an ordered group has no torsion.

9. (1 point) Show that there is a field K such that K is elementarily equivalent to the real numbers and has the property that there are $a, b \in K$ with $b > n \cdot a$ for all $n \in \mathbb{Z}$.

   Note: Such a field is called non-Archimedian, a term which is used in a different way to describe a valuation on a field.

10. (1 point) Show that any torsion free abelian group can be made into an ordered group.