HW2, Math 506, Fall 2016, Due 9/7

August 31, 2016

1. (1 point) Let $\mathcal{L} = \{s\}$ be a single unary function. Let T be a theory which stipulates that s is a bijection and s with no cycles. For which cardinals is T categorical?

2. (1 point) Let $\mathcal{L} = \{R\}$ where R is a binary relation symbol. Conside the theory containing the graph (with at least two vertices) axioms,

$$\forall x \neg R(x,x) \text{ and } \forall x \forall y R(x,y) \to R(y,x) \text{ and } \exists x \exists y (x \neq y)$$

along with the following sentence,

$$\psi_n := \forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n \left(\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} x_i \neq y_j \to \exists z \bigwedge_{i=1}^{n} (R(x_i,z) \wedge \neg R(y_i,z)) \right).$$

Now let T be the theory consisting of the above axioms as n ranges over $\mathbb{N}$. Prove that T is satisfiable.

3. (1 point) Let T be the theory of abelian groups with every element having order two. Show that T is κ-categorical for all infinite cardinals. Is T complete? If not, find $T' \supseteq T$ such that T and T' have the same infinite models.

4. (1 point) (Ultrafilters) Let I be a set and let $P(I)$ be the power set of I. Then a filter on I is a collection D so that $I \in D$, $\emptyset \notin D$, $A, B \in D$ implies $A \cap B \in D$, and $A \subseteq B$ and $A \in D$ implies $B \in D$.

    1. Show that the set of subsets X of $\mathbb{R}$ such that $\mathbb{R} \backslash X$ has Lebesgue measure zero is a filter.
    2. Let κ be an infinite cardinal. Show that the set of cofinite subsets of κ is a filter. This is called the Frechet filter.
    3. Show that if $x \in I$ and $D = \{Y \subseteq I \,|\, x \in Y\}$ is a filter. This is called a principal filter.
    4. A filter D on I is called an ultrafilter on I if for all $X \subseteq I$ either $X \in D$ or $X \notin D$. Show that every filter is contained in an ultrafilter (you will *have* to use the axiom of choice or Zorn's lemma).

5. Apply the previous result to the Frechet filter. Conclude that the resulting ultrafilter is nonprincipal.

5. (2 points) Let $\mathcal{U}$ be an ultrafilter on I. For each $i \in I$, let $\mathcal{M}_i$ be an $\mathcal{L}$-structure. Let $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i / \sim$ where if $f, g \in \prod_{i\in I} \mathcal{M}_i$, $f \sim g$ if $\{i \in I \mid f(i) = g(i)\} \in \mathcal{U}$. For $a \in \prod_{i\in I} \mathcal{M}_i$, we sometimes write a^* for its $\sim$ class. Show that $\mathcal{M}$ is an $\mathcal{L}$-structure under the following interpretations:

   1. For each $c \in \mathcal{C}$, $c^{\mathcal{M}} = (c^{\mathcal{M}_i})_{i\in I} / \sim$
   2. For $r \in \mathcal{R}$, and $g_1, \dots g_n \in \prod_{i\in I} \mathcal{M}_i$, let $\mathcal{M} \models r(g_1^*, \dots, g_n^*)$ if $\{i \in I \mid \mathcal{M}_i \models r(g_1(i), \dots, g_n(i))\} \in \mathcal{U}$.
   3. For $f \in \mathcal{F}$, and $g, g_1, \dots g_n \in \prod_{i\in I} \mathcal{M}_i$, let $f^{\mathcal{M}}(g_1^*, \dots, g_n^*) = g^*$ if $\{i \in I \mid f^{\mathcal{M}_i}(g_1(i), \dots, g_n(i)) = g(i)\} \in \mathcal{U}$.

   You need to make sure these definitions give a well-defined $\mathcal{L}$-structure. Notation: sometimes people write $\prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ or $\prod_{\mathcal{U}} \mathcal{M}_i$ for the *ultraproduct* we just built. If all the structures $\mathcal{M}_i$ are the same, then we call the construction an *ultrapower*, and in that case if $\mathcal{M}_i = \mathcal{N}$ for each $i \in I$, one often writes $\mathcal{N}^{\mathcal{U}}$.

   Let $\phi(v)$ be an $\mathcal{L}$-formula. Show that

   $$\mathcal{M} \models \phi(g_1^*, \dots, g_n^*) \text{ if and only if } \{i \in I \mid \mathcal{M}_i \models \phi(g_1(i), \dots, g_n(i))\} \in \mathcal{U}.$$

6. (2 points) Let $\mathbb{F}_p$ denote the field with p elements. Let $\mathcal{L} = \mathcal{L}_{rings}$ and let D be a nonprincipal ultrafilter on the set of prime numbers. Let $K = \prod_D \mathbb{F}_p$.

   1. Is K a field?
   2. What is the characteristic of K?
   3. Is $K \equiv \mathbb{R}$?
   4. Does K contain any irrational algebraic number?
   5. Is $K \equiv \mathbb{C}$
   6. Show K has a unique algebraic extension of each degree.
   7. Is there a solution to $x^2 + 1 = 0$ in K?
   8. (Need to know something about number theory for this part) Show that there are infinitely many solutions to the equation $y^3 = x^8 - x^3 + 1$.

7. (1 point) Let S, T be $\mathcal{L}$-structures, and let $\mathcal{D}$ be an ultrafilter on some set. Suppose that $S \subset T$. Is $S^{\mathcal{D}} \subseteq T^{\mathcal{D}}$? If not, explain why this is almost true.