HW4, Math 506, Fall 2016, Due 9/23

September 14, 2016

1 Do each of the first three problems:

1. (3 points) You should use last week's homework problems on the following exercise.

    1. Show that if $\mathcal{M} \subset \mathcal{N}$ are models of T and $\mathcal{M}$ is existentially closed, then there is $\mathcal{M}_1$ with $\mathcal{M} \subset \mathcal{N} \subset M_1$ with $\mathcal{M} \prec \mathcal{M}_1$.
    2. Show that T is model-complete if and only if every model of T is existentially closed.
    3. Suppose that T is $\forall\exists$-axiomatizable and has infinite models. Suppose that T is κ-categorical for some infinite cardinal κ. Show that T is model-complete.
    4. Show that T is model-complete if and only if for any formula $\varphi(\bar{v})$ there is a quantifier-free formula such that $\psi(\bar{v}, \bar{w})$ such that $T \models \varphi(\bar{v}) \leftrightarrow \exists \bar{w} \psi(\bar{v}, \bar{w})$.
    5. Show that any model complete theory has a $\forall\exists$-axiomatization.

2. (2 points) Let T and T' be $\mathcal{L}$-theories. Then T' is a model companion of T if T' is model-complete, every model of T has an extension which is a model of T' and every model of T' has an extension of T.

    1. Show that any theory has at most one model companion.
    2. Find the model companion of the theory of discrete linear orders.
    3. Suppose that T is $\forall\exists$-axiomatizable. Then show that the model companion (should it exist) of T must be the theory of existentially closed models of T.
    4. Observe something about the previous two parts of this problem.

3. (2 points) Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\mathcal{U}$ be a nonprinciple ultrafilter on a set I. Let $\mathcal{M}^*$ denote the ultrapower $\prod_{\mathcal{U}} \mathcal{M}$.

    - We call a set $S \subseteq \mathcal{M}^*$ *internal* if there are $W_i \subset \mathcal{M}$ for $i \in I$ such that for any $(a_i)_{i \in I} \in \prod_I \mathcal{M}$, $(a_i)_{i \in I}/\mathcal{U} \in S$ if and only if $\{i \in I \mid a_i \in W_i\} \in \mathcal{U}$. Give an example of a non-internal subset of an ultraproduct. The sets W_i are called the components of an internal set.
    - Consider $\mathcal{M}$ as a substructure of $\mathcal{M}^*$ under the natural injection. Which subsets of $\mathcal{M}$ are internal in $\mathcal{M}^*$?

- Let K be a field and let $\bar{x}$ be a finite tuple of variables. The *ring of internal polynomials over* K is defined:

$$K[\bar{x}]_{int} := \prod_I K[\bar{x}]/\mathcal{U}.$$

An *internal ideal* of $K[\bar{x}]_{int}$ is an internal subset such that almost every (with respect to $\mathcal{U}$) component is an ideal of $K[\bar{x}]$. Show that an internal ideal is an ideal.

2 Do at least two of the following problems:

4. (2 points)
   1. Let s be the successor function on $\mathbb{Z}$. Show that the theory of $(\mathbb{Z}, s)$ has quantifier elimination.
   2. Consider the structure $\mathbb{Z}$ in the language $+, 0, 1$ along with predicates P_n which cut out the set of elements divisible by n (for each $n \in \mathbb{N}$). Show that the theory of this structure has quantifier elimination.

5. (2 points) A structure has definable Skolem functions if its theory has definable Skolem functions. Consider the structure $\mathbb{N}$ considered as a structure in the language $(<, +, *, 0, 1)$. Show that $\mathbb{N}$ has definable Skolem functions.

   Recall that a theory T has definable Skolem functions if for any formula $\varphi(\bar{x}, y)$, there is a definable function f such that

$$T \models \forall \bar{x}(\exists y \varphi(\bar{x}, y) \to \varphi(\bar{x}, f(\bar{x})).$$

6. (2 points) Let $\mathcal{M}$ be an elementary substructure of $\mathcal{N}$. Let $A \subset \mathcal{M}$ be a subset. Show that $acl_{\mathcal{M}}(A) = acl_{\mathcal{N}}(A)$. Give an example showing that this is not true for $M \subset N$ even under the assumption that $\mathcal{M} \equiv \mathcal{N}$.