HW5, Math 506, Fall 2016, Due 10/7

September 29, 2016

1. (2 points) Recall that if $\mathcal{M}$ is a structure in a language including the binary relation $<$ interpreted as a linear order on $\mathcal{M}$, we say that $\mathcal{M}$ is o-minimal if every definable subset of $\mathcal{M}$ is a boolean combination of points and intervals (whose endpoints are in $\mathcal{M}$). Consider the topology generated by open boxes on $\mathcal{M}^n$ for $n \in \mathbb{N}$. Let $X \subset \mathcal{M}^n$ be definable. Show that the closure of X and the interior of X are definable.

2. (2 points) Assume $\mathcal{M}$ is o-minimal. Let $X \subseteq \mathcal{M}^{n+m}$. For $\bar{a} \in \mathcal{M}^m$, let $X_{\bar{a}} = \{\bar{b} \in \mathcal{M}^n \mid (\bar{b}, \bar{a}) \in X\}$. Show that $\{\bar{a} \mid X_{\bar{a}} \text{ is open}\}$ is a definable set.

3. (3 points) Let $\mathcal{G}$ be an o-minimal ordered group.

    1. Show that the only definable subgroups of $\mathcal{G}$ are $1_{\mathcal{G}}$ and $\mathcal{G}$.
    2. Show that $\mathcal{G}$ is abelian (use part 1).
    3. Show that $\mathcal{G}$ is divisible (use part 1).

4. (2 points) Fix a nonprinciple ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let k be a field. Recall from the previous homework, the ring of internal polynomials, denoted $k^*[\bar{x}]_{int}$. Let S be an internal set. Let $(S)_{int}$ denote the internal ideal whose r^{th} component is given by the ideal generated by the r^{th} component of S. That is, the r^{th} component is given by (S_r). Show that $(S)_{int}$ is contained in any ideal of $k^*[\bar{x}]_{int}$ which contains S.

5. (2 points) Let $(\sqrt{S})_{int}$ denote the internal ideal whose r^{th} component is given by the radical ideal generated by the r^{th} component of S. That is, the r^{th} component is given by $\sqrt{S_r}$. Show that $(\sqrt{S})_{int}$ is contained in any internal radical ideal of $k^*[\bar{x}]_{int}$ which contains S.

6. (1 point) Show that if $tp(a/A) = tp(b/A)$ then $tp(a/dcl(A)) = tp(b/dcl(A))$.