HW6, Math 506, Fall 2016, Due 10/21

October 20, 2016

1. (3 points) Let T be the theory of $\mathbb{Z}$ in the signature $\{s\}$ where s is interpreted as the sucessor function. What are the principle types in $S_n(\emptyset)$?

2. (3 points) Let T be the theory of $\mathbb{Z}$ in the signature $\{s, <\}$ where s is interpreted as the sucessor function and $<$ is the linear order. What are the principle types in $S_n(\emptyset)$?

3. (3 points) Let $\bar{a}, \bar{b} \in \mathcal{M} \models T$. Show that $tp(\bar{a}, \bar{b}/\emptyset)$ is principle if and only if $tp(\bar{a}/\emptyset)$ is principle and $tp(\bar{b}/\bar{a})$ is principle.

4. (3 points) Give an example for which $tp(\bar{b}/\emptyset)$ is principle and $tp(\bar{a}/\emptyset)$ is principle, but $tp(\bar{a}, \bar{b}/\emptyset)$ is not principle.

5. (3 points) We say that $\mathcal{M} \models T$ is minimal if $\mathcal{M}$ has no proper elementary submodels. Give an example of a theory with a minimal model which is infinite.

   Give an example of a theory with a prime model which is not minimal.

6. (10 points) Suppose T has a prime model which is not minimal, $\mathcal{M}$. Assume T is countable.

   1. Show that there is some elementary embedding $f : \mathcal{M} \to \mathcal{M}$ so that $f(\mathcal{M}) \subsetneq \mathcal{M}$.
   2. Show that there is $\mathcal{N}$ with $\mathcal{M}$ an elementary submodel and $\mathcal{M}$ isomorphic to $\mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$.
   3. Build a chain of elementary submodels $\mathcal{M}_i$ for $i \in \mathbb{N}$ with $\mathcal{M}_i$ isomorphic to $\mathcal{M}$. Conclude that the union is isomorphic to $\mathcal{M}$.
   4. Now build an elementary chain of size ω_1 so that $\mathcal{M}_\alpha$ is isomorphic to $\mathcal{M}$ for all $\alpha \in \omega_1$ and $\mathcal{M}_\alpha \subsetneq \mathcal{M}_{\alpha+1}$. Show that $\mathcal{M}'$ is atomic and size $\aleph_1$.
   5. Now, if T is not $\aleph_0$-categorical, then T has a non-atomic model of size $\aleph_1$.
   6. Prove that if T is $\aleph_1$-categorical and not $\aleph_0$-categorical, then prime models are minimal.

7. (6 points) 1. Let $R \subset S$ be commutative rings. The following are equivalent:
   - For any equation, $\sum_{i=1}^{l} f_i y_i = 0$, $f_i \in R$, the solutions in S^l are S-linear combinations of the solutions in R^l.

- For any system of equations: $\sum_{i=1}^{l} f_{ji} y_i = 0$, for $j = 1, \dots, k$, $f_{ji} \in R$, the solutions in S^l are S-linear combinations of the solutions in R^l.
- S is flat over R.

Show that $k^*[\bar{x}]_{int}$ is a flat extension of $k^*[\bar{x}]$. Note that we are using the notation of the previous homework.

2. Given $n, d, k \in \mathbb{N}$, show there is $\alpha \in \mathbb{N}$ such that for any field k and any system $\sum_{i=1}^{l} f_{ji} y_i = 0$, for $j = 1, \dots, k$ where $f_{ji} \in k[\bar{x}]$, $deg(f_{ji}) \leq d$, the submodule of $k[\bar{x}]^l$-solutions is generated by solutions $(g_1, \dots, g_l)$ where the g_i have degree at most α.