HW7, Math 506, Fall 2016, Due 11/4

October 24, 2016

Throughout the assignment, assume T is a complete theory in a countable language which has infinite models.

1. (5 points) Recall the Keisler-Shelah theorem, which says that if

$$f_T(\kappa) := sup\{|S_1(\mathcal{M})| : \mathcal{M} \models T, |\mathcal{M}| = \kappa\},$$

then $f_T(\kappa)$ is one of the following:

$$\kappa, \kappa + 2^{\aleph_0}, ded(\kappa), ded(\kappa)^{\aleph_0}, 2^\kappa,$$

where

$$ded(\kappa) := sup\{|I| : I \text{ is a linear order with a dense set of size } \kappa\}.$$

Give specific examples which show each of these possibilities can occur.

2. (1 point) Show that $ded(\kappa) \leq 2^\kappa$.

3. (5 points) Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Let $(\mathcal{M}_i)_{i \in \mathbb{N}}$ be $\mathcal{L}$-structures and let $\mathcal{M}^* = \prod \mathcal{M}_i/\mathcal{U}$. Let $A \subseteq \mathcal{M}^*$ be a countable set. For any $a \in A$, pick some $f_a \in \prod \mathcal{M}_i$ for which $a = f_a/\sim_{\mathcal{U}}$. Let $\Gamma(v) = \{\varphi_i(v) : i \in \mathbb{N}\}$. be any set of $\mathcal{L}_A$-formulas so that $\Gamma(v) \cup Th_A(\mathcal{M}^*)$ is satisfiable. Let $D_i = \{n \in \mathbb{N} \mid \mathcal{M}_n \models \exists v \varphi_{i,n}(v)\}$ where we form $\varphi_{i,n}$ from φ_i using the map $a \mapsto f_a$. Specifically, replace each instance of a with $f_a(n)$ for each $a \in A$ to form $\varphi_{i,n}$ from φ.

   - Show that $D_i \in \mathcal{U}$.
   - Show that there is $g \in \prod \mathcal{M}_i$ so that when $i \leq n$, and $n \in D_i$,

   $$\mathcal{M}_n \models \varphi_{i,n}(g(n)).$$

   - Show that g realizes the partial type $\Gamma(v)$.
   - Explain why $\mathcal{M}^*$ is $\aleph_1$-saturated.

4. (3 points) Show that if $\mathcal{M}$ is an infinite $\mathcal{L}$-structure, there is a descending chain of elementary extensions $\mathcal{N}_n$ of $\mathcal{M}$ so that $\cap \mathcal{N}_n = \mathcal{M}$.

5. (3 points) (requires some set theory) Let κ be an uncountable regular cardinal. Let $\mathcal{M}$ be an $\mathcal{L}$-structure with $|\mathcal{M}| = \kappa$. Suppose that $\mathcal{M} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$ with $|\mathcal{M}_\alpha| < \kappa$. Show that

$$\{\alpha < \kappa \mid \mathcal{M}_\alpha \prec \mathcal{M}\}$$

is closed and unbounded.

6. (6 points) A module M over a commutative ring R is *faithfully flat* if M is flat over R and for any R-module N, if $M \otimes_R N = 0$ then it must be the case that $N = 0$.

    1. Show that a ring $S \supset R$ is faithfully flat iff $\mathfrak{m}S \neq S$ for any $\mathfrak{m} \subset R$ a maximal ideal iff S is flat and any system

    $$\sum_{i=1}^{l} f_{ji} y_i = g_j$$

    for $j = 1, \ldots, k$ with $f_{ji}, g_j \in R$ with a solution in S^l has a solution in R^l.

    2. Show that $k^*[\bar{x}]_{int}$ is a faithfully flat extension of $k^*[\bar{x}]$.