Short overview of papers of Shmuel Friedland organized by subjects

1. *Elasticity*: 1,2, 5.

These papers were done while I was an undergraduate and graduate student, working part time for Y. Stavsky, Professor in Aeronautics Department in Technion. I worked on obtaining exact solutions to ODE in theory of bucklings of shells.

2. *Theory of Analytic Functions of One Complex Variable and Related Subjects*: 3, 4, 7, 11, 12, 20, 22, 27.

These papers deal mostly with various coefficient problems of certain families of analytic functions of one complex variable.

In 3 (my Master thesis) I proved the Robertson conjecture for $n = 4$, which implies the Bieberbach conjecture for $n = 4$ (verified first by Z.Charzynski and Schiffer). It turns out that this is a special case ($n = 4$) of Milin's inequality, which were proved in full by L. de Brange a few years later. (They yield the Bieberbach conjecture.)

Jointly with Z. Nehari we generalized in 4 Nehari's famous condition for the univalence. This work inspired me to consider problems in 8 (my D.Sc. thesis).

Jointly with D. Aharonov we proved in 7, 11, the sharp cofficient bounds for the functions of bounded boundary rotations.

Jointly with M. Schiffer we obtained in 20, 22 some global results on the coefficients of univalent functions using control theory. These results were recently discussed in 1998 Ph.D. thesis of Oliver Roth, Bayerischen Julius-Maximilian-University, Wurzburg, under the title the Friedland-Schiffer equation. (See also O. Roth, Pontryagin maximum principle in geometric function theory, *Complex Variables* 41 (2000), 391-426.) I extended these ideas to some global principle in control theory and differential games in 27.

3. *Nonoscillation and Disconjugacy*: 9, 16, 18.

In the Memoir 18 I treat in a uniform way the classical notions of nonoscillation and disconjugacy as certain variational problems. I derive the corresponding Euler-Lagrange equations and show that they possess a solution. I found also some new sharp constants. My constants were shown to yield the uniqueness for nonlinear multipoint boundary-value problems. C.J. Amick: *J. London Math. Soc.* 21(1980), 304-310.

4. *Eigenvalue Problems and Eigenvalue Inequalities*: 8, 14, 17, 19, 29, 30, 32, 42, 44, 50, 51, 62, 65, 71, 74, 75, 84, 86, 95, 96, 98, 101, 103, 106, 114, 122, 124, 128, 135.

Jointly with S. Karlin we give some basic inequality in 14 for the spectral radius of a nonnegative matrix. This inequality turned out to be fundamental in many pure and applied areas. It can be viewed as a discrete explicit version of the famous Donsker-Varadhan formula, *Proc. Acad. Sci. USA* 72(1975), 780-783.

In 17 together with R. Loewy we find the minimal dimension for any subspace of $n \times n$ real symmetric matrices to contain a nonzero matrix whose first eigenvalue is of multiplicity m at least. This result gives a proof to an old theorem of Bohnenblust. Recently it was used by L. Lovácz: Semidefinite programs and combinatorial optimization, Lecture Notes 2001.

Jointly with W. Hayman we generalize in 19 Ahlfors theorem to subharmonic functions in any number of variables using some eigenvalue inequalities. These inequalities are obtained by using the symmetrization principle. Recently this paper became of interest to researchers in PDE, e.g. Caffarelli, Luis A.; Jerison, David; Kenig, Carlos E. Some new monotonicity theorems with applications to free boundary problems. Ann. of Math. (2) 155 (2002), no. 2, 369–404. (My paper 19 has 6 reference citations and 1 review citation in MathSciNet.)

In 24, 29, 32 I show that the extremal solutions to many eigenvalue problems satisfy the bang-bang principle.

In 44, 50, 84, 86 we bound the variation of certain spectral functions using norms.

In a joint paper with J. Cohen, T. Kato and F. Kelly, we derive in 51 various versions of the Golden-Thompson inequalities (in statistical mechanics). This paper was used in the recent work of C. O'Cinneide: Markov additive processes and Perron-Frobenius eigenvalue inequalities., Ann. Probab. 28 (2000), 1230–1258.

Together with my former Ph.D. student G. Porta we give in 136 a version of Golden-Thompson inequality for self adjoint operators.

In 62, 75, 96, 98, 103, 122 we derive some inequalities for the first eigenvalue of 0-1 matrices which correspond to the first eigenvalue of certain graphs. Some of these inequalities are sharp. 62 has found a recent application in J. Snellman: Bounds for the entropy of graded algebras, *arXiv:math.RA/0209080* v2 10 Sep 2002. (See also J. Snellman, The maximal spectral radius of a digraph with $(M+1)^2 - S$ edges, *Elec. Lin. Alg.* 10 (2003), 179-189.)

In 95, 101, 114, 128 we discuss the upper bounds for the second eigenvalue of non-negative matrices, which is of cardinal importance in the rate of convergence of Markov chains. (The Cheeger inequality.)

Jointly with L. Elsner we give in 106 a generalization of T. Kato inequality (for self adjoint operators) to normal operators.

In 124 I generalize Klyachko's theorem, which solved the famous Horn conjecture, to hermitian matrices and to selfadjoint, positive, compact operators. My result on hermitian matrices was improved substantially by W. Fulton: Eigenvalues of majorized Hermitian matrices and Littlewood-Richardson coefficients, *Linear Algebra Appl.* 319 (2000), 23-36.

5. *Inverse Eigenvalue Problems* (IEP): 6, 15, 21, 23, 26, 33, 70, 73, 94

In 21, generalizing the results of 6 and 15, I showed how to combine the methods of algebraic geometry with matrix theory to produce the fundamental paper on IEP, which was cited many times. (Reference citations 3 and Review citations 4 in MathSciNet.) An exposition of some of these results were given by J.C. Alexander: Matrices, eigenvalues and complex projective spaces, *Amer. Math. Monthly* 85(1978), 727-733.

In 26 I study systematically the IEP for nonnegative and eventually nonnegative matrices. A complete solution of this problem was given by M. Boyle and D. Handle: The spectra of nonnegative matrices via symbolic dynamics, *Annals of Math.*, 133(1991), 249-316, for which they received the second H. Schneider prize.

In 70, 73 together with Nocedal and Overton we continue to apply the ideas of 21 to obtain variuos numerical methods to solve IEP.

In 94 I treat the IEP for symmetric Toeplitz matrices using complex and real algebraic geometry. My methods were applied successfully by J.H. Landau: The inverse eigenvalue problem for real symmetric Toeplitz matrices, *Journal of AMS* 7(1994), 749-767, to prove the long standing conjecture that any real spectrum is realized by a real symmetric Toeplitz matrix.

6. *Permanents and Applications*: 10, 28, 31, 35, 46, 76, 78, 157, 158.

In 35 I give the correct exponential lower bound for the permanents of doubly stochastic matrices. This result enabled me to solve one of the Erdös-Rényi conjectures. My lower bound was improved by Egorichev and Falikman in 1980 by proving the van der Waerden conjecture. I used their methods to prove the Tverberg conjecture in 46.

In 76 together with Li and Schneider we discuss additive decomposition of nonnegative matrices with applications to scaling and permanents. It turned out that these ideas were used by N. Linial, Samorodnitsky and A. Wigderson: A deterministic strongly polynomial algorithm for matrix scaling and approximate permanents, *Combinatorica* 20(2000), 545-568, to study the complexity of permanent computation of certain nonnegative matrices.

In 78 I use the van der Waerden conjecture to give a simple proof that any 7-regular digraph contains an even cycle. This result is cited in R.L. Graham, M. Grotschel and L. Lovácz: Handbook of Combinatorics, North-Holland, 1996. C. Thomassen: The even cycle problem for directed graphs, *Journal of AMS* 5(1992), 217-229, proved Lovász conjecture that any 3-regular digraph contains an even cycle.

Together with Gurvits we give lower bounds for partial matchings in regular bipartite graphs in [157]. These results can be viewed as an analog of the famous A. Schrijver lower bound on counting 1-factors in regular bipartite graphs. We show that these results are asymptotically sharp in certain cases. We give applications to the monomer-dimer entropy. Further improvements of these results are given in [158].

7. *Invariants of Classes of Matrices and Applications*: 36, 38, 39, 40, 43, 47, 53, 54, 63, 67, 85, 89.

In 36, 39 and 40 I studied problems of analytic similarity of matrices, which originated from the work of W. Wasow: "Asymptotic Expansions for Ordinary Differential Equations", Krieger, N.Y. 1976. I generalized many known results.

Jointly with N. Moiseyev we applied these ideas and methods to certain problems of resonanse states in Molecular Physics in 38 and 43. These results were successfuly continued by Moiseyev and his group in theoretical chemistry.

In 54 I treated the notorious "wild problem" of simultaneous similarity of matrices considered by many outstanding mathematicians, e.g. I. Gelfand: I.M. Gelfand and V.A. Ponomarev, Remarks on the classification of a pair of commuting linear transformations in a finite dimensional vector space, *Functional Anal. Appl.* 325-326; I.M. Gelfand, The cohomology of infinite dimensional Lie algebras, some questions of integtal geometry, in "*Actes, Congrés Intern. Math.*", vol. 1 (1970),95-111.

I showed that this wild problem can be reasonably classified by using algebraic geometry.

In 63 and 67 I showed how to apply the methods of 54 to find the invariants and the canonical forms which appear in the standard (linear) models of control theory.

8. *Stability Problems in Matrices*: 25, 45, 52, 56, 57, 59, 112, 144.

Together with Michelli we discuss in 25 comparison theorems for countable number of difference equations and apply then to splines. In a joint work with de Boor and Pinkus we prove in 52 Michelli conjecture on spline interpolation.

In 45, 56, 57, 59 we discuss the stability criterias for sets of matrices, which are of fundamental importance in the theory of (stable) convergent schemes of hyperbolic equations. The most remarkable paper of these four papers is a joint paper with Zenger (57), which shows that a unit ball of the spectrally dominant norm is a stable set.

Together with Elsner we give in 112 necessary and sufficients conditions for convergence of all infinite products in two given matrices. This condition is important in wavelets.

In [144] I give a simple necessary condition for convergence of products of matrices to rank one matrix in the projective space.

9. *Nonnegative Matrices and Operators*: 37, 41, 58, 66, 72, 77, 83, 88, 91, 93, 102, 108, 123, 126.

In 37, 66, 77, 91 we discuss various properties of powers of nonnegative and related matrices.

In 41 I describe convex spectral functions of hermitian, nonnegative and totally positive matrices.

In 88 I give an elegant characterization of the spectral radius of any nonnegative operator, with respect to the natural cone of nonnegative self adjoint elements of C^* algebra. See 83 and 93 for the finite dimensional version of this theorem.

Together with L. Elsner we give in 108 a certain inequality for the Hadamard square of the matrix, which has applications to graphs, permanents and eigenvalue perturbations.

Jointly with L. Elsner we generalize in 123 and 126 some classical results on nonnegative, asymetric, infinite Toeplitz matrices to nonnegative, asymetric, infinite, block Toeplitz matrices.

10. *Topological and Algebraic Geometry Methods in Matrices*: 55, 68, 119, 129, 131.

In a joint work with Robbin and Sylvester we give in 55 the crossing rule: the precise version of the famous (heuristic) Wigner-von Neumann noncrossing rule. It was very well received by theoretical physicists and by mathematicians working in stability of hyperbolic systems.

Together with Berger we extend in 68 some of the results in 55. Further generalizations were given by K.Y. Lam and P.Y.H. Yiu: Sums of squares formulae near Hurwitz-Radon range, Contemporary Math. 58, II(1987), 51-56.

In 119 and 129 we discuss the minimal dimension of subspaces of $n \times n$ symmetric matrices containing a nonzero matrix of rank m at most. In 129 we use results of Harris-Tu on determinantal varieties to find the above minimal dimension for a given m and an infinite family of n (depending on m). This result genearlizes the famous result in P. Lax: The mulltiplicity of eigenvalues, *Bull. Amer. Math. Soc.* 6(1982), 213-215, for $m = 2$.

Jointly with A. Libgober in 131 we answer an open case in 129 using the tools of algebraic geometry.

11. *Algebraic and Number Theoretic Problems in Matrices and Applications to Graphs and Graph Isomorphism Problem*: 60, 61, 64, 80, 81, 90, 99, 104, 115, 137, 153, 154.

In 61, together with Alon and Kalai, we show (using Chevalley theorem) that every 4-regular graph plus one edge contains a 3-regular subgraph. This is a variation of the Berge-Sauer conjecture proved by V.A. Tashkinov: Regular subgraphs of regular graphs, *Soviet Math. Dokl.* 26(1982), 37-38. More general results are given in 60.

In 80 and 90 I discuss the connections between the coherent algebras and quadratic forms respectively and the graph isomorphism problem (g.i.p.). The complexity of g.i.p. is one of the main unclassified problems in theoretical computer science.

In 99 I give natural conditions for rational orthogonal similarity of rational symmetric matrices. It was generalized by W.C. Waterhouse: Orthogonal similarity and pairs of quadratic forms, *Linear Algebra Appl.* 231(1995), 175-179.

Together with Brualdi and Pothen we solve in 104 the sparse basis problem (of importance in computer science) using multilinear algebra.

In 115 I show that for $n >> 1$ the monomial group has the maximal order among all finite subgroups of $GL(n, \mathbf{Q})$. This result was generalized by W. Feit, The orders of finite linear groups, preprint 1995. This result is discussed in J. Kuzmanovich and A. Pavlichenkov, Finite groups of matrices whose entries are integers, *Monthly*, 109(2002), 173-186.

Together with Kim, Peled, Pless and Perepelitsa we consider explicit constructions of LDPC Ccodes which are serious contenders to Turbo codes. The constructions of these codes are geometric-algebraic. The proofs that these codes have a large weight are number theoretical.

In [153], together with C. Krattenthaler we bound the the exponent of 2, which divides certain ratios of products of factorials. We solve the conjecture stated in [129], and apply to the dimension of subspaces of matrices containing a nonzero matrix of maximal prescribed rank.

In [154], together with Bhattacharya and Peled we characterize the extreme points of the polytope of dual degree partitions.

12. *Discrete Groups and Ergodic Theory*: 97, 107, 109, 116, 132-134, 141.

In 97 together with my an informal Ph.D. student S. Hersonsky we generalize Jorgensen's inequality to discrete groups in normed algebras.

In 107 I discuss the connection between Auslander's conjecture and invariant measures on groups of homeomorphisms.

In 109 I study discrete groups of unitary isometries in finite dimensional vector spaces and their applications to lower bounds on radii of balls in hyperbolic manifolds.

In 116 I discuss properly discontinuous groups on certain matrix homogenous spaces. Some of my results were generalized by Y. Benoist: Action propres sur les espace homogenes reductifs, *Annals Math.* 144(1996), 315-347.

Together with my former Ph.D. student P. Freitas in 132-134 we make a fundamental study of Busemann compcatifications on the homogeneous space $GL(n, C)/U_n$ and of discrete subgroups of biholomorphisms of Siegel upper half plane.

In 141 together with B. Weiss we study generalized inrterval exchanges and apply these results to new versions of the 2-3 conjecture of H. Furstenberg.

13. *Dynamical Systems*: 79, 87, 92, 105, 110, 113, 117, 118, 121, 130, 138, 139, 156.

The joint work with J. Milnor 79 stimulated many works on the subject of complex dynamics of plane polynomial diffeomorphisms. It has 17 references citations and 12 citations in MathSciNet. Bedford and Smillie have produced at least 7 papers on this subject, e.g. E. Bedford, M. Lyubich and J. Smillie, Polynomial differomorphisms of $\mathbf{C}^2$ IV: The measure of maximal entropy and laminar currents, Inventiones Math. 112(1993), 77-125.

In 87, 92, 105, 110, 156 I discuss the notion of entropy for various maps arising in the context of algebraic geometry: rational maps, multivalued algebraic maps, groups and semi-groups of finitely generated algebraic maps. The connecting theme in these papers is the use of graphs. In 87 I showed that for the self holomorphic maps of smooth projective variety to itself the entropy is given by the spectral radius of the induced map on the homology. In 105 I extend this result to holomorphic self maps of compact Kähler manifolds. These results show the validity of the entropy conjecture for these maps in the tightest form. (M. Shub, Dynamical systems, filtrations and entropy, *Bulletin AMS* 80(1974), 27-41.) This result is mentioned in C.T. McMullen: Dynamics on K3 surfaces, Salem numbers and Siegel disks, J. Reine Angew. Math. 545 (2002), 201-233.

In 113 I discuss in detail the use of matrices to calculate the entropy of $\mathbf{Z}^d$ subshifts of finite type. In particular I prove that $d-1$ symmetries imply that the topological entropy is equal to the periodic entropy. This result has a siginificant implication to some problems of statistical mechanics, even for the case $d = 2$: E.H. Lieb: Residual entropy of square ice, *Phys. Rev.* 162(1967), 162-172. Surprisingly, this topic became fashionable in IEEE circles under the name multi-dimensional capacity. I gave two hours lecture on this topic in the recent MTNS-02 conference August 2002 in Notre Dame Univ. A survey paper 130, which also contains many new resutls on this subject, will appear in the proceeings of this conference. Together with my colleague Uri Peled we compute in 138-139 the famous entropies of two and three dimensional monomer-dimer configurations with high precision.

In 117 I show how to use some results in matrices to compute the Hausdorff dimension of subshifts, which yields a generalization of the Mauldin-Williams formula: R.D. Mauldin and S.C. Williams, Hausdorff dimension in graph directed constructions, *Trans. AMS* 309(1988), 811-829. I apply these results to compute the Hausdorff dimension of the limit sets of finitely generated free groups of isometries of inifinite trees.

In a joint paper with Ochs 118 we discuss in detail the existence of Gibbs measures whose Hausdorff dimension is equal to the Hausdorff dimension of the nonwandering set. Some results of this paper are improved in recent works of C. Wolf, e.g.: Dimension of Julia sets of polynomial automorphisms of C^2, *Michigan Math. J.* 47 (2000), 585-600.

In 121 I introduce the notion of discrete Lyapunov exponent for certain subshifts of finite types. I show that the Hausdorff dimension (with respect to a suitable metric) has a variational characterization analogous to the characterization for complex hyperbolic polynomial maps. This characterzation yields a lower bound for the Hausdorff dimension which is sharp in some clasical examples, e.g. geometrically finite Kleinian groups.

14. *The limit of certain matrices and operators:* 100, 136.

In s joint work with W. So 100 we show that certain one parameter family of hermitian matrices have a limit.

In a joint work with my former Ph.D. student G. Porta in 136 we extended the above results to one parameter family of operators in two selfadjoint, positive, compact operators.

15. *The Jacobian conjecture:* 120, 125

In 120 I published the results of my work on the Jacobian conjecture. (It is one of the suggested problems for 21 century by S. Smale: *Math. Intelligencer* 20 1998), #2.) It deals with the monodromy action on the affine algebraic curves, Gauss-Manin connection and their role for polynomial maps with constant nonzero Jacobian.

In 125 I show that the Jacobian conjecture is equivalent to finiteness of certain cohomologies.

16. *Random Matrices:* 135.

Together with B. Ryder and O. Zeitouni we show that the permanent of positive matrices with entries in $[a, b]$ can be efficiently computed for large n, using the determinants of random matrices with corresponding independent Gaussian entries.

17. *Singular Value Decomposiiton and Genomics:* 143, 145-147, 152, 155.

In 143 I study theoretical and numerical aspects of singular value decomposition with applications to DNA microarrays. This study is continued in joint works with Torokhti and Howlett in [152] and [155]. In [146], together with Kaveh, Niknejad and Zare we give an algorithm for fast Monte-Carlo low rank approximations for matrices.

In 145, together with L. Chihara and A. Niknejad we suggest a new algorithm to complete the missing entries in DNA microarrays. A variant of this algorithm is considered in [147].