On Cheeger types inequalities for weighted graphs

Shmuel Friedland
University of Illinois at Chicago
Department of Mathematics
851 S. Morgan St., Chicago, Illinois 60607-7045
USA

Reinhard Nabben
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
33 501 Bielefeld
Germany

Abstract

We give several bounds on the second smallest eigenvalue of the weighted Laplacian matrix of a finite graph and on the second largest eigenvalue of its weighted adjacency matrix. We establish relations between the given Cheeger-type bounds here and the known bounds in the literature. We show that one of our bounds is the best Cheeger-type bound available.

1 Introduction

Let $G = (V, E(G))$ be a finite, undirected, simple, connected graph on n vertices $< n >:= \{1, \dots, n\}$. (Loops are not allowed.) We will call such G a *graph* G. The Laplacian matrix $L = [l_{i,j}]$ of G is defined by

$$l_{i,j} \quad = \quad \begin{cases} \delta_i & \text{if } i = j, \\ -1 & \text{if} (i,j) \in E(G), \\ 0 & \text{otherwise}, \end{cases}$$

where δ_i is the degree of vertex i. With $\Delta = diag(\delta_1, \dots, \delta_n)$, the matrix $A = \tilde{\Delta} - L$ is called the adjacency matrix of G. Let

$$\begin{aligned} 0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n, \\ \mu_1 \leq \mu_2 \leq \dots \leq \mu_n, \end{aligned}$$

be the eigenvalues of L and A respectively. The spectra of both of these matrices can be used to obtain several information about the graph, e.g. estimates for the diameter ot the graph (see the surveys by Merris [Me] and Mohar [Mo3]). In particular estimates or bounds for λ_2 and μ_{n-1} are of great interest. Recently, discrete analogs of Cheeger's [Che] fundamental inequality for λ_2 of the Laplacian on the compact Riemannian manifold are established ([A], [D], [DK], [L], [Mo1], [Chu]). In these works, several discrete versions of Cheeger constants are used. We bring three versions of Cheeger constants. Let the Cheeger constant $h(G)$ of a finite graph G be:

$$h(G) = \min_{\emptyset \neq U \subset V} \frac{|E(U,\bar{U})|}{\min(|U|,|\bar{U}|)} = \min_{\substack{\emptyset \neq U \subset V \\ 0 \neq |U| \leq \lfloor \frac{n}{2} \rfloor}} \frac{|E(U,\bar{U})|}{|U|}, \tag{1.1}$$

where $E(U,\bar{U})$ are the edges connecting vertices in U with vertices in $\bar{U}$, the complement of U. The constant $h(G)$ is called the isoperimetric number of G by Mohar [Mo1] and the modified Cheeger constant by Chung in [Chu] (p.34).

Let

$$i(G) = \min_{\emptyset \neq U \subset V} \frac{|E(U,\bar{U})|}{\min(|E(U)|,|E(\bar{U})|)} = \min_{\substack{\emptyset \neq U \subset V \\ |E(U)| \leq \frac{1}{2}|E(V)|}} \frac{|E(U,\bar{U})|}{|E(U)|}, \tag{1.2}$$

where $|E(U)| = \sum_{i\in U} \delta_i$. The constant $i(G)$ is called the transition isoperimetric number of G by Mohar [Mo2] and the Cheeger constant of G by Chung in [Chu]. Obviously one has

$$\underline{\delta} i(G) \leq h(G) \leq \bar{\delta} i(G), \tag{1.3}$$

where

$$\underline{\delta} = \min_i \delta_i \quad \text{and} \quad \bar{\delta} = \max_i \delta_i.$$

Friedland used in [Fr] a slightly different constant

$$\epsilon(G) = \min_{\substack{\emptyset \neq U \subset V \\ |U| \leq \lfloor \frac{n}{2} \rfloor}} \frac{|E(U,\bar{U})|}{|E(U)|}. \tag{1.4}$$

It is straightforward to see that

$$\epsilon(G) \leq i(G). \tag{1.5}$$

Furthermore

$$\underline{\delta}\epsilon(G) \leq h(G) \leq \bar{\delta}\epsilon(G). \tag{1.6}$$

For δ-regular graphs, i.e. graphs for which the degree of all vertices is δ, the three constants satisfy

$$h(G) = \delta i(G) = \delta \epsilon(G).$$

All these constants have a lot of various interpretations and applications. They can serve as measures of the connectivity of graphs and are clearly related to the so-called bisection width problem. (see Mohar [Mo1] for more applications). It seems that the computation of $h(G), i(G), \epsilon(G)$ are not polynomial in n. These constants can be bounded by λ_2 or μ_{n-1}. We mention some of these (Cheeger type) inequalities. Alon proved in [A] (see also [D]) the following inequality

$$\lambda_2 \geq \frac{1}{2\bar{\delta}} h(G)^2. \tag{1.7}$$

The related inequality using $i(G)$ instead of $h(G)$ is proved in [DK]. From results by Mohar [Mo1] (see also [BZ]) we obtain

$$\lambda_2 \geq \bar{\delta} - \sqrt{\bar{\delta}^2 - h(G)^2}, \tag{1.8}$$

if $G \neq K_1, K_2, K_3$. (K_n is the complete graph on n vertices.) Note that the (1.8) improves (1.7). Moreover Mohar [Mo1] showed

$$\mu_{n-1} \leq \sqrt{\bar{\delta}^2 - h(G)^2}. \tag{1.9}$$

One can extend the above results to positively weighted graphs (possibly with loops). Let G be a graph. Let $w_{i,j} > 0$ for $i \neq j$ be the weight of the edge $(i,j) \in E(G)$, where $w_{i,j} = 0$ if $i \neq j$ and $(i,j) \notin E(G)$. Let $w_{i,i} \geq 0$ be the weight of the loop (i,i). (According to our definition $(i,i) \notin E(G)$. Thus an undirected, simple, connected graph with loops G' is G_w for a corresponding graph G with $0-1$ weights.) Define the reduced weighted degree δ_i and the weighted degree $\tilde{\delta}_i$ of vertex i as

$$\delta_i = \sum_{(i,j)\in E(G)} w_{i,j},$$
$$\tilde{\delta}_i = \tilde{\delta}_i + w_{i,i}.$$

The weighted Laplacian $L_w = [l_{i,j}]$ is then given by

$$\begin{aligned} l_{i,j} &= -w_{i,j} \quad \text{for} \quad i \neq j \\ l_{i,i} &= \delta_i. \end{aligned}$$

Similarly we define $A_w := \tilde{\Delta}_w - L_w$, where

$$\tilde{\Delta}_w = diag(\tilde{\delta}_1, \ldots, \tilde{\delta}_n)$$
$$\Delta_w = diag(\delta_1, \ldots, \delta_n).$$

It is worth to mention that weighted loops do not influence L_w in contrast to A_w. (This is the standard convention, e.g. [Chu], §1.4. A different definition of the Laplacian of G_w, which takes in account weighted loops, is given in [Fr].) Hence when studying the eigenvalues of the Laplacian matrix we may assume that $w_{i,i} = 0$, i.e. the weighted graph is loopless. That is the reason we introduced the reduced degree δ_i of vertex i of G_w. With

$$|E(U)| = \sum_{i \in U} \delta_i \quad \text{and} \quad |E(U, \bar{U})| = \sum_{i \in U, j \notin U} w_{i,j}$$

one can define the constants $h(G_w)$, $i(G_w)$ and $\epsilon(G_w)$ as in (1.1), (1.2), and (1.4) for the weighted graph G_w.

Friedland established in [Fr] that for any diagonal matrix $D = diag(d_1, \ldots, d_n)$ with $d_i > 0$,

$$\lambda_2(D^{-1}L_w) \geq \min_i \frac{\delta_i}{2d_i} \epsilon(G_w)^2.$$

Recently, Berman and Zhang proved in [BZ] that

$$\lambda_2(D^{-1}L_w) \geq \frac{1}{\max_i d_i}(\bar{\delta} - \sqrt{\bar{\delta}^2 - h(G_w)^2}). \tag{1.10}$$

Another constant for weighted graphs is introduced by Fiedler in [Fi].

$$g(G_w) = \min_{\emptyset \neq U \subset V} \frac{|E(U, \bar{U})|}{|E(U)||E(\bar{U})|}. \tag{1.11}$$

This constant is called in [Fi] the *averaged minimal cut* of G. Fiedler proved that

$$\lambda_2(L_w) \geq 2g(G_w). \tag{1.12}$$

It seems that $g(G_w)$ is not related to a discrete version of Cheeger's inequality. Indeed, assume that G is any connected δ-regular graph on n vertices. Choose in the right hand side of (1.11) $U = \{i\}$. Then

$$g(G) \leq \frac{1}{\delta(n-1)}.$$

Thus (1.12) fails to detect expanders for which discrete Cheeger constants were introduced originally. (Expanders are an infinite family of δ-regular graphs whose second eigenvalues of the Laplacian is bounded below by a fixed $a > 0$, e.g. [A]).

In the previous paper [FN] the authors gave the following bounds for λ_2 and μ_{n-1} for weighted graphs. Let

$$\tau_m(L_w) = \min_{U \subset V, |U| = m} \lambda_1(L_w(U)), \tag{1.13}$$

where $L_w(U)$ is the principal submatrix of L_w whose rows and columns are in U. Similarly,

$$\rho_m(A_w) = \max_{U \subset V, |U|=m} \mu_m(A_w(U)). \tag{1.14}$$

It is shown in [FN] that

$$\lambda_2 \geq \tau_{\lfloor \frac{n}{2} \rfloor}(L_w) \quad \text{and} \quad \mu_{n-1} \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A_w). \tag{1.15}$$

Thus the second smallest eigenvalue of the Laplacian and the second largest eigenvalue of the adjacency matrix of a graph can be bounded by considering eigenvalues of subgraphs of the half size.

The object of this paper is twofold. First, we establish new Cheeger type bounds on λ_2 for weighted graphs, using the constants $i(G_w)$ and $\epsilon(G_w)$. These bounds generalize and improve a bound by Chung and a bound by Friedland mentioned above. We also give a new bound for μ_{n-1} of a weighted graph. This bound improves the bound given by Mohar.

Second, we establish a new type of bound, which is better than all the above discussed Cheeger type bounds. Let

$$\sigma(L_w) = \min_{U \subset V, |U| \leq \lfloor \frac{n}{2} \rfloor} \max(\lambda_1(L_w(U)), \lambda_1(L_w(\bar{U}))), \tag{1.16}$$

$$\nu(A_w) = \max_{U \subset V, |U| \leq \lfloor \frac{n}{2} \rfloor} \min(\mu_{|U|}(A_w(U)), \mu_{|\bar{U}|}(A_w(\bar{U}))), \tag{1.17}$$

Then

$$\lambda_2 \geq \sigma(L_w) \quad \text{and} \quad \mu_{n-1} \leq \nu(A_w). \tag{1.18}$$

2 Improvement and comparison of known inequalities

Let G_w be a weighted graph. As explained in §1, L_w is not influenced by the weighted loops. Hence, in our proofs for the inequalities involving the eigenvalues of the Laplacian matrix of a weighted graph G_w multiplied by a positive diagonal matrix, we will always assume that $w_{i,i} = 0, i = 1, \ldots, n$.

Theorem 2.1 *Let G_w be a weighted, undirected, connected graph and $V =< n >$. Let $D = diag(d_1, \ldots, d_n)$ be a positive diagonal matrix. Then*

$$\lambda_2(D^{-1}L_w) \geq \min_i \frac{\delta_i}{d_i}(1 - \sqrt{1 - i(G_w)^2}). \tag{2.1}$$

Proof. Let $v = [v_i]$ be the eigenvector corresponding to $\lambda_2(D^{-1}L_w)$. Let $U = \{i : v_i > 0\}$ and $\bar{U} = \{i : v_i \leq 0\}$. Rename the vertices of G such that $U =< t >$. In the proof of Theorem 2.3 in [FN] it is proved that

$$\lambda_2(D^{-1}L_w) \geq \max(\lambda_1(D^{-1}L_w(U)), \lambda_1(D^{-1}L_w(\bar{U}))). \tag{2.2}$$

Here $D^{-1}L_w(U)$ stands for a principal submatrix of $D^{-1}L_w$ whose rows and columns are in U. Without loss of generality (w.l.o.g.) we can assume that

$$\sum_{i\in U} \delta_i \leq \sum_{j\in \bar{U}} \delta_j. \tag{2.3}$$

Let $\tilde{\lambda}_1 := \lambda_1(D^{-1}L_w(U))$. Since $D^{-1}L_w(U)$ is an M–matrix (see [BP]), there exists a positive eigenvector x corresponding to $\tilde{\lambda}_1$, i.e.

$$D^{-1}L_w(U)x = \tilde{\lambda}_1 x. \tag{2.4}$$

W.l.o.g. we rename the vertices of G so that

$$x_i \geq x_{i+1} \quad \text{for} \quad i < t-1.$$

Set

$$\alpha = \min_{1\leq k\leq n-1} \frac{\sum_{(i,j)\in E(G), i\leq k<j} w_{i,j}}{\min(\sum_{i\leq k}\delta_k, \sum_{i>k}\delta_k)}.$$

Clearly, $\alpha \geq i(G_w)$. Let $y \in \mathrm{IR}^n$ given by

$$y_i = \begin{cases} x_i & \text{if } i \leq t \\ 0 & \text{else.} \end{cases}$$

and define

$$R = \frac{\sum_{(i,j)\in E(G)} w_{i,j}(y_i - y_j)^2}{\sum_{i\in V} y_i^2 \delta_i}.$$

Rewriting the denominator of R, and using the Schwarz inequality and the definition of α we obtain

$$\begin{aligned} R &= \frac{(\sum_{(i,j)\in E(G)} w_{i,j}(y_i - y_j)^2)(\sum_{(i,j)\in E(G)} w_{i,j}(y_i + y_j)^2)}{(\sum_{i\in V} y_i^2\delta_i)(\sum_{(i,j)\in E(G)} w_{i,j}(y_i + y_j)^2)} \\ &\geq \frac{(\sum_{(i,j)\in E(G)} w_{i,j}|y_i^2 - y_j^2|)^2}{(\sum_{i\in V} y_i^2\delta_i)(2\sum_{i\in V} y_i^2\delta_i - R\sum_{i\in V} y_i^2\delta_i)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(\sum_{k=1}^{n} \sum_{(i,j)\in E(G), i\le k, j>k} w_{i,j}|y_k^2 - y_{k+1}^2|)^2}{(2-R)\sum_k y_k^2 \delta_k} \\
&\geq \frac{(\sum_{k=1}^{n} (y_k^2 - y_{k+1}^2)\alpha * \sum_{j\le k} \delta_j)^2}{(2-R)\sum_k y_k^2 \delta_k} \\
&= \frac{\alpha^2}{2-R}.
\end{aligned}
$$

Hence

$$R^2 - 2R + \alpha^2 \leq 0.$$

Thus

$$
\begin{aligned}
R &\geq 1 - \sqrt{1-\alpha^2} \\
&\geq 1 - \sqrt{1 - i(G_w)^2}.
\end{aligned}
$$

Multiplying (2.4) with $x^T D(U)$ we get

$$x^T L_w(U) x = \tilde{\lambda}_1 x^T D(U) x.$$

Hence

$$
\begin{aligned}
\frac{\max_i d_i}{\underline{\delta}} \lambda_2(D^{-1}L_w) x^T x &\geq \frac{1}{\underline{\delta}} \lambda_2(D^{-1}L_w) x^T D(U) x \\
&\geq \frac{1}{\underline{\delta}} \tilde{\lambda}_1 x^T D(U) x \\
&= \frac{1}{\underline{\delta}} \frac{x^T L_w(U) x}{x^T x} x^T x \\
&\geq \frac{x^T L(U) x}{x^T \Delta(U) x} x^T x \\
&= R x^t x \\
&\geq (1 - \sqrt{1 - i(G_w)^2}) x^T x.
\end{aligned}
$$

Thus

$$\lambda_2(D^{-1}L_w) \geq \min_i \frac{\delta_i}{d_i}(1 - \sqrt{1 - i(G_w)^2}).$$

□

As an immediate corollary we obtain the result for the weighted Laplacian matrix of a graph.

Corollary 2.2 *Let G be a weighted, undirected, connected graph. Then*

$$\lambda_2(L_w) \geq \underline{\delta}(1 - \sqrt{1 - i(G_w)^2}).$$

Now, as in [Fr] for $U \subseteq V$ let

$$\epsilon(U) := \min_{\emptyset \neq W \subset U} \frac{|E(W, \bar{W})|}{|E(W)|} \tag{2.5}$$

and similarly

$$i(U) := \min_{\emptyset \neq W \subset U} \frac{|E(W, \bar{W})|}{\min(|E(W)|, |E(\bar{W})|)}. \tag{2.6}$$

Lemma 2.3 *Let G_w be a weighted, undirected, connected graph and $V =< n >$. Let $D = diag(d_1, \ldots, d_n)$ be a positive diagonal matrix. Then for any nonempty subset U of V*

$$\max(\lambda_1(D^{-1}L_w(U)), \lambda_1(D^{-1}L_w(\bar{U}))) \geq \min_{i \in U} \frac{\delta_i}{d_i}(1 - \sqrt{1 - i(U)^2}). \tag{2.7}$$

$$\lambda_1(D^{-1}L_w(U)) \geq \min_{i \in U} \frac{\delta_i}{d_i}(1 - \sqrt{1 - \epsilon(U)^2}). \tag{2.8}$$

Proof. It is clear that either U or $\bar{U}$ satisfies (2.3) and we obtain (2.7) as in the proof of Theorem 2.1. To prove (2.8) we can not assume in general (2.3). Thus we define

$$\tilde{\alpha} = \min_{1 \leq k \leq |U|} \frac{\sum_{(i,j) \in E(G), i \leq k < j} w_{i,j}}{\sum_{i \leq k} \delta_k}.$$

Use the arguments of the proof of Theorem 2.1 for $\tilde{\alpha}$ to obtain (2.8). □

Theorem 2.4 *Let G_w be a weighted, undirected, connected graph and $V =< n >$. Let $D = diag(d_1, \ldots, d_n)$ be a positive diagonal matrix. Then*

$$\lambda_2(D^{-1}L_w) \geq \sigma(D^{-1}L_w). \tag{2.9}$$

Proof. Recall that $D^{-1}L_w$ has a real spectrum. Let $D^{-1}L_w v = \lambda_2(D^{-1}L_w)v$, where v is a nonzero vector. Since G is a connected graph it follows that v has at least one positive and one negative coordinate. Assume that $U \subset< n >$ is the set where all the coordinates of v are positive. Then $\bar{U}$ is the set where all the coordiantes of v are nonpositive. Then (2.2) holds. The min max characterization of $\sigma(D^{-1}L_w)$ yields (2.9). □

The next Theorem gives a comparison of our new bounds with the well-known bounds mentioned in the previous section.

Theorem 2.5 *Let G_w be a weighted, undirected, connected graph and $V =< n >$. Let $D = diag(d_1, \ldots, d_n)$ be a positive diagonal matrix. Then*

$$\begin{aligned} \lambda_2(D^{-1}L_w) &\geq \sigma(D^{-1}L_w) \\ &\geq \tau_{\lfloor \frac{n}{2} \rfloor}(D^{-1}L_w) \\ &\geq \min_i \frac{\delta_i}{d_i}\left(1 - \sqrt{1 - \epsilon(G_w)^2}\right) \\ &\geq \min_i \frac{\delta_i}{2d_i}\epsilon(G_w)^2. \end{aligned}$$

Furthermore

$$\begin{aligned} \lambda_2(D^{-1}L_w) &\geq \sigma(D^{-1}L_w) \\ &\geq \min_i \frac{\delta_i}{d_i}\left(1 - \sqrt{1 - i(G_w)^2}\right) \\ &\geq \min_i \frac{\delta_i}{d_i}\left(1 - \sqrt{1 - \epsilon(G_w)^2}\right) \\ &\geq \min_i \frac{\delta_i}{2d_i}\epsilon(G_w)^2. \end{aligned}$$

Proof. The first inequality is given in the previous Theorem. To prove the second assume that

$$\sigma(D^{-1}L_w) = \max(\lambda_1(D^{-1}L_w(W)), \lambda_1(D^{-1}L_w(\bar{W}))), \quad |W| \leq \lfloor \frac{n}{2} \rfloor. \tag{2.10}$$

Let U_1 be any subset of V such that $|U_1| = \lfloor \frac{n}{2} \rfloor$ and $W \subseteq U_1$. Then

$$\tau_{\lfloor \frac{n}{2} \rfloor}(D^{-1}L_w) \leq \lambda_1(D^{-1}L_w(U_1)) \leq \lambda_1(D^{-1}L_w(W)) \leq \sigma(D^{-1}L_w).$$

The third inequality follows immediately from Lemma 2.3. Moreover, since

$$1 - \sqrt{1 - \epsilon(G_w)^2} \geq 1 - 1 + \frac{1}{2}\epsilon(G_w)^2 = \frac{1}{2}\epsilon(G_w)^2,$$

we obtain the last inequality.
The first inequality of the second set is again Theorem 2.4. For the second, let W be as in (2.10). Then choose U be either W or $\bar{W}$ such that (2.3) holds. The arguments of the proof of Theorem 2.1 yield

$$\lambda_1(D^{-1}L_w(U)) \geq \min_i \frac{\delta_i}{d_i}(1-\sqrt{1-i(G_w)^2})).$$

The third inequality uses (1.5) and the last is proved above. $\square$

Thus Theorem 2.5 shows that $\sigma(D^{-1}L_w)$ is better than the bound $\tau_{\lfloor \frac{n}{2} \rfloor}$ and better than the Cheeger-type bounds using the quantity $\epsilon(G_w)$ and $i(G)$. As (2.3) does not hold in general, $\tau_{\lfloor \frac{n}{2} \rfloor}$ is not comparable with the bound (2.1).

Note that with $D = I$ we obtain the inequalities for the weighted Laplacian matrix.

Next we compare our new bounds with the bound (1.10) by Berman and Zhang, which uses $h(G)$. For δ-regular graphs our Cheeger-type bound (2.1) and the bound (1.10) are the same. However, we will see in the next section that for arbitrary graphs it is not possible to compare these bounds. Consider the matrix $\Delta^{-\frac{1}{2}}L_w\Delta^{-\frac{1}{2}}$, which is closely related to the matrix $\tilde{\Delta}^{-\frac{1}{2}}L_w\tilde{\Delta}^{-\frac{1}{2}}$ considered in [Chu]. (Note that $\tilde{\Delta}^{-\frac{1}{2}}L_w\tilde{\Delta}^{-\frac{1}{2}} \leq \Delta^{-\frac{1}{2}}L_w\Delta^{-\frac{1}{2}}$.) We easily obtain the following comparison of our new bound (2.1) and (1.10).

Corollary 2.6 *Let G be a graph. Then*

$$\begin{aligned}\lambda_2(\Delta^{-\frac{1}{2}}L\Delta^{-\frac{1}{2}}) &\geq 1-\sqrt{1-i(G)^2)} \\ &\geq 1-\sqrt{1-\frac{h(G)^2}{\bar{\delta}^2}}.\end{aligned}$$

Proof. We use $D = \Delta$ in Theorem 2.1 and in (1.10) and use that the fact that the eigenvalues of $\Delta^{-\frac{1}{2}}L\Delta^{-\frac{1}{2}}$ and $\Delta^{-1}L$ are the same. The second inequality follows with (1.3). $\square$

Note that the first inequality in Corollarry 2.6 was proved by Chung in [Chu].

Morover, following the proof of Theorem 2.2 in [BZ] we obtain

Theorem 2.7 *Let G_w be a weighted, undirected, connected graph and $V =< n >$. Let $D = diag(d_1, \ldots, d_n)$ be a positive diagonal matrix. Then*

$$\lambda_2(D^{-1}L_w) \geq \sigma(D^{-1}L_w) \geq \tau_{\lfloor \frac{n}{2} \rfloor}(D^{-1}L_w) \geq \frac{1}{\max_i d_i}\left(\bar{\delta}-\sqrt{\bar{\delta}^2-h(G_w)^2}\right). \tag{2.11}$$

We now estimate from above the second largest eigenvalue of the weighted adjacency matrix of G_w. Our next Theorem is mainly based on Theorem 2.1. Since $D^{-1}L_w$ does not take in account the weighted loops we introduce

$$\omega = \max_{i \in <n>} w_{i,i}.$$

Theorem 2.8 *Let G_w be a weighted, undirected, connected graph. Let $V =< n >$. Then*

$$\begin{aligned}
\mu_{n-1}(A_w) &\leq \nu(A_w) && (2.12)\\
&\leq \rho_{\lfloor \frac{n}{2} \rfloor}(A_w) && (2.13)\\
&\leq \bar{\delta}\sqrt{1-\epsilon(G_w)^2} + \omega && (2.14)\\
&\leq \sqrt{\bar{\delta}^2 - h(G_w)^2} + \omega. && (2.15)
\end{aligned}$$

Furthermore

$$\begin{aligned}
\mu_{n-1}(A_w) &\leq \nu(A_w) && (2.16)\\
&\leq \bar{\delta}\sqrt{1-i(G_w)^2} + \omega && (2.17)\\
&\leq \bar{\delta}\sqrt{1-\epsilon(G_w)^2} + \omega && (2.18)\\
&\leq \sqrt{\bar{\delta}^2 - h(G_w)^2} + \omega. && (2.19)
\end{aligned}$$

Proof. Then the proof of the first inequality of 2.12 is essentially the same as the proof of Theorem 2.4. The second inequality of 2.12 follows by similar arguments as in the proof of Theorem 2.5. To prove the third inequality of 2.12 we assume first that $\omega = 0$, i.e. G_w does not have weighted loops. We now use Lemma 2.3 with $D = \Delta$. Thus, for every subset U with $|U| \leq \lfloor \frac{n}{2} \rfloor$

$$\lambda_1(\Delta^{-1}L_w(U)) \geq 1 - \sqrt{1-\epsilon(U)^2}.$$

Let I denote the identity matrix. Hence

$$\mu_{|U|}((\Delta^{-1}A_w - I)(U)) \leq -1 + \sqrt{1-\epsilon(U)^2}.$$

Therefore

$$\mu_{|U|}((\Delta^{-\frac{1}{2}}A_w\Delta^{-\frac{1}{2}})(U)) = \mu_{|U|}((\Delta^{-1}A_w)(U)) \leq \sqrt{1-\epsilon(U)^2}.$$

Then

$$(\sqrt{1-\epsilon(U)^2})I - (\Delta^{-\frac{1}{2}}A_w\Delta^{-\frac{1}{2}})(U))$$

is a positive definite matrix. Hence

$$(\sqrt{1-\epsilon(U)^2})\Delta(U) - A_w(U)$$

is also positive definite. Therefore

$$\begin{aligned}\mu_{|U|}(A_w(U)) &\leq \max_{i\in U} \delta_i(\sqrt{1-\epsilon(U)^2}) \\ &\leq \bar{\delta}(\sqrt{1-\epsilon(U)^2}).\end{aligned}$$

As $\epsilon(G_w) \leq \epsilon(U)$ we obtain

$$\rho_{\lfloor \frac{n}{2} \rfloor}(A) \leq \bar{\delta}(\sqrt{1-\epsilon(G_w)^2}).$$

The last inequality of 2.12 from the rihgt-hand side of (1.6).
Assume now that G_w has weighted loops. Let

$$B = A_w - \tilde{\Delta} + \Delta.$$

Then $B = A_{w'}$ for a corresponding weighted graph $G_{w'}$, which was obtained from G_w by deleting weighted loops. Clearly, $A_w \leq A_{w'} + \omega I$. Hence

$$\rho_{\lfloor \frac{n}{2} \rfloor}(A_w) \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A_{w'}) + \omega.$$

Use the validity of 2.12 for $G_{w'}$ and the above inequality to deduce 2.12 for G_w. The proof of 2.16 is similar to the proof of 2.12. □

Thus, similarly to the weighted Laplacian matrix we see that for the weighted adjacency matrix, $\nu(A_w)$ is the best of Cheeger type bounds. Theorem 2.8 improves the bounds by Mohar (1.9) and generalizes the bound in [FN] for regular graphs. Note we have also established two new Cheeger-type bounds, namely $\bar{\delta}\sqrt{1-i(G_w)^2}+\omega$ and $\bar{\delta}\sqrt{1-\epsilon(G_w)^2}+\omega$, which both improve the well-known bound by Mohar.

3 Examples

In this section we will compare the bounds stated in the previous section.

Example 1: Let G_1 be the graph given as in Figure 1 and let G_2 obtained from G_1 by deleting vertex 5 and edge $(2,5)$. The related Laplacian matrices are

$$L_1 = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

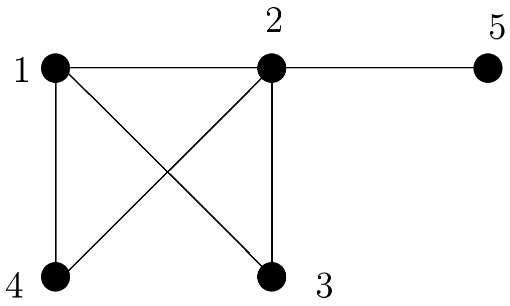

Figure 1

The constants for this graphs are as follows

	G_1	G_2
$h(G)$	1	1.5
$i(G)$	0.6	0.6
$\epsilon(G)$	0.6	0.6
$g(G)$	0.25	2/3

We obtain the following bounds

	L_1	L_2
λ_2	1	2
$\sigma(L)$	1	1.3820
$\tau_{\lfloor \frac{n}{2} \rfloor}(L)$	0.6972	1.3820
$2g(L)$	0.5	1.3333
$\underline{\delta}\left(1-\sqrt{1-i(G_w)^2}\right)$	0.2	0.4
$\underline{\delta}\left(1-\sqrt{1-\epsilon(G_w)^2}\right)$	0.2	0.4
$\frac{\underline{\delta}}{2}\epsilon(G_w)^2$	0.18	0.36
$\left(\bar{\delta}-\sqrt{\bar{\delta}^2-h(G_w)^2}\right)$	0.127	0.401
$\frac{1}{2\bar{\delta}}h(G)^2$	0.125	0.3750

Example 2: Consider a cycle with $n = 2m$ vertices. We get the constants

$$h(G) = \frac{2}{m} \qquad i(G) = \epsilon(G) = \frac{1}{m} \qquad g(G) = \frac{2}{m^2}.$$

The eigenvalues of the related Laplacian matrix are

$$\lambda_j = 2 - 2cos\frac{2\pi j}{n} \quad j = 0, \ldots, n-1.$$

For large n we get

$$\lambda_2 \approx \frac{\pi^2}{m^2}$$

and

$$\sigma(L) = \tau_{\lfloor \frac{n}{2} \rfloor}(L) \approx \frac{\pi^2}{m^2}.$$

Since this graph is 2-regular we obtain

$$\underline{\delta}\left(1 - \sqrt{1 - i(G_w)^2}\right) = \underline{\delta}\left(1 - \sqrt{1 - \epsilon(G_w)^2}\right) = \left(\bar{\delta} - \sqrt{\bar{\delta}^2 - h(G_w)^2}\right) = 2 - \sqrt{4 - \frac{4}{m^2}}.$$

Moreover,

$$2g(L) = \frac{4}{m^2} \qquad \frac{\underline{\delta}}{2}\epsilon(G_w)^2 = \frac{1}{2\bar{\delta}}h(G)^2 = \frac{1}{m^2}.$$

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