Boundedness of Singular Radon Transforms on L^p Spaces Under a Finite-Type Condition

Michael Greenblatt

1. Introduction

Singular Radon transforms are a type of singular integral operator, a prototype for which is the Hilbert transform along curves. Let $\gamma : [-1, 1] \rightarrow \mathbf{R}^n$ be a curve with $\gamma(0) = 0$, and let f be a Schwarz function on $\mathbf{R}^{n+1}$. The Hilbert transform of f along $(y, \gamma(y))$ is defined by

$$Hf(x,t) = \int_{-1}^{1} f(x-y, t-\gamma(y))\frac{1}{y}dy \tag{1.1}$$

Here x and y are in $\mathbf{R}$, t is in $\mathbf{R}^n$. The corresponding maximal operator is defined by

$$Mf(x,t) = \sup_{0<r<1} \frac{1}{2r}\int_{-r}^{r} |f(x-y, t-\gamma(y))|dy \tag{1.2}$$

Thus $Hf(x,t)$ is a type of average of f along a curve containing (x,t), and $Mf(x,t)$ is a corresponding maximal operator. The question asked here is, in analogy to the standard Hilbert transform and maximal operator, for which curves γ and for which $1 < p < \infty$ are these operators bounded on L^p. Note the operators are of a local character. Although there are methods that deal with global properties of singular and maximal Radon transforms, for the purposes of this paper we will only consider local properties and hence we always assume the function f is supported on a fixed compact set.

Example 1.1: Suppose $\gamma(y) = 0$ for all y. Then our operators reduce to the standard Hilbert transform and maximal operator in the x and y variables and are thus bounded on L^p for all $1 < p < \infty$.

Example 1.2: Suppose $n = 1$ and $\gamma(y) = y^k$ where k is an integer greater than 1. L^2 boundedness of H for $k = 2$ was shown by Fabes [6], and later for all $1 < p < \infty$ by Nagel, Riviere, and Wainger [12]. L^p boundedness of M for all $1 < p < \infty$ was shown by Stein and Wainger [25]. The crucial property of γ here is its "curvature". To give a rough idea of what curvature means here and how it relates to our operators, note that for our $\gamma(y) = y^k$ we may rewrite (1.1) as

$$\int_{-1}^{1}\int_{\mathbf{R}} f(x-y, t-u)\delta(u-y^k)\frac{1}{y}du\,dy \tag{1.3}$$

Notice that $g(u,y) = \delta(u-y^k)\frac{1}{y}$ satisfies $g(s^k u, sy) = s^{-k-1}g(u,y)$ and thus if $\phi(t)$ is an appropriate cutoff function, and $\bar{\phi}(t)$ denotes $\phi(t) - \phi(t/2)$, we may rewrite

$$\delta(u-y^k)\frac{1}{y} = K_0(u,y) + \sum_{j=1}^{\infty} 2^{(k+1)j} K_1(2^{kj}u, 2^j y) \tag{1.4}$$

Here $K_0(u,y) = \phi(y)\delta(u-y^k)\frac{1}{y}$ and $K_1(u,y) = \bar{\phi}(y)\delta(u-y^k)\frac{1}{y}$. As a result, the term $2^{(k+1)j}K_1(2^{kj}u, 2^j y)$ may be viewed as the portion of $\delta(u-y^k)\frac{1}{y}$ supported where $y \sim 2^{-j}$. Since K_0 and K_1 are distributions on $\mathbf{R}^2$ whose Fourier transforms at 0 are 0, for $|\xi| \leq 1$ we have that

$$|\hat{K}_0(\xi)|, |\hat{K}_1(\xi)| < C|\xi| \tag{1.5}$$

Furthermore, for $K = K_0$ or K_1 and $\psi = \phi$ or $\bar{\phi}$ we have that $\hat{K}(\xi_1, \xi_2)$ is of the form

$$\int_{\mathbf{R}} e^{i(y\xi_1 + y^k \xi_2)}\psi(y)dy$$

As a result, the method of stationary phase gives us that $|\hat{K}_0(\xi)|$, $|\hat{K}_1(\xi)| < C|\xi|^{-1/k}$. (See for example [23] chapter 8.) It is here that we are using the curvature of γ. In conjunction with (1.5) this implies that

$$|\hat{K}_0(\xi_1, \xi_2) + \sum_{j=1}^{\infty} \hat{K}_1(2^{-kj}\xi_1, 2^{-j}\xi_2)| < C$$

In other words the Fourier transform of the expression in (1.5) is a bounded function. But this means that the operator H is a convolution operator whose kernel has bounded Fourier transform, and hence is bounded on L^2.

One may obtain L^p estimates for H by embedding H in an analytic family of operators and then using interpolation. More specifically, for $Re(s) < 1/2$, let $\alpha_s(t)$ be the distribution on $\mathbf{R}$ whose Fourier transform is $(1+\xi^2)^{s/2}$. Define the operator H_s by

$$H_s f(t,x) = \int_{-1}^{1}\int_{\mathbf{R}} f(x-y, t-u)\alpha_s(|y|^{-k}(u-y^k))|y|^{-k}\frac{1}{y}du\,dy \tag{1.6}$$

Then H_0 is our original operator. For $Re(s) < 1/k$, L^2 boundedness of H_s follows from the argument used to prove L^2 boundedness of H; here we get the estimate $|\hat{K}_0(\xi)|$, $|\hat{K}_1(\xi)| < C|\xi|^{-1/k+Re(s)}$. In other words, the increase at infinity of the Fourier tranform of α_s is more than compensated by the decrease at infinity of the Fourier transform of a measure smoothly supported on a curved surface. Furthermore, for $Re(s) < 0$, the operator H_s resembles a singular integral operator adapted to a metric induced by the scalings $(u,y) \rightarrow (s^k u, sy)$, given by $B(t,x;r) = \{(s,y) \in \mathbf{R}^2 : |s-t| < Cr^k, |x-y| < r\}$. (The $|y|^{-k}$ factors in (1.6) were introduced for compatibility with that metric.) As a result, the natural generalizations of the Calderon-Zygmund theorem to such metrics (see for example Chapter 1 of [23]) may be used to prove L^p boundedness of H_s for $Re(s) < 0$ for all $1 < p < \infty$. Using analytic interpolation, L^p boundedness of $H = H_0$ then follows.

As for the maximal function, L^2 and L^p boundedness are shown in a related fashion involving the introduction of an appropriate square function.

After the initial results concerning prototypical operators such as those of Example 1.2, there was interest in extending these results to a more general class of curves. In [25], methods very similar to those outlined above, involving the use of scalings and the metrics induced by them were used by Stein and Wainger to prove L^p boundedness of H and M when γ satisfies a finite-type condition, namely that for some positive integer N the vectors $\gamma'(0)$, $\gamma''(0)$, ... , $\gamma^{(N)}(0)$ span the whole space $\mathbf{R}^{n+1}$.

The next question one might ask is how to generalize the above results when instead of averaging along curves we average along submanifolds of some positive codimension, again assuming the submanifolds satisfy an appropriate finite-type condition. Examples over special curves on nilpotent Lie groups were treated in [2]. Using a lifting argument, Ricci and Stein [19] were able to prove L^p boundedness of H and M in such circumstances, again in the setting of nilpotent Lie groups. Also see [18] for some model cases.

Generalizing further, it is natural to ask what generalizations may be made to the situation where the submanifolds being integrated over vary from point to point (the "variable-coefficient" case). This is not only of intrinsic interest but also has applications to other parts of mathematics, such as integral geometry and $\bar{\delta}$-Neumann problems. See [14] for an overview of several such applications. This situation is more complicated than the translation-invariant case for two primary reasons. First, the use of the Fourier transform as above is made much more difficult by the fact that the operator is no longer translation-invariant. Secondly, there is a relatively obvious notion of finite-type only in the translation-invariant situation, and thus a natural generalization is needed for the nontranslation-invariant case.

In view of the dyadic decomposition used in the model cases above, one might expect that in generalizations to nontranslation-invariant situations almost-orthogonality would be used in some analogous way. This would have to be done in conjunction with a curvature condition, in other words, a way of dealing with the second difficulty. One natural way of doing this is to elaborate on the method of lifting. This was successfully done in a sequence of papers culminating in [3]. In [3], a curvature condition generalizing the notion of finite-type was established. By using a theory of lifting to nilpotent Lie groups, a given singular Radon transform satisfying this curvature condition can be lifted to a singular Radon transform on the nilpotent Lie group. On this nilpotent Lie group, one can use generalizations of scalings such as in (1.4) in conjunction with almost-orthogonality to prove L^2 estimates. L^p and maximal estimates can then be proved without substantial additional difficulty.

Although a theory of singular and maximal Radon transforms under a finite-type condition was developed in [3], the use of lifting to nilpotent Lie groups made the proof indirect and for the sake of applications a more direct approach was sought. Clues on how to do this were given by papers such as Greenleaf and Uhlmann [10] as well as Phong and Stein [15] and its extension in the thesis of Cuccagna [5]. In the latter two papers, one writes singular Radon transforms over submanifolds of dimension m and of codimension n in the form

$$Tf(t,x) = \int_{\mathbf{R}^n} f(t + S(t,x,y), y) K(t,x,y) \psi(t,x,y) dy \tag{1.7}$$

Here $t \in \mathbf{R}^m$, x, $y \in R^n$, $S(t,x,x) = 0$ for all x, $K(t,x,y)$ is some kernel like $\frac{x_1 - y_1}{|x-y|^{n+1}}$ that satisfies an appropriate cancellation condition, and ψ is some cutoff function. A strong curvature condition on S is assumed; in an appropriate coordinate system the curvature condition here is that for each $\lambda \in \mathbf{R}^n$, $|\lambda| = 1$, at least one mixed partial of $\lambda \cdot S$ in the x and y variables is nonzero. In this situation, the ball $B(t,x;r)$ can be taken to be $\{(s,y) \in \mathbf{R}^{m+n} : |y-x| < r, |s - t - S(t,x,y)| < r^2\}$.

One may rewrite (1.7) as

$$Tf(t,x) = \int_{\mathbf{R}^{m+n}} \hat{f}(\lambda, y) e^{i\lambda\cdot(t+S(t,x,y))} K(t,x,y)\psi(t,x,y) dy\, d\lambda \tag{1.8}$$

the Fourier transform being in the left variable only. T is then written as $T_1 + T_2$, where

$$T_1 f(t,x) = \int_{\mathbf{R}^{m+n}} \hat{f}(\lambda, y) e^{i\lambda\cdot(t+S(t,x,y))} \phi(|\lambda||x-y|^2) K(t,x,y)\psi(t,x,y) dy\, d\lambda$$

$$T_2 f(t,x) = \int_{\mathbf{R}^{m+n}} \hat{f}(\lambda, y) e^{i\lambda\cdot(t+S(t,x,y))} (1-\phi)(|\lambda||x-y|^2) K(t,x,y)\psi(t,x,y) dy\, d\lambda \tag{1.9}$$

Here ϕ is our bump function from before. $T_1 f(t,x)$ can be rewritten as

$$\int_{\mathbf{R}^{m+n}} \frac{1}{|x-y|^{2m}} \hat{\phi}\Big(\frac{t + S(t,x,y) - s}{|x-y|^2}\Big) K(t,x,y)\psi(t,x,y) f(s,y) ds\, dy$$

T_1 is thus a singular integral under the metric associated with T and the proof of L^p boundedness for all $1 < p < \infty$ is straightforward. To deal with $T_2 = T - T_1$, almost-orthogonality is used in the λ variable (implicitly in [15], but more explicitly in [5]) to reduce the L^2 theory of T_2 to proving uniform L^2 bounds on the following operators, where $\bar{\phi}(t) = \phi(t) - \phi(t/2)$ as before:

$$T_2^j f(t,x) = \int_{\mathbf{R}^{m+n}} \hat{f}(\lambda, y) e^{i\lambda\cdot(t+S(t,x,y))} (1-\phi)(|\lambda||x-y|^2) \bar{\phi}(2^{-j}|\lambda|) K(t,x,y)\psi(t,x,y) dy\, d\lambda \tag{1.10}$$

This is achieved via a TT^* argument involving an appropriate integration by parts that uses the curvature condition in its manifestation in terms of mixed partials of S.

A similar approach to attaining these uniform L^2 estimates is to write $T_2^j = \sum_{i=0}^{\infty} T_2^{j,i}$, where $T_2^{j,i} f(t,x) =$

$$\int_{\mathbf{R}^{m+n}} \hat{f}(\lambda, y) e^{i\lambda\cdot(t+S(t,x,y))} \bar{\phi}(2^{-i}|\lambda||x-y|^2)\phi(2^{-j}|\lambda|)K(t,x,y)\psi(t,x,y)dy\, d\lambda \quad (1.11)$$

Then by using a TT^* argument one may show that as an operator on L^2, $||T_2^{j,i}|| < C2^{-\delta i}$ for some small δ, and $||T_2^j|| < C$ follows from addition. A related argument is used to obtain estimates on oscillatory integrals with singular kernel and polynomial phase in a paper by Ricci and Stein [19]. It might occur to one to sum the operators $T_2^{j,i}$ in j to obtain an operator T_i and then directly show that $||T_i|| < C2^{-\delta i}$. Inspired thus, we define the operator

$$T_i f(t,x) = \int_{\mathbf{R}^{m+n}} \frac{2^{im}}{|x-y|^{2m}} \hat{\bar{\phi}}(2^i \frac{t+S(t,x,y)-s}{|x-y|^2})K(t,x,y)\psi(t,x,y)f(s,y)ds\, dy \quad (1.12)$$

The almost-orthogonality used above translates into dyadically dividing the operator T_i into the portions where $|x-y| \sim 2^{-j}$ for various j. This is analogous to the decomposition in (1.4). A similar decomposition also appears in [2]. Boundedness on L^2 of a given dyadic piece again follows via a TT^* argument, giving $||T_i f||_2 < C2^{-\delta i}||f||_2$. Then the method of the Calderon-Zygmund theorem applied to the metric associated to T gives $||T_i f||_p < Ci||f||_p$ for all $1 < p < \infty$. Real interpolation then implies $||T_i f||_p < C_p 2^{-\delta' i}||f||_p$, and summing gives $||T_2 f||_p < C'_p||f||_p$ for our original operator T_2. Combining with the estimates on T_1 we conclude $||Tf||_p < C''_p||f||_p$, the estimate we seek.

In the author's thesis [8] this set-up was used in codimension 1 and was generalized to the case where T satisfied a weaker curvature condition equivalent to that of [3]. Real-variable methods were used in the TT^* arguments, and the use of the Fourier transform was avoided. To make the L^2 estimates in the generalization, it was necessary to introduce a rather technical definition for the metric associated with T, and it was difficult to extend the arguments to codimension greater than 1. Eventually, the author found geometric interpretations for these technicalities, which eventually led to the current paper. In this paper, a more natural definition for the metric associated with T is introduced. New L^2 methods are described, and L^p boundedness of singular Radon transforms for $1 < p < \infty$ is proved under rather general finite-type circumstances. It is worth noting that the methods of [8] do have at least one advantage over the ones here; with simple modifications they can be used to prove L^p boundedness of maximal Radon transforms (in codimension 1). The singular integrals of this paper are a bit too singular for the same to be true for methods here; more will be said about this in the final section of this paper.

Acknowledgments: The author would like to thank Eli Stein for his many helpful suggestions during the development of [8]. It would be difficult to overstate his impact on the subject of singular Radon transforms, this paper included. The author would also like to thank David Jerison for a number of helpful discussions.

2. Statement of Theorems and Outline of Proof

We start by switching to a notation resembling that of [3]. Suppose $D_1 \subset D_1'$, $D_1 \neq D_1'$ are closed balls in R^n, and D_2 is a closed ball in R^k centered at the origin. Suppose we have a C^1 function $K(x,t) : D_1' \times D_2 - \{0\} \rightarrow \mathbf{R}$ which is a distribution in the t variable for fixed x, and suppose we have a C^∞ function $\gamma(x,t) : D_1 \times D_2 \rightarrow D_1'$ such that $\gamma(x,0) = x$. Define the singular Radon transform $T_{\gamma,K}$ on a Schwarz function $f(x)$ by

$$T_{\gamma,K}f(x) = \int_{R^k} f(\gamma(x,t))K(x,t)\,dt \tag{2.1}$$

We will suppress the subscripts on T when it is clear which singular Radon transform is being discussed. $T_{\gamma,K}f(x)$ is to be viewed as averaging f in the t variable over the surface $\gamma(x,t)$ with respect to the kernel $K(x,t)$, which is a singular integral kernel in the t variable such as that of the Riesz transform, $t_i/|t|^{k+1}$. Note that γ need not even be a local immersion in the t variable for fixed x, but when it is an embedding the operator truly is a singular averaging operator over submanifolds of dimension k.

To motivate the curvature condition of this paper and its associated noneuclidean metric in analogy to Example 1.2, we ask what conditions are needed on a metric in order for the method of the Calderon-Zygmund theorem to apply to singular integrals with respect to the metric. Since we are to be working under a finite-type assumption, we would expect there to be a doubling condition. A natural set of conditions, the result of a number of papers by various people including Fabes, Riviere, Stein, Coffman and Weiss (See p. 48 of [23] for more details) is given by the following four conditions, where $|\circ|$ denotes Lebesgue measure. These conditions are essentially those of Coffman and Weiss [4]: (see p. 8-9 of [23])

$$|B(x,2r)| \leq M|B(x,r)| \tag{$CZ1$}$$

$$x \in D_1, \;\Rightarrow\; D_1 \subset \cup_r B(x,r),\; \cap_r B(x,r) = \{x\} \tag{$CZ2$}$$

$$B(x_1,r) \cap B(x_2,r) \neq \emptyset \Rightarrow B(x_2,r) \subset B(x_1,Mr) \tag{$CZ3$}$$

$$\forall r > 0,\; \forall \text{ open sets } U,\; \text{the map } x \rightarrow |B(x,r) \cap U| \text{ is continuous.} \tag{$CZ4$}$$

Furthermore, from past experience we expect that $B(x,r)$ contains the portion of radius r of the surface being integrated over at x; in other words we expect that

$$\{\gamma(x,t) : |t| < r\} \subset B(x,r) \tag{S}$$

(S) and $(CZ3)$ together imply that if $|t_1|$, $|t_2| < r$ and $\gamma(x_1,t_1) = \gamma(x_2,t_2)$, then $x_2 \in B(x_1,Mr)$. In other words if the piece of the graph of $t \rightarrow \gamma(x_1,t)$ of radius r intersects the corresponding piece of the graph of $t \rightarrow \gamma(x_2,y)$, then $x_2 \in B(x_1,Mr)$.

In view of this, if our balls are to be defined by a metric, it might be reasonable to define the distance $d(x,y)$ from x to y by

$$d(x,y) = \inf\{r : \text{ there exist } N\,,\; \{x_i\}_{i=1}^N, \{t_i\}_{i=1}^{N-1}, \{u_i\}_{i=1}^{N-1} \text{ such that}$$

$$x_1 = x, x_N = y, \sum_{i=0}^{N-1} |t_i| + |u_i| < r, \text{ for all } i\, \gamma(x_i, t_i) = \gamma(x_{i+1}, u_i) \}$$

In other words $B(x,r)$ consists of those points that can be joined by our surfaces in such a way that the total distance travelled in this joining is at most r. This definition evidently satisfies $(CZ3)$ and (S), and furthermore will turn out to be essentially the definition needed for the L^2 methods of this paper. Note the similarity to the Carnot-Caratheodory and other similar metrics useful in geometry and complex analysis. There may also be a relation to the metrics of [13]. However, it is crucial for our arguments that the metric be defined via broken paths as done here.

As for the conditions $(CZ1)$ and $(CZ2)$, note that if we assume the balls $B(x,r)$ are open, by compactness $(CZ2)$ implies that at each x, $D_1 \subset B(x, r_x)$ for some r_x. Thus if r_0 is one of these r_x's, $D_1 \subset B(y, 2r_0)$ for each $y \in D_1$. So if we apply $(CZ1)$, successively taking $r = r_0$, $\frac{r_0}{2}$, $\frac{r_0}{4}$, etc, we obtain

$$x \in D_1 \Rightarrow m(B(x,r)) > Mr^Q \tag{C}$$

Here Q is independent of x. (C) is equivalent to the curvature condition of [3], and once we have developed the technical lemmas of Section 3 we will see (Lemma 3.11) that if (C) is satisfied, the metric we have defined is equivalent to one that satisfies $(CZ1)$-$(CZ4)$. Namely, define $B_N(x,r)$ by

$$B_N(x,r) = \{y : \text{ there exist } \{x_i\}_{i=1}^{N+1}, \{t_i\}_{i=1}^{N}, \{u_i\}_{i=1}^{N} \text{ such that}$$
$$x_1 = x, x_{N+1} = y, \sum_{i=0}^{N} |t_i| + |u_i| < r, \text{ and for each } i,\ \gamma(x_i, t_i) = \gamma(x_{i+1}, u_i)\} \tag{2.2}$$

We will see that the metric induced by the balls $B_{2n}(x,r)$ is equivalent to the metric induced by $B(x,r)$ and furthermore satisfies $(CZ1) - (CZ4)$. The purpose of switching to this metric is to simplify the verification of $(CZ4)$. Notice that the condition (C) also resembles the condition introduced by Fefferman and Phong [7] in their study of subelliptic problems; see also the discussion in Section 7.

We will show that for each x and r there is an open set $U_{x,r}$ containing $B(x,r)$ and a C^1 map $Z_{x,r} : U_{x,r} \longrightarrow [-Cr, Cr]^n$, such that if x' and x'' are in $U_{x,r}$, then $d(x', x'') < C|Z_{x,r}(x') - Z_{x,r}(x'')|$. Furthermore the Jacobian determinant of $Z_{x,r}$ is bounded above and below uniformly in x and r; thus $Z_{x,r}$ provides a coordinate system of sorts for $B(x,r)$. This coordinate system will prove useful in describing x-derivative conditions for $K(x,t)$.

We return to our operator (2.1). We now state a few conditions on γ and K needed for our arguments. To start with, writing $\gamma(x,t) = (\gamma_1(x,t), ..., \gamma_n(x,t))$, we assume that M is sufficiently large that for a large N appropriate for our arguments we have, for every i:

$$\sup_{|\alpha|+|\beta|<N,\ (x,t)\in D_1' \times D_2} |\partial_x^\alpha \partial_t^\beta \gamma_i(x,t)| < M \tag{G1}$$

Next, we make the assumptions that, for each $(x,t) \in D_1' \times D_2$ we have:

$$\gamma(x,0) = x \text{ and } |\det d_x\gamma(x,t)| > 1/2 \tag{$G2$}$$

The latter type of condition is necessary for L^p boundedness for any p; see the proof of the Proposition below. $K(x,t)$ is assumed to satisfy the size estimates

$$|\nabla_t K(x,t)| < M|t|^{-k-|\alpha|} \tag{$K1$}$$

We will need further conditions on K which we will discuss in a moment. First, we introduce some notation. As in [3], we define $\gamma^{-1}(x,t)$ by $\gamma(\gamma^{-1}(x,t),t) = x$.

We now iteratively define $\beta^{2m-1}(x,t_1,t_1',t_2,t_2',...,t_m)$ and $\beta^{2m}(x,t_1,t_1',t_2,t_2',...,t_m,t_m')$ by

$$\begin{gathered}\beta^1(x,t) = \gamma(x,t)\\ \beta^2(x,t_1,t_1') = \gamma^{-1}(\gamma(x,t_1),t_1')\\ \beta^{2m}(x,t_1,t_1',...,t_m,t_m') = \gamma^{-1}(\beta^{2m-1}(x,t_1,t_1',...,t_m),t_m')\\ \beta^{2m+1}(x,t_1,t_1',...,t_m,t_m',t_{m+1}) = \gamma(\beta^{2m}(x,t_1,t_1',...,t_m,t_m'),t_{m+1})\end{gathered}$$

Let ϕ be our bump function on $\mathbf{R}$ from before, and let $\phi_0(t) = \phi(t) - \phi(2t)$. Define $K_j(x,t) = \phi_0(2^j t)K(x,t)$, and let j_0 be some integer such that 2^{-j_0} is larger than the radius of D_2. We define the singular integral $\tilde{T}$ associated to T by

$$\tilde{T}f(x) = \sum_{j \geq j_0} 2^{(2n-2)kj} \int \prod_{m=2}^{n} \phi(2^j|t_m|) \prod_{m=1}^{n-1} \phi(2^j|t_m'|) f(\beta^{2n-1}(x,t_1,t_1',...,t_n))K_j(x,t_1)$$

$$\prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m' \tag{2.3}$$

For each term j, the iteration function β "fills" the ball of radius 2^{-j}, while whatever cancellation $K_j(x,t)$ has is still preserved.

It is easy to verify that $T^*f(x) =$

$$\int f(\gamma^{-1}(x,t))\det(d_x\gamma^{-1})(x,t)K(\gamma^{-1}(x,t),t)dt$$

Due to the condition $|\det d_x\gamma(x,t)| > 1/2$, the function $\det(d_x\gamma^{-1})(x,t)$ is C^∞ and can safely be ignored. As a result, given a condition on $K(x,t)$, the corresponding assumption on T^* is given by assuming that condition on $K(\gamma^{-1}(x,t),t)$.

We now present our main theorem, or rather, two main theorems with closely related proofs. For the first we assume that for each x, K satisfies

$$x, x'' \in U_{x,2^{-j}} \Rightarrow |K_j(x',t) - K_j(x'',t)| < M2^{j+1}|Z_{x,2^{-j}}(x') - Z_{x,2^{-j}}(x'')| \tag{$K2$}$$

Here $Z_{x,r}$ is the coordinate system on $B(x,r)$ mentioned earlier. We also assume cancellation conditions on $K(x,t)$ and the kernel corresponding to T^*, $K(\gamma^{-1}(x,t),t)$:

$$|\int K_j(x,t)dt|\,,\;|\int K_j(\gamma^{-1}(x,t),t)dt| < M2^{-\epsilon j} \text{ for some } \epsilon > 0. \qquad (K3)$$

The cancellation condition $(K3)$ was chosen for convenience for our arguments; no attempt is being made here to find a general (or even coordinate-invariant) cancellation condition. The most natural and general cancellation condition would most likely be an analogue to the cancellation conditions of the $T(1)$ theorem for the singular integral we will associate to T; working under such assumptions however is beyond the scope of this paper.

Our main theorem for such K is

Theorem 2.1: If γ satisfies (C), $(G1)$, and $(G2)$, and K satisfies $(K1)-(K3)$, then T is bounded on L^p for all $1 < p < \infty$.

Suppose we assume a stronger x-derivative condition on K at each x:

$$x', x'' \in U_{x,2^{-j}} \Rightarrow |K_j(x',t) - K_j(x'',t)| < M2^{j+1-\epsilon}\,|Z_{x,2^{-j}}(x') - Z_{x,2^{-j}}(x'')| \qquad (K2')$$

Then without any cancellation conditions on $K(x,t)$ or $K(\gamma^{-1}(x,t),t)$ we have the following theorem.

Theorem 2.2: If γ satisfies (C), $(G1)$, and $(G2)$, and K satisfies $(K1)$, $(K2')$, then for any $1 < p < \infty$, T is bounded on L^p if and only if $\tilde{T}$ is.

In both theorems, the bounds obtained will depend on M, n, k, Q, and p. Throughout this paper, we use the symbols C or δ to denote any constant depending on M, n, k, Q, and p. We sometimes use the the symbols more than once in a given argument; in such a situation C denotes the maximum of the several C's appearing, and δ denotes the minimum of the several δ's appearing. A subscripted C_i or δ_i or a primed C', C'', δ', δ'', etc. denotes a number which is fixed during a given argument.

In the arguments of this paper, it will often be necessary to assume that $|t|$ is less than some fixed δ. Since $K(x,t)$ is bounded on $\{|t| > \delta\}$ for each $\delta > 0$, the condition that $|\det\, d_x\gamma(x,t)| > \frac{1}{2}$ ensures that by simple estimates the portion of T supported on any set $\{|t| > \delta\}$ is bounded on L^2. Namely, as before we suppose $\phi(t)$ is a compactly supported nonnegative cut-off function identically equal to 1 on a neighborhood of the origin, and we write $K(x,t) = K_{in}(x,t) + K_{out}(x,t)$, where $K_{in}(x,t) = K(x,t)\phi(\frac{t}{\delta})$ and $K_{out}(x,t) = K(x,t)(1-\phi(\frac{t}{\delta}))$. Correspondingly we define

$$T_{in}f(x) = \int f(\gamma(x,t))K_{in}(x,t)dt,\ T_{out}f(x) = \int f(\gamma(x,t))K_{out}(x,t)dt$$

We then have the following:

Proposition: $||T_{out}||_{L^2 \to L^2} < C_1$, where C_1 depends on δ and the parameters of T.

Proof: We have

$$\int |T_{out} f(x)|^p dx = \int_{\mathbf{R}^n} | \int_{\mathbf{R}^k} f(\gamma(x,t)) K_{out}(x,t) dt|^p dx$$

$$\leq C \int_{D_1' \times D_2} |f(\gamma(x,t))|^p dt\, dx$$

But since $|\det\, d_x \gamma(x,t)| > 1/2$, in the x integration for a fixed t we may make the coordinate change $x' = \gamma(x,t)$, and the above is at most

$$C \int_{\mathbf{R}^n \times D_2} |f(x)|^p dx\, dt < C||f||_p^p$$

Thus T_{out} is bounded on L^p and we are done.

In view of the Proposition, in what follows we will replace T by T_{in} for appropriately small δ when necessary. It is easy to check that $K_{in}(x,t)$ satisfies $(K1) - (K3)$ when $K(x,t)$; therefore, when we do such a replacement we may still assume $(K1) - (K3)$ are satisfied.

For $j \geq j_0$, define the operator $T_{0,j}$ by $T_{0,j} f(x) =$

$$2^{2(n-1)kj} \int \prod_{m=2}^{n} \phi(2^j |t_m|) \prod_{m=1}^{n-1} \phi(2^j |t_m'|) f(\beta^{2n-1}(x, t_1, t_1', ..., t_n)) K_j(x, t_1) \prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m' \tag{2.4}$$

(This is the jth term in the expression defining $\tilde{T} f(x)$). For $i > 0$, $j \geq j_0$, define the operator $T_{i,j}$ by $T_{i,j} f(x) =$

$$\int 2^{2(n-1)k(i+j)} [\prod_{m=2}^{n} \phi(2^{i+j} |t_m|) \prod_{m=1}^{n-1} \phi(2^{i+j} |t_m'|) - 2^{-2(n-1)k} \prod_{m=2}^{n} \phi(2^{i+j-1} |t_m|)$$

$$\prod_{m=1}^{n-1} \phi(2^{i+j-1} |t_m'|)] f(\beta^{2n-1}(x, t_1, t_1', ..., t_{n-1}', t_n)) K_j(x, t_1) \prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m' \tag{2.5}$$

Thus $\sum_{i \geq 0} T_{i,j} f(x) = \int f(\gamma(x,t)) K_j(x,t) dt$ (i.e. the portion of T where $|t| \sim 2^{-j}$) If we define $T_i = \sum_{j \geq j_0} T_{i,j}$, we then have $T = \tilde{T} + \sum_{i > 0} T_i$. This decomposition can be viewed as a Littlewood-Paley decomposition adapted to the surfaces defined by $\gamma(x,t)$. T_i is in some sense a hybrid of T and $\tilde{T}$, of "thickness" 2^{-i} times that of $\tilde{T}$ and having a cancellation arising from the bracketed expression in (2.5); thus T_i is analogous to the operator (1.12).

The proof of Theorems 2.1 and 2.2 will proceed as follows. In Section 3 we will prove a number of geometric and analytic facts needed for our arguments. In Section 4 we will show that under the hypotheses of Theorem 2.1 or 2.2, the finite-type condition (C) implies that

$$\forall i > 0 \ ||T_{i,j}||_{2,2} < C2^{-\delta i}$$

(The estimate $||T_{0,j}|| < C$ is straightforward and doesn't require (C)). In Section 5 we will prove, for $i \geq 0$ under the hypotheses of Theorem (2.1), and for $i \geq 1$ under the hypotheses of Theorem 2.2, the following almost-orthogonality relations.

$$||T_{i,j_1}T^*_{i,j_2}||_{2,2}\ ,\ ||T^*_{i,j_1}T_{i,j_2}||_{2,2} < C2^{-\delta(i+|j_1-j_2|)} \tag{2.6}$$

Note that if $\frac{i}{|j_1-j_2|} \geq \delta_0$ for some δ_0 these relations follow from the estimates $||T_{i,j}||_{2,2} < C2^{-\delta i}$. Thus it suffices to show that

$$\frac{i}{|j_1-j_2|} < \delta_0 \Longrightarrow ||T_{i,j_1}T^*_{i,j_2}||_{2,2}\ ,\ ||T^*_{i,j_1}T_{i,j_2}||_{2,2} < C2^{-\delta(|j_1-j_2|)} \tag{2.7}$$

We will see that if δ_0 is chosen appropriately small, one may prove (2.6) by an elaboration of the type of argument that works for the analogous relations in the proof of the L^2 boundedness of singular integrals like the Hilbert transform, etc.

Thus by Cotlar-Stein almost-orthogonality (see [19] p.279-281), the results of Sections 4 and 5 imply that under the hypotheses of Theorems 2.1 or 2.2, we have $||T_i||_{2,2} < C2^{-\delta i}$ for all $i > 0$, and that under the hypotheses of Theorem 2.1, we have that $||\tilde{T}||_{2,2} < C$ as well.

In Section 6 we show that the general version of the Calderon-Zygmund theorem applied to our metric implies that under the hypotheses of Theorems 2.1 or 2.2, for all $1 < p < \infty$ and for all $i > 0$ we have $||T_i||_{p,p} < C(i+1)$. Moreover, the same will be shown to be true for $i = 0$ under the hypotheses of Theorem 2.1. So by real interpolation, under the hypotheses of Theorems 2.1 or 2.2, for all $i > 0$, we have $||T_i||_{p,p} < C'2^{-\delta' i}$, and furthermore, under the hypotheses of Theorem 2.1 we have that $||\tilde{T}||_{p,p} < C'$. Summing over $i > 0$ shows $||T - \tilde{T}||_{p,p} < C$ under the hypotheses of Theorems 2.1 and 2.2, thus completing the proofs of both theorems.

This paper will conclude with a discussion of several problems related to those addressed in this paper.

3. Lemmas and Constructions

The results of this section are analytic and geometric facts that hold under $(G1)$, $(G2)$, and (C); in particular they hold under the hypotheses of Theorems 2.1 or 2.2. As in [3] we make extensive use of Jacobian determinants of functions deriving from the iteration functions β^n. We start with the following variant of a Van der Corput lemma of Christ (Lemma (3.1) of [2])

Lemma 3.1: If $I \subset R$ is an interval, and $f : I \to \mathbf{R}$ is a function with $|f^{(q)}(t)| \geq \epsilon_1$ for each $t \in I$, then $|\{t : |f(t)| \leq \epsilon_2\}| \leq C_q(\epsilon_2/\epsilon_1)^{1/q}$. In addition, $\sup_{t\in I} |f(t)| \geq \frac{1}{2}(C_q)^{-q}\epsilon_1|I|^q$.

Proof: We consider the first statement first. The proof is similar in spirit to that of Van der Corput's lemma. We first consider the case where $\epsilon_1 = \epsilon_2 = 1$. By downwards induction on $r \leq q$, we show that there is a collection F_r of at most $2^{q-r+1} - 1$ subintervals of I of length ≤ 2 such that $|f^r(t)| > 1$ outside the union of the intervals in F_r. For $r = q$ this is obvious. Suppose we are at step r of the induction. By induction hypothesis, $I - \{$ the union of the intervals in $F_{r+1}\}$ is the union of at most 2^{q-r} intervals, on each of which we have $|f^{(r+1)}| \geq 1$. Suppose J is one of these intervals. Then there is a subinterval J' of J of length at most 2 such that $|f^{(r)}| \geq 1$ outside this interval. We define F_r to be the union of F_{r+1} and the J's corresponding to a J. Induction follows.

The case where ϵ_1 and ϵ_2 are not both 1 follows by applying the $\epsilon_1 = \epsilon_2 = 1$ case to the function $1/\epsilon_2\ f((\epsilon_2/\epsilon_1)^{1/q}t)$.

To prove the latter statement of this lemma, set $\epsilon_2 = \frac{1}{2}(C_q)^{-q}\epsilon_1|I|^q$ in the first statement. Then $|\{t \in I : |f(t)| \leq \epsilon_2\} \leq 2^{-\frac{1}{q}}|I|$. Thus $\exists t \in I$ with $|f(t)| \geq \epsilon_2$, and we are done.

Lemma 3.2: Let $f : [-r, r]^m \to \mathbf{R}$ satisfy $A_f = \sup_{x\in[-r,r]^m} |f(x)| \geq C_1 r^l$ for some positive integer l. Then there exist positive δ_1 and C_2 depending on C_1, l, m, $||f||_{C^l([-r,r]^m)}$, $\sup_{|\alpha|=l+1,\ x\in[-r,r]^m} |\partial^\alpha f(x)|$, such that, as long as $r < \delta_1$ we have

a) For any $\epsilon < 1$, $|\{x \in [-r, r]^m : |f(x)| \leq \epsilon A_f\}| \leq C_2\epsilon^{\frac{1}{l}} r^m$.

b) For any x_1, $x_2 \in [-r, r]^m$, $|f(x_2) - f(x_1)| < C_2\frac{|x_2-x_1|}{r}A_f$.

Proof: Our first observation is that it suffices to prove the theorem under the assumption that $\sup_x |\nabla f(x)| > \frac{1}{4\sqrt{m}r}A_f$. For otherwise $\inf_x |f(x)| \geq \sup_x |f(x)| - \sup_x |\nabla f(x)| \times$ diameter of $[-r, r]^m \geq \frac{1}{2}\sup_x |f(x)| = \frac{A_f}{2}$ and conclusion a) is trivial. Furthermore $|f(x_1) - f(x_2)| \leq \sup_x |\nabla f(x)||x_1 - x_2| \leq \frac{|x_1-x_2|}{4\sqrt{m}r}A_f$ and b) follows as well. So we may assume that $\sup_x |\nabla f(x)| > \frac{1}{4\sqrt{m}r}A_f$.

Let $c > 0$. Define $P_c f(x) = \sum_{1\leq|\alpha|\leq l} |\partial^\alpha f(x)|(cr)^{|\alpha|}$, and let $B_f = \sup_{x\in[-r,r]^m}$ $|P_c f(x)|$. Let $x_0 \in [-r, r]^m$ such that $|P_c f(x_0)| = B_f$. Note that by above, $B_f \geq \frac{c}{4\sqrt{m}}A_f$. Let α' be such that $1 \leq |\alpha'| \leq l$ be such that $|\partial^{\alpha'} f(x_0)(cr)^{|\alpha'|}| \geq \frac{1}{l^m}P_c f(x_0)$. So there exists a $\delta > 0$ and a vector $z \in \mathbf{R}^m$, $|z| = 1$, such that $|(\sum_{i=1}^m z_i\partial_{x_i})^{|\alpha'|}f(x_0)(cr)^{\alpha'}| \geq \frac{\delta}{l^m}P_c f(x_0)$. (See [23] p.343) In what follows we use the notation ∂_z to denote $\sum_{i=1}^m z_i\partial_{x_i}$.

Taylor expanding $\partial_z^{|\alpha'|} f(x)$, we see that

$$|\partial_z^{|\alpha'|} f(x)(cr)^{\alpha'} - \partial_z^{|\alpha'|} f(x_0)(cr)^{\alpha'}|$$

$$\leq l^m \sum_{l \geq |\beta| > |\alpha'|} |\partial^\beta f(x_0)|(cr)^{|\alpha'|}(2\sqrt{m}r)^{\beta - |\alpha'|} + C(cr)^{|\alpha'|}(2\sqrt{m}r)^{l+1-|\alpha'|}$$

$$\leq \frac{(2m)^{\frac{1}{2}} l^m}{c} P_c f(x_0) + C' c^{|\alpha'|} r^{l+1}$$

We choose $c = \frac{l^{2m+1}(2m)^{\frac{1}{2}}}{\delta}$, δ as above. Since $P_c f(x_0) \geq Cr^l$, if $r < \delta_1$ for an appropriately chosen δ_1, we have $C' c^{|\alpha'|} r^{l+1} < \frac{(2m)^{\frac{1}{2}} l^m}{2c} P_c f(x_0)$, and as a result

$$|\partial_z^{|\alpha'|} f(x)(cr)^{\alpha'} - \partial_z^{|\alpha'|} f(x_0)(cr)^{\alpha'}| \leq \frac{(2m)^{\frac{1}{2}} l^m}{2c} P_c f(x_0)$$

$$\leq \frac{1}{2} |\partial_z^{|\alpha'|} f(x_0)(cr)^{\alpha'}|$$

Therefore if $x \in [-r, r]^m$, $|\partial_z^{|\alpha'|} f(x)(cr)^{\alpha'}| \geq \frac{1}{2} |\partial_z^{|\alpha'|} f(x_0)\ (cr)^{\alpha'}| \geq \frac{\delta}{2l^m} P_c f(x_0)$ $= \frac{\delta}{2l^m} B_f$.

So if we apply Lemma 3.1 in the z direction, if $\epsilon \leq 1$, $|\{x \in [-r, r]^m : |f(x)| \leq \epsilon A_f\} \leq |\{x \in [-r, r]^m : |f(x)| \leq \frac{4\sqrt{m}\epsilon}{c} B_f\} \leq C \epsilon^{\frac{1}{|\alpha'|}} r^m \leq C r^m \epsilon^{\frac{1}{l}}$, giving us a). Lemma 3.1 also tells us that $A_f = \sup_{x \in [-r,r]^m} |f(x)| > \delta' B_f$ for some $\delta' > 0$. In other words we have $\frac{A_f}{B_f} \sim 1$.

Turning to conclusion b) of this lemma, if x_1, $x_2 \in [-r, r]^m$ then

$$|f(x_2) - f(x_1)| < \sum_{1 \leq |\alpha| \leq l} |\partial^\alpha f(x_1)||x_2 - x_1|^\alpha + C|x_2 - x_1|^{l+1}$$

$$\leq \frac{|x_2 - x_1|}{r} (\sum_{1 \leq |\alpha| \leq l} |\partial^\alpha f(x_1)|(2\sqrt{m}r)^\alpha + Cr^{l+1})$$

$$< \frac{|x_2 - x_1|}{r} \frac{2(2\sqrt{m})^l}{c^l} B_f$$

$$< C' \frac{|x_2 - x_1|}{r} A_f$$

(assuming $r < \delta_1$ for appropriately small δ_1). Thus b) is proven as well, and we are done.

Lemma 3.3: Suppose A is a subset of D_1 homeomorphic to the l-dimensional closed unit ball, and S is a surface in $\mathbf{R}^n$ that is parameterized by a C^1 function $f : A \longrightarrow \mathbf{R}^n$ such that $f(x) = (x_1, ..., x_l, \Phi_1(x_1, ..., x_l), ..., \Phi_{n-l}(x_1, ..., x_l))$. We further suppose that there exists a constant $C_1 > \frac{1}{2}$ such that if x and y are in A, then $|f(x) - f(y)| < C_1|x - y|$.

Then there exists a $\delta_1 > 0$ depending on l, n, C_1, and bounds on finitely many derivatives of γ, such that if $\{t_p\}_{p=1}^{3n^2}$,$\{t'_p\}_{p=1}^{3n^2}$ are in $\mathbf{R}^k$ with $|t_p|, |t'_p| < \delta_1$ for each p and S_t denotes the surface $\{\beta(x, t_1, ..., t'_{3n^2}) : x \in S\}$, then there exists an A_t homeomorphic to the closed unit ball of $\mathbf{R}^l$ such that S_t can be parameterized by $f_t : A_t \rightarrow \mathbf{R}^n$, where $f(x_1, ..., x_l) = (x_1, ..., x_l, \Phi_1^t(x_1, ..., x_l), ..., \Phi_{n-l}^t(x_1, ..., x_l))$. Furthermore we may assume $f_t(x)$ is C^1 in t and x and satisfies $|f_t(x) - f_t(y)| < 4C_1|x - y|$.

Note: the number $3n^2$ happens to be what is needed for our later arguments and is not critical to the proof.

Proof: If t_p, $t'_p = 0$ for all p, the map $x \longrightarrow \beta^{6n^2}(x, t_1, ..., t'_{3n^2})$ is just the identity map. Thus by smoothness, we may let $\delta_1 > 0$ such that for any x, $y \in D'_1$ (not necessarily in A), any $(t_1, ..., t'_{3n^2})$ satisfying $|t_p|, |t'_p| < \delta_1$ for all p, then we have

$$\beta^{6n^2}(y, t_1, ..., t'_{3n^2}) = \beta^{6n^2}(x, t_1, ..., t'_{3n^2}) + (y - x) + E$$

Here $|E| < \frac{1}{2C_1}|y-x|$. Thus if $P : R^n \longrightarrow R^l$ denotes projection onto the first l coordinates and x, $y \in S$ we have that

$$|P(\beta^{6n^2}(y, t_1, ..., t'_{3n^2})) - P(\beta^{6n^2}(x, t_1, ..., t'_{3n^2}))| > |Py - Px| - \frac{1}{2C_1}|y - x|$$

$$> \frac{1}{C_1}|y - x| - \frac{1}{2C_1}|y - x| = \frac{1}{2C_1}|y - x|$$

$$> \frac{1}{4C_1}|\beta^{6n^2}(y, t_1, ..., t'_{3n^2}) - \beta^{6n^2}(x, t_1, ..., t'_{3n^2})|$$

The conclusions of this lemma now follow in a straightforward fashion.

Lemma 3.4: Suppose $g : [-r, r]^l \times [a, b] \rightarrow \mathbf{R}^m$ is a C^1 function. Letting $P : \mathbf{R}^m \rightarrow \mathbf{R}^l$ be projection on the first l coordinates, we suppose that the vectors $\{P\partial_{x_p} g(x, t)\}_{p=1}^l$ span $\mathbf{R}^l$ for all $(x, t) \in [-r, r]^l \times [a, b]$. We further suppose that

$$(x, t) \in [-r, r]^l \times [a, b] \ \Rightarrow \ P\partial_t g(x, t) \in \{\sum_{p=1}^l v_p P\partial_{x_p} g(x, t) : |v_p| < C_1\} \tag{3.1}$$

Then if $a_1 < b_1 \in [a, b]$ with $|a_1 - b_1| < \frac{r}{4C_1}$, we have $\cup_{a_1 \leq t \leq b_1} Pg([-\frac{r}{2}, \frac{r}{2}]^l \times \{t\}) \subset \cap_{a_1 \leq t \leq b_1} Pg([-r, r]^l \times \{t\})$.

Proof: Suppose a_2, $b_2 \in [a_1, b_1]$ and suppose $x \in [-\frac{r}{2}, \frac{r}{2}]^l$. If $a_2 + \epsilon \in [a_1, b_1]$ and ϵ is sufficiently small, by (3.1) we have $Pg(x + y(x, \epsilon), a_2 + \epsilon) = Pg(x, a_2)$ for some $|y(x, \epsilon)| <$

$2C_1\epsilon$. Dividing $[a_2, b_2]$ (or $[b_2, a_2]$ as the case may be) into intervals $[a_2, a_2 + \epsilon_1]$, $[a_2 + \epsilon_1, a_2 + \epsilon_1 + \epsilon_2]$,...,$[b_2 - \epsilon_N, b_2]$ in this fashion, we obtain $Pg(x+y, b_2) = Pg(x, a_2)$ for some $|y| < \frac{r}{2}$. We thus conclude that $Pg([-\frac{r}{2}, \frac{r}{2}]^l \times \{a_2\}) \subset Pg([-r, r]^l \times \{b_2\})$. Since a_2 and b_2 were arbitrary, by taking unions over a_2 and intersections over b_2, the lemma follows.

Lemma 3.5:

a) Suppose $m > l$ are positive integers, and $\{v_p\}_{p=1}^m$ are vectors in $\mathbf{R}^l$. Then there is a constant $C_{l,m}$ such that we can always find a subset $\{p_1, ..., p_l\}$ of $\{1, ..., m\}$ such that $\{\sum_{q=1}^m c_q v_q : |c_q| \leq 1\} \subset \{\sum_{q=1}^l c_q v_{p_q} : |c_q| \leq C_{l,m}\}$.

b) Suppose $\{v_p\}_{p=1}^m$ are vectors spanning $\mathbf{R}^m$, and v_{m+1} is another vector such that the measure of $\{\sum_{q=1}^{m+1} c_q v_q : |c_i| < 1\}$ has measure less than C_1 times the measure of $\{\sum_{q=1}^m c_q v_q : |c_q| < 1\}$. Then for some constant D_m, v_{m+1} can be written as $\sum_{q=1}^m c_q v_q$ with $|c_q| < C_1 D_m$ for each q.

Proof: a) follows from an easy induction on $m \geq l$. For $m = l$ the conclusion is trivial. For $m > l$, we can find $c_1, ..., c_m$ not all zero such that $\sum_{q=1}^m c_q v_q = 0$. So if q_0 is such that c_{q_0} is of maximal magnitude, $v_{q_0} = \sum_{q \neq q_0} \frac{c_q}{c_{q_0}} v_q$ and each $\frac{c_q}{c_{q_0}}$ is of magnitude at most 1. Hence by applying the induction hypothesis to $\{v_q : q \neq q_0\}$, a) follows.

To see b), just change coordinates to ones in which v_q are the basic unit coordinate vectors $(0, ..., 0, 1, 0, ..., 0)$ in $\mathbf{R}^m$, in which case the result is evident.

For a function $f : [a_1, b_1] \times ... \times [a_m, b_m] \rightarrow \mathbf{R}^l$, and a vector $v = (v_1, ..., v_m) \in \mathbf{R}^m$ with $|v| = 1$, denote $\sum_{i=1}^m v_i \partial_{y_i} f(y_1, ..., y_m)$ by $\partial_v f(y_1, ..., y_m)$. For vectors ξ_1,...,ξ_p in $\mathbf{R}^m$ of norm 1, we define $Jac_{\xi_1,..,\xi_p} f(y_1, ..., y_m)$ to be the $\min\{l, p\}$-dimensional Hausdorff measure of the set $\{\sum_{q=1}^p v_q \partial_{\xi_q} f(y_1, ..., y_m) : |v_q| < 1 \, \forall q\}$. (The notation arises from the resemblance to Jacobian determinants.)

Notice that since $\gamma(x, t)$ has Jacobian in x equal to the identity matrix when $t = 0$, there is a $\delta > 0$ such that if $q_1 + q_2 \leq 3n^2$ and $|t_i|, |t_i'| < \delta$ for $0 \leq i \leq q_1 + q_2$, then for any vectors ξ_1,..., ξ_p in the $(2q_1 + 1)k$-dimensional $(t_1, t_1', ..., t_{q_1})$ space we have:

$$\frac{1}{2} < \frac{Jac_{\xi_1,...,\xi_p} \beta^{2q_1+2q_2-1}(x, t_1, t_1', ..., t_{q_1+q_2})}{Jac_{\xi_1,...,\xi_p} \beta^{2q_1-1}(x, t_1, t_1', ..., t_{q_1})} < 2 \tag{3.2a}$$

$$\frac{1}{2} < \frac{Jac_{\xi_1,...,\xi_p} \beta^{2q_1+2q_2-1}(x, t_1, t_1', ..., t_{q_1+q_2})}{Jac_{\xi_1,...,\xi_p} \beta^{2q_1}(x, t_1, t_1', ..., t_{q_1}')} < 2 \tag{3.2b}$$

For an l-dimensional hyperplane A in $\mathbf{R}^n$ going through the origin, let $P_A : \mathbf{R}^n \rightarrow \mathbf{R}^l$ be the usual projection. Where ξ_1,...,ξ_m are vectors in $\mathbf{R}^k$ each of which lies in the k-dimensional space corresponding to a t_q or t_q', for $m \geq l$ we define $M_{l,m}(x, r) =$

$$\sup_{dim A = l, |\xi_1|, ..., |\xi_m| = 1, |t_q|\,,\,|t_q'| < r \, \forall q} Jac_{\xi_1, \,...\,, \xi_m} P_A \beta^{2m}(x, t_1, t_1', ..., t_{m-1}', t_m, t_m')$$

By Lemma 3.5a), this is at most

$$C \times sup_{dim A=l,|\xi_1|,...,|\xi_l|=1,|t_q|\,,\,|t'_q|<r\,\forall q} Jac_{\xi_1,...,\xi_l} P_A \beta^{2m}(x, t_1, t'_1, ..., t'_{m-1}, t_m, t'_m)$$

We now come to the chief technical lemma concerning the functions β^m.

Lemma 3.6: There exists an $\epsilon > 0$ and constants $\epsilon_1, ..., \epsilon_n$ (depending on the usual parameters of T), such that for each $x \in D_1$ and each $0 < r < \epsilon$ we have:

a)If $M_{m,m}(x,r) > \epsilon_m r^{q_m}$, where q_m is a positive integer that depends on m, the integer Q of (C), and the parameters M, n, and k.

b)If $m \leq p \leq 6n^2$, then $M_{m,p}(x,r) < CM_{m,m}(x,r)$.

c)There exist unit vectors ξ_1,...,ξ_n $\in \mathbf{R}^k$, and a vector $(\bar{t_1}, \bar{t'_1}, ..., \bar{t'_{n-1}}, \bar{t_n}) \in [-r,r]^{k(2n-1)}$ such that for each $s = (s_1, ..., s_m)$ with $|s| < \epsilon_m r$ and for each $l \leq m$ we have

$$Jac_{\xi_1,...,\xi_l} \beta^{2m-1}(x, (\bar{t_1}, \bar{t'_1}, ..., \bar{t_m}) + \sum_{p=1}^{m} s_p \xi_p) > \epsilon_m M_{l,l}$$

Here each ξ_l is a unit vector in the t_l variable (the components of ξ_l in the other t and t' directions are all zero.)

d)Furthermore, for some $q \leq 6n^2$, suppose $(\dot{t_1}, \dot{t'_1}, ..., \dot{t_{q-1}}, \dot{t'_{q-1}}, \dot{t_q}) \in [-r,r]^{k(2q-1)}$ with $Jac_{\xi_1,...,\xi_l} \beta^{2q-1}(x, \dot{t_1}, \dot{t'_1}, ..., \dot{t_q}) > \delta_1 M_{l,l}$ for each $l \leq m$, where each ξ_p is a k-dimensional unit vector in one of the $2q-1$ t or t' directions. For fixed s_{l+1},...,s_m, let $S_{(s_{l+1},...,s_m)}$ denote the image of $\{(s_1, ..., s_l) : |s_p| < \epsilon_m \delta_1^p r$ for all $p\}$ under the map $(s_1, ..., s_l) \rightarrow \beta^{2q-1}(x, (\dot{t_1}, \dot{t'_1}, ..., \dot{t_q}) + \sum_{p=1}^{m} s_p \xi_p)$, and let S denote the entire image of $\{(s_1, .., s_m) : |s_p| < \epsilon_m \delta_1^p r$ for all p$\}$ under $(s_1, .., s_m) \rightarrow \beta^{2q-1}(x, (\dot{t_1}, \dot{t'_1}, ., \dot{t_q}) + \sum_{p=1}^{m} s_p \xi_p)$.

Then there is a sequence $A_1 \subset ... \subset A_m$ of hyperplanes through the origin, A_l of dimension l, such that if $P_l : \mathbf{R}^n \rightarrow A_l$ denotes the usual projection, then for each $(s_{l+1}$,...,$s_m)$, $P_l : S_{(s_{l+1},...,s_m)} \rightarrow A_l$ has an inverse $\Psi_{(s_{l+1},...,s_m)} : U_{(s_{l+1},...,s_m)} \subset A_l \rightarrow S_{(s_{l+1},...,s_m)}$ such that

$$\forall y\,,\, \forall z \in U_{(s_{l+1},...,s_m)}\ |\Psi_{(s_{l+1},...,s_m)}(y) - \Psi_{(s_{l+1},...,s_m)}(z)| < C|y - z| \tag{3.3}$$

for each y and z. Similarly, $P_m : S \rightarrow A_m$ has an inverse $\Psi : U_m \subset A_m \rightarrow S$ such that

$$\forall y\,,\, \forall z \in U_m\ |\Psi(y) - \Psi(z)| < C|y - z| \tag{3.4}$$

Furthermore, there are positive integers η_1, ...,η_n and positive constants c_1,...,c_n depending on δ_1 and the parameters of T such that U_m contains a (Euclidean) ball of radius $c_m r^{\eta_m}$.

Note: Again the number $6n^2$ happens to be what is useful for our arguments and is not critical for the proof of this theorem.

Proof: We induct on m. We first prove a), then we simultaneously prove b) and construct the $(\xi_1, ..., \xi_m)$ and $(t_1, t_1', ..., t_{m-1}, t_{m-1}', t_m)$ of c), and at the end we prove d). We assume $m > 1$ throughout; the case $m = 1$ is always an easier version of the arguments for $m > 1$, with the role of manifolds constructed in earlier stages of the induction being played by x , and the role of hyperplanes A_m from earlier stages in the induction being played by $\{0\}$.

So suppose we are at stage $m + 1$ of the induction. Let S be the surface corresponding to m in d) of this lemma, where $\tilde{t_l}$ and $\tilde{t_l'}$ are equal to the corresponding $\bar{t_l}$ and $\bar{t_l'}$ of c) of this lemma, and $\delta_1 = \epsilon_m$ of c) of this lemma. Rotating coordinates if necessary, assume for $1 \leq l \leq m$ that $A_l = \mathbf{R}^l$ (i.e. the last $n-l$ coordinates of points in A_l are zero.) By a) of induction hypothesis and (3.3), for some $\delta > 0$ the image $P_m(S)$ contains a (Euclidean) disk $D(x', \delta r^{q_m})$. (If $m = 0$, take P_0 to be projection onto the origin.) Let $\delta' > 0$ such that $\frac{\delta'}{\delta} < \frac{1}{2} \sup_{(x,t)\in D_1' \times D_2, 1\leq p \leq n} |\nabla_t \gamma_p(x,t)|$ (pth component of γ). For $q_0 > 0$, let B_{q_0} denote the set $\{y \in \mathbf{R}^n : P_m y \in D(x', \delta' r^{q_m}),\ |\Psi(P_m y) - y| \leq r^{q_0}\}$, Ψ as in d) of the induction hypothesis. Thus B_{q_0} has measure less than $Cr^{q_0(n-m)+mq_m}$.

Let $y' \in S$ such that $P_m y' = x'$. By the finite-type condition (C), $B(y', \delta' r^{q_m})$ (the ball in our metric) has measure at least $Cr^{q_m Q}$. Hence if the integer q_0 is chosen sufficiently large and r is sufficiently small (which we may assume by the Proposition of section 2), then $B(y', \delta' r^{q_m}) \not\subset B_{q_0}$. Hence there exist $\{x_l\}_{l=1}^N$ with $x_1 = y'$, $x_N \notin B_{q_0}$, $\{t_l\}_{l=1}^{N-1}$, $\{u_l\}_{l=1}^{N-1}$ with $\sum_{l=1}^{N-1} |t_l| + \sum_{l=1}^{N-1} |u_l| \leq \delta' r^{q_m}$, such that $\gamma(x_l, t_l) = \gamma(x_{l+1}, u_l)$ for each $1 \leq l \leq N - 1$. Note by definition of δ' that $P_m\gamma(x_l, t_l)$ and $P_m\gamma(x_{l+1}, u_l)$ $\in D(x', \delta r^{q_m})$. Define $y_l = \gamma(x_l, t_l)$ in what follows.

Since $x_N \notin B_{q_0}$, we have that $|x_N - \Psi P_m x_N| > r^{q_0}$. Let $\frac{|(x_{p+1} - \Psi P_m x_{p+1}) - (y_p - \Psi P_m y_p)|}{|u_p|}$ be the largest number in $\{\frac{|(x_l - \Psi P_m x_l) - (y_l - \Psi P_m y_l)|}{|t_l|}\}_{l=1}^{N-1} \cup \{\frac{|(x_{l+1} - \Psi P_m x_{l+1}) - (y_l - \Psi P_m y_l)|}{|u_l|}\}_{l=1}^{N-1}$. Although the largest number in this set might be of the form $\frac{|(x_p - \Psi P_m x_p) - (y_p - \Psi P_m y_p)|}{|t_p|}$, the following arguments are virtually identical in the 2 situations and thus we assume the former situation. We have

$$\frac{|(x_{p+1} - \Psi P_m x_{p+1}) - (y_p - \Psi P_m y_p)|}{|u_p|} \geq \frac{|(x_{p+1} - \Psi P_m x_{p+1})}{\sum_{l\leq p}(|t_l| + |u_l|)} \tag{3.5}$$

This is because the righthand expression, which is equal to $\frac{|(x_{p+1} - \Psi P_m x_{p+1}) - (x_1 - \Psi P_m x_1)|}{\sum_{l\leq p} |t_l| + |u_l|}$, is at most $\sum_{l=1}^{p} \frac{|(x_l - \Psi P_m x_l) - (y_l - \Psi P_m y_l)|}{\sum_{l\leq p} |t_l| + |u_l|} + \frac{|(x_{l+1} - \Psi P_m x_{l+1}) - (y_l - \Psi P_m y_l)|}{\sum_{l\leq p} |t_l| + |u_l|}$, a weighted average of several numbers whose maximum is $\frac{|(x_{p+1} - \Psi P_m x_{p+1}) - (y_p - \Psi P_m y_p)|}{|u_p|}$. In a similar vein, we have

$$\frac{|(x_{p+1} - \Psi P_m x_{p+1}) - (y_p - \Psi P_m y_p)|}{|u_p|} \geq \frac{|(x_N - \Psi P_m x_N)}{\sum_{l<N}(|t_l| + |u_l|)} > r^{q_0} \tag{3.6}$$

Let $x'' = \Psi P_m x_{p+1}$. Then, expanding in the first component, $\gamma(x'', u_p) = \gamma(x_{p+1} + \Psi P_m x_{p+1} - x_{p+1}, u_p) = y_p + \Psi P_m x_{p+1} - x_{p+1} + E$, where $|E|$ is bounded above by

$C \sup_{x \in D_1'} |d_x \gamma(x, u_p)| \, |\Psi P_m x_{p+1} - x_{p+1}|$. Since $\gamma(x,0) = x$ for each x, $\sup_{x \in D_1'} |d_x \gamma(x, u_p)| < C|u_p|$. So by (3.5) we have

$$E \leq C|(x_{p+1} - \Psi P_m x_{p+1}) - (y_p - \Psi P_m y_p)| \times \sum_{l \leq p} |t_l| + |u_l|$$

Define the function Θ by $\Theta(y) = \Psi P_m y - y$. Then $\Theta(y_p + \Psi P_m x_{p+1} - x_{p+1}) = \Theta(y_p) - \Theta(x_{p+1})$. Hence $\Theta(\gamma(x'', u_p)) = \Theta(y_p) - \Theta(x_{p+1}) + E'$, where since $|d_y(\Theta)| < C$, $|E'| < C|\Theta(x_{p+1}) - \Theta(y_p)| \sum_{l \leq p}(|t_l| + |u_l|)$. Note however that $\sum_{l \leq p}(|t_l| + |u_l|) < \delta' r^{q_m}$, thus if r is sufficiently small, by (3.5) we have

$$\Theta(\gamma(x'', u_p)) > \frac{1}{2}(\Theta(y_p) - \Theta(x_{p+1})) > \frac{1}{2}|u_p| r^{q_0}$$

In other words, while $x'' \in S$, the "vertical" distance between $\gamma(x'', u_p)$ and S is at least $\frac{1}{2} r^{q_0}$. If $t \in \mathbf{R}^k$ with $|t| \leq |u_p|$, let the surface S_t be $\{\gamma(x,t) : x \in S\}$. By Lemma 3.3, each S_t can be parameterized by $\alpha_t : U_t \subset \mathbf{R}_k \rightarrow \mathbf{R}^n$, where $\alpha_t(x_1, ..., x_m) = (x_1, ..., x_m, \alpha_t^1(x_1, ...x_m), ..., \alpha_t^{n-m}(x_1, ..., x_m))$ with $|\nabla_x \alpha_t^p| < 2C$ for each p. We can assume δ and δ' were chosen so that we have $D(x', \delta r^{q_m}) \subset U_t$ for all t. Since $P_m \gamma(x'', t) \in D(x', \delta r^{q_m})$ for each $|t| \leq |u_p|$ and $|\gamma(x'', u_p) - \Psi P_m \gamma(x'', u_p)| > \frac{1}{2} r^{q_0} |u_p|$, we conclude that there is a $|u| < \delta' r^{q_0}$, a $(x_1, ..., x_m) \in D(x', \delta r^{q_m})$, a component $(t)_s$ of t, and an $l \in \{1, ..., n-m\}$ such that

$$|\partial_{(t)_s} \alpha_u^l (x_1, ..., x_m)| > \delta'' r^{q_0}$$

But if $(t_1, t_1', t_2, ..., t_m, t_m', u)$ is such that $\beta^{2m-1}(x, t_1, t_1', t_2, ..., t_m, t_m', u) = \alpha_0(x_1, ..., x_m)$, and P_{m+1} denotes projection onto the plane spanned by $\mathbf{R}^k$ and the t_{m+1} direction, we have

$$\frac{|Jac_{\xi_1, ..., \xi_m, (t)_s} P_{m+1} \beta^{2m+1}(x, t_1, t_1', ..., t_m, t_m', u)|}{|Jac_{\xi_1, ..., \xi_m} P_m \beta^{2m-1}(x, t_1, t_1', ..., t_m, t_m')|} > C r^{q_0}$$

Thus a) of the induction follows.

Turning to b) and c), which we will do simultaneously: Let $q = 6n^2$, and let $(\hat{t_1}, \hat{t_1'}, ..., \hat{t_{q-1}'}, \hat{t_q}) \in [-r, r]^{k(2q-1)}$ and $\xi_1', ..., \xi_{m+1}'$ be unit vectors in a t or t' direction such that for some $(m+1)$- dimensional hyperplane A_{m+1}, $Jac_{\xi_1', ..., \xi_{m+1}'} P_{m+1} \beta^{2q-1}(x, \hat{t_1}, \hat{t_1'}, ..., \hat{t_q}) = M_{m+1,q}(x, r)$, where P_{m+1} denotes projection onto A_{m+1}. Note that we have

$$Jac_{\xi_1', .., \xi_{m+1}'} P_{m+1} \beta^{2q-1}(x, \hat{t_1}, \hat{t_1'}, ..., \hat{t_q}) = Jac_{\xi_1', .., \xi_{m+1}'} P_{m+1} \beta^{2q-1+2m}(x, 0, ..., 0, \hat{t_1}, \hat{t_1'}, .., \hat{t_q})$$

Let $\bar{t_l}$, $\bar{t_l'}$ be as in the case m of c) of the induction hypothesis. If we apply Lemma 3.2a) to the determinant of the $m+1$ by $m+1$ matrix whose ith row is given by $P_{m+1} \partial_{x_i} \beta^{2q-1+2m}(x, t_1, t_1', ..., t_{q+m-1}', t_{q+m})$ and we denote $\sup_{(t_1, t_1', .., t_{q+m}) \in [-r,r]^{(2q+2m-1)k}}$ $Jac_{\xi_1', ..., \xi_{m+1}'} P_{m+1} \beta^{2q+2m-1}(t_1, t_1', ..., t_{q+m})$ by $\bar{M}_{m+1,q}(x, r)$, then for a δ_0 which we will choose later, there exists a $(\tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}})$ such that for $s \leq m$, $|\tilde{t_s} - \bar{t_s}| < \delta_0 r$, for $s \leq m$ we have $|\tilde{t'_s}| < \delta_0 r$, for $s > m$, $|\tilde{t_s}| < \delta_0 r$ and such that for some δ_2 we have

$$Jac_{\xi'_1,...,\xi'_{m+1}} P_{m+1}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) > \delta_2 \bar{M}_{m+1,q}(x,r) \geq M_{m+1,q}(x,r)$$

Again using Lemma 3.2a, if δ_0 were chosen appropriately small and ξ_1,...,ξ_m are as in c) of the induction hypothesis, then $Jac_{\xi_1,...,\xi_m}\beta^{2m-1}(\tilde{t_1}, \tilde{t_1}', ..., \tilde{t_m}) > \frac{\epsilon_m}{2} M_{m,m}(x,r)$. By eq. 3.2a) and 3.2b), $Jac_{\xi_1,...,\xi_m}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) > \frac{\epsilon_m}{4} M_{m,m}(x,r)$. If P denotes projection onto the m-dimensional hyperplane A spanned by $\{\partial_{\xi_p}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) : 1 \leq p \leq m\}$, then by b) of the induction hypothesis and Lemma 3.5b), for each s $P\partial_{\xi'_s}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}$,..., $\tilde{t_{q+m}})$ is contained in $\{\sum_{p=1}^m v_p \partial_{\xi_p}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) : \forall p\, |v_p| < C'\}$ for appropriate C'. Thus we may find $\xi' \in \{\xi'_1, ..., \xi'_{m+1}\}$ such that we have:

$$|\partial_{\xi'}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) - P\partial_{\xi'}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}})| > \delta' \frac{M_{m+1,q}(x,r)}{M_{m,m}(x,r)}$$

Consequently, we may conclude that

$$Jac_{\xi_1,...,\xi_m,\xi'}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) > \delta'_2 M_{m+1,q}(x,r)$$

Thus denoting $\sup_{(t_1,t'_1,..,t_{q+m})\in[-r,r]^{(2q+2m-1)k}} Jac_{\xi_1,...,\xi_m,\xi'} P_{m+1}\beta^{2q+2m-1}(t_1, t'_1, .. ., t_{q+m})$ by $\hat{M}_{m+1,q}(x,r)$, by Lemma 3.2a) and changing $(\tilde{t_1}, \tilde{t'_1}$,...,$\tilde{t_{q+m}})$ if necessary, we can assume that the following holds:

$$Jac_{\xi_1,...,\xi_m,\xi'}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) > \delta''_2 \hat{M}_{m+1,q}(x,r) \geq \delta''_2 M_{m+1,q}(x,r)$$

Notice that there is some flexibility in choosing the hyperplane A_m corresponding to m and $(\tilde{t_1}, \tilde{t_1}', ..., \tilde{t_m})$ in d) of the induction hypothesis; replacing C by $2C$ in (3.3) and (3.4) any hyperplane making an angle $< \frac{1}{3C}$ with the original A_m works. In particular, we may choose A_m to be contained in an A_{m+1} satisfying

$$Jac_{\xi_1,...,\xi_m,\xi'} P_{m+1}\beta^{2q-1+2m}(x, \tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) > \delta'''_2 \hat{M}_{m+1,q}(x,r) \geq \delta'''_2 M_{m+1,q}(x,r) \tag{3.7}$$

Here P_{m+1} denotes projection onto A_{m+1}. Note by Lemma 3.2b) that, replacing δ'''_2 by $\frac{\delta'''_2}{2}$ if necessary, we may let $\delta_3 > 0$ such that (3.7) holds with $(\tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}})$ replaced by $(u_1, u'_1, ..., u_{q+m})$ whenever each $|u_i - \tilde{t_i}|$, $|u'_i - \tilde{t'_i}| < \delta_3 r$. Also, shrinking δ_3 if necessary, we may assume that for such u_i and u'_i we have $|u_i - \bar{t_i}|$,$|u'_i - \bar{t'_i}|$ are small enough so that for $1 \leq l \leq m$, $Jac_{\xi_1,...,\xi_l} P_m\beta^{2q-1+2m}(x, u_1, ..., u_{q+m}) > \frac{\epsilon_m}{8C} M_{l,l}(x,r)$. Rotating coordinates if necessary, in what follows we take A_m to be $\mathbf{R}^m$ and A_{m+1} to be $\mathbf{R}^{m+1}$.

Now define the map $\alpha : [-\delta_3 r, \delta_3 r]^{m+1} \to \mathbf{R}^n$ by

$$\alpha(s_1, .., s_{m+1}) = \beta^{2q-1+2m}(x, (\tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_{q+m}}) + \sum_{i=1}^{m+1} s_i \xi_i)$$

Define the surfaces S_s and T_s by

$$S_s = \alpha([-\delta_3 r, \delta_3 r]^m \times \{s\}), T_s = \alpha([\frac{-\delta_3 r}{2}, \frac{\delta_3 r}{2}]^m \times \{s\})$$

Using Lemma 3.3 and d) of the induction hypothesis, if δ_3 is appropriately small, we may let $U_s \subset \mathbf{R}^k$ and $\Psi_s : U_s \to \mathbf{R}^n$ parameterize S_s, where $\Psi_s(x_1, ..., x_m) = (x_1, ..., x_m, \Psi_s^{m+1}(x_1, ..., x_m), ..., \Psi_s^n(x_1, ..., x_m))$.

By Lemma 3.5b) and b) of the induction hypothesis, $P_m \partial_{s_{m+1}} \alpha(s_1, ..., s_{m+1}) \in \{\sum_{i=1}^m v_i P_m \partial_{s_i} \alpha(s_1, ..., s_{m+1}) : |v_i| < C\}$. As a result, by Lemma 3.4, we may find $\delta_4 > 0$ such that

$$\cup_{|s|<\delta_3 r} P_m T_s \subset \cap_{|s|<\delta_4 r} P_m S_s \tag{3.8}$$

By the condition (3.7) we have that $\partial_s \Psi_s^{m+1}(x_1, ..., x_m) > C\frac{M_{m+1,q}(x,r)}{M_{m,m}(x,r)}$ for all $(x_1, .., x_m) \in \cup_{|s|<\delta_3 r} P_m T_s$, so by (3.8) we have

$$(x_1, ..., x_m) \in \cup_{|s|<\delta_3 r} P_m T_s \longrightarrow |\Psi_{\delta_4 r}^{m+1}(x_1, ..., x_m) - \Psi_0^{m+1}(x_1, ..., x_m)| > \delta_5 \frac{M_{m+1,q}(x,r)}{M_{m,m}(x,r)} r \tag{3.9}$$

If $\tilde{S}$ denotes the surface $\{\beta^{2m-1}(x, (\tilde{t_1}, \tilde{t_1}', \tilde{t_2}, ..., \tilde{t'_{m-1}}, \tilde{t_m}) + \sum_{p=1}^m c_p \xi_p) : |c_p| < \frac{\epsilon_m^{m+1}}{2^m} r\}$, by d) of induction hypothesis, $\tilde{S}$ can be parameterized by $\Psi_0 : U \subset \mathbf{R}^m \to \mathbf{R}^n$, and furthermore we know that $P_m T_{\delta_4 r} \cup P_m T_0$ is in the domain of Ψ_0 if δ_4 is appropriately small by using b) of the induction hypothesis and applying Lemma 3.4. Thus by (3.8) there is a point y' on $T_{\delta_4 r}$ or T_0, of the form $\beta^{2q-1+2m}(x, (\tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_m}) + \sum_{p=1}^m c'_p \xi_p, (\tilde{t'_m}, ..., \tilde{t_{q+m}}) + \epsilon \xi')$ for some $|c'_p| < \delta_3 \, \forall p$, $\epsilon = 0$ or δ_4, such that y' satisfies $|y' - \Psi_0 P_m y'| > \delta_5 \frac{M_{m+1,q}}{M_{m,m}} r$.

Notice however that $x' = \beta^{2m}(x, (\tilde{t_1}, \tilde{t'_1}, ..., \tilde{t_m}) + \sum_{p=1}^m c'_p \xi_p) \in \tilde{S}$. So what the above says is that x' can be "linked" to y' in at most q steps with each link being of length $< \delta_3 r + \delta_4 r$, and the drop from y' to its projection on $\tilde{S}$ is at least $\delta_5 \frac{M_{m+1,q}(x,r)}{M_{m,m}(x,r)} r$. Thus by the exact argument used in part a) of this induction (for brevity it will not be included), if $y'' = \Psi_0 P_m y'$, then there is a $|t| < \delta_3 r + \delta_4 r$ such that

$$|\gamma(y'', t) - \Psi_0 P_m \gamma(y'', t)| > \delta_6 \frac{M_{m+1,q}(x,r)}{M_{m,m}(x,r)} |t| \tag{3.10}$$

Then if V_t denotes the surface $\{\gamma(y, t) : y \in \tilde{S}\}$, and V_t is parameterized by $(x_1, ..., x_m) \to \Upsilon^t(x_1, ..., x_m) = (x_1, ..., x_m, \Upsilon_{m+1}^t(x_1, ..., x_m), ..., \Upsilon_{n-m}^t(x_1, ..., x_m))$, by Lemma 3.3 we have that each $|\nabla_x(\Upsilon_p)| < C$, and by (3.10) (implicitly using a condition like (3.8) applied to the Υ_p^t) we have $|\partial_{(t)_s} \Upsilon_p^u(x_1, ..., x_m)| > \delta_6 \frac{M_{m+1,q}(x,r)}{M_{m,m}(x,r)}$ at some $(x_1, ..., x_m)$ and u, for some component $(t)_s$. If $(t_1, ..., t_m)$ are such that $\beta^{2m+1}(x, t_1, t'_1, ..., t_m, t'_m, u) = \Upsilon^u(x_1, ..., x_m)$, using that $|\nabla_x(\Upsilon_p)| < C$ for all p this means $Jac_{\xi_1, ..., \xi_{m+1}} \beta^{2m+1}(x, t_1, t'_1, ..., t'_m, u)$ is at least $\delta_7 M_{m+1,q}(x,r)$. In other words, $M_{m+1,m+1}(x,r) > \delta_7 M_{m+1,q}(x,r)$. Since by definition (set some ts and t's to zero) for $p \geq m$ we have $M_{m+1,p}(x,r) \geq M_{m+1,m+1}(x,r)$, this proves b) of the induction. Furthermore, this $(t_1, t_1', ...t_m, t_m', u)$ gives us c) of the induction as well.

We proceed to d). Suppose $(\dot{t_1}, \dot{t_1'}, .., \dot{t_{q-1}'}, \dot{t_q}) \in [-r,r]^{k(2q-1)}$ with $Jac_{\xi_1,..,\xi_l}\beta^{2q-1}(x, \dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) > \delta_1 M_{l,l}$ for each $l \leq m+1$. By the case m of the induction hypothesis, and using Lemma 3.3, we have hyperplanes $A_1 \subset ... \subset A_m$ through the origin such that if $l < m+1$, for each $s_{l+1},...,s_{m+1}$, $P_l : S_{(s_{l+1},...,s_{m+1})} \to A_l$ has an inverse $\Psi_{(s_{l+1},...,s_{m+1})} : U_{(s_{l+1},...,s_{m+1})} \subset A_l \to S_{(s_{l+1},...,s_{m+1})}$ satisfying

$$\forall y\,,\; z \in U_{(s_{l+1},...,s_{m+1})}\; |\Psi_{(s_{l+1},...,s_{m+1})}(y) - \Psi_{(s_{l+1},...,s_{m+1})}(z)| < C|y-z| \tag{3.11}$$

Notice that if we replace the A_l's by a sequence of nested hyperplanes $A_1' \subset ... \subset A_m'$ such that each A_l' makes an angle less than $\frac{1}{2C}$ with its corresponding A_l, then (3.11) still holds with C perhaps replaced by $3C$. Hence, replacing A_l's by A_l''s if necessary, we may assume $A_m \subset A_{m+1}$ for some A_{m+1} with $Jac_{\xi_1,...,\xi_{m+1}} P_{A_{m+1}}\beta^{2q-1}(x, \dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) > \delta\delta_1 M_{m+1,m+1}(x,r)$. Rotating coordinates and changing A_{m+1} if necessary, assume that

$$\forall l \leq m\; A_l = \mathbf{R}^l \text{ and } \partial_{\xi_{m+1}}\beta^{2q-1}(x, \dot{t_1}, \dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) \in A_{m+1} = \mathbf{R}^{m+1} \tag{3.12}$$

Thus we may take $A_{m+1} = \mathbf{R}^{m+1}$. Denote by $P_l : \mathbf{R}^n \to \mathbf{R}^l$ the projection onto the first l coordinates. So we have that for each $l \leq m+1$, $Jac_{\xi_1,...,\xi_l} P_l \beta^{2q-1}(x, \dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) > \delta_2\delta_1 M_{l,l}$ for an appropriate δ_2. So assuming δ_3 is appropriately small, by Lemma 3.2b we have

$$\forall l \leq m+1\; |s_p| < \delta_3\delta_1 r \to Jac_{\xi_1,...\xi_l} P_l \beta^{2q-1}(x, (\dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) + \sum_{p=1}^{m} s_p\xi_p) > \frac{\delta_2}{2}\delta_1 M_{l,l} \tag{3.13}$$

Shrinking ϵ_m if necessary, we may assume (3.13) holds for $|s_l| < \epsilon_m \delta_1^l r$ as well.

For $s \in [-\epsilon_m\delta_1^m r, \epsilon_m\delta_1^m r]$, define the surface S_s by

$$S_s = \{\beta^{2q-1}(x, (\dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) + \sum_{p=1}^{m} s_p\xi_p + s\xi_{m+1}) : |s_l| < \epsilon_m\delta_1^l r\}$$

By Lemma 3.3, for each such S_s, we can find a $\Psi_s : U_s \subset \mathbf{R}^m \to \mathbf{R}^n$ parameterizing S_s, with $\Psi_s(x_1, ..., x_m) = (x_1, ..., x_m, \Psi_s^1(x_1, ..., x_m), ..., \Psi_s^{n-m}(x_1\;, ..., x_m))$ with each $|\nabla_x \Psi_s^p| < 6C$. If $(s_1, ..., s_m)$ is such that $\beta^{2q-1}(x, (\dot{t_1}, \dot{t_1'}, .., \dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1}) = \Psi_s(x_1, ..., x_m)$, then we have

$$|\partial_s \Psi_s^1(x_1, ..., x_m)| > \frac{Jac_{\xi_1,...,\xi_{m+1}} P_{m+1}\beta^{2q-1}(x, (\dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1})}{Jac_{\xi_1,...,\xi_m} P_m \beta^{2q-1}(x, (\dot{t_1}, \dot{t_1'}, ..., \dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1})}$$

By (3.13) the numerator of this expression is at least $\frac{\delta_2}{2}\delta_1 M_{m+1,m+1}(x,r)$. Similarly, for $l \neq 1$ if $P_{m,l}$ denotes projection onto the hyperplane generated by $\mathbf{R}^m$ and the $(l+m)$th unit coordinate vector, we have

$$|\partial_s \Psi_s^l(x_1,...,x_m)| < 6C\frac{Jac_{\xi_1,...,\xi_{m+1}} P_{m,l}\beta^{2q-1}(x,(\dot{t_1},\dot{t_1'},...,\dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1})}{Jac_{\xi_1,...,\xi_m} P_m \beta^{2q-1}(x,(\dot{t_1},\dot{t_1'},,...,\dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1})}$$

However, the numerator of this expression is 0 when $s_l = 0$ for all $l \leq m$ by our choice of coordinate system (3.12). So by Lemma 3.2b), this numerator is at most $C'\delta_1 M_{m+1,m+1}(x,r)$. Thus we have

$$\forall l > 1 \; \frac{|\partial_s \Psi_s^l(x_1,...,x_m)|}{|\partial_s \Psi_s^1(x_1,...,x_m)|} < C'' \tag{3.14}$$

Define the surface T_s by

$$T_s = \{\beta^{2q-1}(x,(\dot{t_1},\dot{t_1'},...,\dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1}) : |s_l| < \frac{\epsilon_m}{2}\delta_1^l r\}$$

Denote $P_m T_s$ by V_s. By (3.13), b) of induction hypothesis, and Lemma 3.5b), for some $|c_p| < C\delta_1^{-1}$ we have

$$P_m \partial_s \beta^{2q-1}(x,(\dot{t_1},\dot{t_1'},...,\dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1})$$

$$= \sum_{p=1}^m c_p P_m \partial_s \beta^{2q-1}(x,(\dot{t_1},\dot{t_1'},...,\dot{t_q}) + \sum_{p=1}^m s_p\xi_p + s\xi_{m+1})$$

Thus by Lemma 3.4, if ϵ_{m+1} is sufficiently small, we have

$$\cup_{|s|<\epsilon_{m+1}\delta_1^{m+1}r} V_s \subset \cap_{|s|<\epsilon_{m+1}\delta_1^{m+1}r} U_s$$

Therefore, since $|\partial_s \Psi_s^1(x_1,...,x_m)| \neq 0$, the sets $P_{m+1}T_s$ are disjoint; they "lie above" a set in $\mathbf{R}^m$ where the Ψ_s are all defined. Hence the projection P_{m+1} is invertible on the set $S = \{\beta^{2q-1}(x,(\dot{t_1},\dot{t_1'},...,\dot{t_q}) + \sum_{p=1}^{m+1} s_p\xi_p) : \forall l \leq m \; |s_l| < \frac{\epsilon_m}{2}\delta_1^l r \, , \, |s_{m+1}| \leq \epsilon_{m+1}\delta_1^{m+1} r\}$. Furthermore, by induction hypothesis each set V_s for $|s| < \epsilon_{m+1}\delta_1^{m+1}r$ contains a Euclidean ball B_m of radius $c_m' r^{\eta_m}$ for some constant c_m' depending on δ_1 and the parameters of T. Therefore, $P_{m+1}S$ contains the cylinder $B_m \times I_m$ for some interval I_m of radius $> d_m r^{\nu_m}$ for some nonnegative d_m and ν_m, where ν_m may depend on the parameters of T and d_m may depend on the parameters of T as well as δ_1. Thus $P_{m+1}S$ contains a ball of some radius $c_{m+1}r^{\eta_{m+1}}$ as advertised.

Let $U = \cap_{|s|<\epsilon_{m+1}\delta_1^{m+1}r} U_s$ The proof of d) of this induction will be complete once we show the inverse Ψ to P_{m+1} satisfies (3.4) on the set $\{(x_1,...,x_m,\Psi_s^1(x_1,...,x_m)) :$

$(x_1,..,x_m) \in U, |s| < \epsilon_{m+1}\delta_1^{m+1}r\}$. Note that $\Psi(x_1,..,x_m,\Psi_s^1(x_1,..,x_m)) = (\Psi_s^2(x_1,..,x_m)$ $,..,\Psi_s^{n-m}(x_1,...,x_m))$. So suppose $(x_1,...,x_m)$, $(x_1',...,x_m') \in U$ and $|s|,|s'| < \epsilon_{m+1}\delta_1^{m+1}$. Then for $l \geq 2$, we have

$$|\Psi_s^l(x_1,...,x_m) - \Psi_{s'}^l(x_1',...,x_m')| < |\Psi_s^l(x_1,...,x_m) - \Psi_{s'}^l(x_1,...,x_m)|$$

$$+|\Psi_{s'}^l(x_1,...,x_m) - \Psi_{s'}^l(x_1',...,x_m')|$$

The right hand term is bounded by $C|(x_1,...,x_m)-(x_1',...,x_m')|$, which is at most a constant times $|(x_1,...,x_m,\Psi_s^1(x_1,...,x_m)) - (x_1',...,x_m',\Psi_s^1(x_1',...,x_m'))|$ and we are done with that term. As for the left hand term, by (3.12)

$$|\Psi_s^l(x_1,...,x_m) - \Psi_{s'}^l(x_1,...,x_m)| < C''|\Psi_s^1(x_1,...,x_m) - \Psi_{s'}^1(x_1,...,x_m)|$$

$$< C''|\Psi_{s'}^1(x_1,...,x_m) - \Psi_{s'}^1(x_1',...,x_m')| + C''|\Psi_{s'}^1(x_1',...,x_m') - \Psi_s^1(x_1,...,x_m)|$$

$$< C|(x_1,...,x_m) - (x_1',...,x_m')| + C''|\Psi_s^1(x_1,...,x_m) \; - \; \Psi_{s'}^1(x_1',...,x_m')|$$

$$< \max{(C',C'')}|(x_1,...,x_m,\Psi_s^1(x_1,...,x_m)) - (x_1',...,x_m',\Psi_s^1(x_1',...,x_m'))|$$

Therefore we are done with the left hand term as well. Part d) of the induction now follows, and we are done with the proof of Lemma 3.6.

Corollary: Each ball $B(x,r)$ of the metric of this paper is an open set with respect to the usual Euclidean topology and furthermore contains a Euclidean ball centered at x of radius cr^α where c and α are independent of x and r. In addition, the same holds for the ball $B_{2n}(x,r)$ defined by (2.2).

Proof: The proof is virtually indentical for $B(x,r)$ and $B_{2n}(x,r)$ so we restrict attention to the case of $B(x,r)$. From c) and d) of Theorem 3.6, we know that the set $A = \{\beta^{2n-1}(x,t_1,0,t_2,0,...,t_n) : |t_1|,...,|t_n| < r\}$ contains a Euclidean ball B of radius c_0r^α for some constants c_0 and α independent of x and r. Since the map $y \rightarrow \beta^{2n}(y,0,t_n,0,t_{n-1},0,...,t_1)$ is a diffeomorphism with Jacobian near the identity for small $|t_1|$,..., $|t_n|$, we can therefore assume that $A' = \{\beta^{4n-2}(x,t_1,0,t_2,0,...,t_n,t_n',0,t_{n-1}',0,...,0,t_1') : |t_1|,..., |t_n| < r\}$ contains the Euclidean ball centered at x of radius c_0r^α, where $(t_1',...,t_n')$ is the point such that $\beta^{2n-1}(x,t_1',0,t_2',0,...,t_n')$ is the center of B. Therefore $B(x,4nr)$ contains the Euclidean ball centered at x of radius c_0r^α. Letting $c = \frac{c_0}{(4n)^\alpha}$ gives us the latter statement of the Corollary. The first statement is now an immediate consequence. For if $y \in B(x,r)$, then with respect to the distance function d of our metric we have $d(x,y) = r' < r$, and thus $B(x,r)$ contains the Euclidean ball centered at y of radius $c(r'-r)^\alpha$.

We now are in a position to show that the curvature condition of this paper is equivalent to that of [3]. Although this fact is not used in the arguments of this paper, it is clearly important to tie together the two curvature conditions.

Proposition: The curvature condition (C) of this paper is equivalent to the curvature condition of [3].

Proof: Assume (C) holds. Then by a) and c) of Lemma 3.6, we may find directions ξ_1,...,ξ_n, a sequence of positive constants $\{r_p\}_{p=1}^{\infty}$ with $r_p \to 0$, and a sequence of vectors $\{(t_1^p, 0, t_2^p, 0, ..., 0, t_n^p)\}_{p=1}^{\infty}$ such that each ξ_s is a component of a t_s, $|t_s^p| < r_p$ for each s and p, and $|Jac_{\xi_1,...,\xi_n}\beta^{2n-1}(x, t_1^p, 0, t_2^p, 0, ..., 0, t_n^p)| > \epsilon_n^2 r^{q_n}$ for each p. This implies that as a function of $(t_1, ..., t_n)$, the function $|Jac_{\xi_1,...,\xi_n}\beta^{2n-1}(x, t_1, 0, t_2, 0, ..., 0, t_n)|$ has a zero of order at most q_n at the origin. But one of the equivalent formulations of the curvature condition of [3] is exactly that this determinant function vanishes to finite order. Hence (C) implies the curvature condition of [3].

Now assume the curvature condition of [3] holds at a point x. By the Jacobian formulation of this condition, as above we may find directions ξ_1,...,ξ_n, a sequence of positive constants $\{r_p\}_{p=1}^{\infty}$ with $r_p \sim 2^{-p}$, and vectors $\{(t_1^p, 0, t_2^p, 0, ..., 0, t_n^p)\}_{p=1}^{\infty}$ such that each ξ_s is a component of a t_s, $|t_s^p| < r_p$ for each s and p, and $|Jac_{\xi_1,....,\xi_n}\beta^{2n-1}(x, t_1^p, 0, t_2^p, 0, ..., 0, t_n^p)|$ $> Cr_p^{\alpha}$ for each p, some constants C and α. Therefore, for each $m < n$ and each p, $|Jac_{\xi_1,...,\xi_m}\beta^{2m-1}(x, t_1^p, 0, t_2^p, 0, ..., 0, t_m^p)| > C' r_p^{\alpha}$ for some constant C'. We now may apply the inductive argument proving d) of Lemma 3.6. For parts a), b) and c) of that lemma hold if we take ϵ_m to be $C' r_p^{\alpha}$ and the constant of b) to be $\frac{C''}{r_p^{\beta}}$ for appropriate $\beta \geq 0$ and C''. (Of course in the proof of Lemma 3.6 these constants did not depend on r, but we may still apply the inductive argument used for proving d) using constants that do depend on r). Similarly, we may take δ_1 in the statement of d) to be $C''' r_p^{\gamma}$ for appropriate C''' and γ, and $(\dot{t_1}, \dot{t_1'}, ..., \dot{t_{n-1}'}, \dot{t_n}) = (t_1^p, 0, t_2^p, 0, ..., 0, t_n^p)$. Since in the proof of d) the constants c_1,...,c_n depend polynomially on ϵ_m, δ_1, and the constant of the statement b), we then have that the ball $D(x, r_p)$ with respect to the metric of this paper has volume at least Cr_p^A for some large A. Therefore the condition (C) holds, and the proof of the proposition is complete.

Lemma 3.7: Suppose for some $a \leq 1$ and some $m \leq 3n^2$, $(\bar{t_1}, \bar{t_1'}, .., \bar{t_m}, \bar{t_m'})$ satisfies

$$Jac_{\xi_1,..,\xi_n}\beta^{2m}(x, \bar{t_1}, \bar{t_1'}, .., \bar{t_m}, \bar{t_m'}) \geq aM_{n,n}(x, r)$$

Further suppose that x' and $(\dot{t_1}, \dot{t_1'}, ..., \dot{t_m}, \dot{t_m'})$ are such that

$$\beta^{2m}(x, \bar{t_1}, \bar{t_1'}, .., \bar{t_m}, \bar{t_m'}) = \beta^{2m}(x', \dot{t_1}, \dot{t_1'}, ..., \dot{t_m}, \dot{t_m'})$$

Then there is a $\delta > 0$ (independent of a) such that if $(t_1, t_1', ..., t_m, t_m')$ satisfies $|t_l|$, $|t_l'| \leq \delta a^2 r$ for each l, then there exists a unique map $R : [0, 1] \to \mathbf{R}^n$ such that for $s \in [0, 1]$,

$$\beta^{2m}(x, (\bar{t_1}, \bar{t_1'}, ..., \bar{t_m}, \bar{t_m'}) + \sum_{l=1}^{n} R_l(s)\xi_l) = \beta^{2m}(x', s(t_1, t_1', ..., t_m, t_m') + (\dot{t_1}, \dot{t_1'}, \ldots, \dot{t_m}, \dot{t_m'}))$$

Proof: Suppose $s_0 \in [0, 1]$ and the conclusion of the lemma holds for s_0. Then by examining $\beta^{4m}(x, s_0(t_1, t_1', ..., t_m, t_m') + (\dot{t_1}, \dot{t_1'}, ..., \dot{t_m}, \dot{t_m'}), (\bar{t_m}, ..., \bar{t_1'}, \bar{t_1}) + \sum_{l=1}^{n} R^l(s)\xi_l)$ and using

Lemma 3.5b) and the fact that $M_{n,2m}(x,r) \sim M_{n,n}(x,r)$, for a unique $(c_1(s_0),...,c_n(s_0)) \in \mathbf{R}^n$ satisfying $|c_l(s_0)| \leq Ca^{-1}$, we have that

$$\partial_v|_{v=0}\beta^{2m}(x',(s_0+\, v)(t_1,t_1',...,t_m,t_m') + (\dot{t_1},\dot{t_1'},...,\dot{t_m},\dot{t_m'}))$$

$$= \sum_{l=1}^{n} c_l(s_0)\partial_v|_{v=0}\beta^{2m}(x,(\bar{t_1},\bar{t_1'},...,\bar{t_m}) + \sum_{l=1}^{n} R^l(s)\xi_l + v\xi_l)$$

Thus for s slightly larger than s_0 we can define the $R_l(s)$ in the statement of this lemma by the differential equation $R_l'(s) = c_l(s)$. If $\delta > 0$ is sufficiently small, the condition that $|t_l|$, $|t_l'| \leq \delta a^2 r$ for all l ensures that $|\sum_{l=1}^n R_l(s)| \leq \delta_2 ar$ for each s, where δ_2 is sufficiently small that $Jac_{\xi_1,...,\xi_n}\beta^{2m}(x,(\bar{t_1},\bar{t_1'},...,\bar{t_m},\bar{t_m'}) + \sum_{l=1}^n R_l(s)\xi_l) > \frac{a}{2}M_{n,n}(x,r)$; such a δ_2 exists by Lemma 3.2b) and the fact that $M_{n,m}(x,r) \sim M_{n,n}(x,r)$. Hence we are done with the proof of the lemma.

We now proceed to the construction of our coordinate system for $B(x,r)$ which we need for our definition of the x-derivative condition on $K(x,t)$. By Lemma 3.6, we may let $(\bar{t_1},\bar{t_1'},...,\bar{t_n},\bar{t_n'})$ with $|t_l|$, $|t_l'| \leq r$, ξ_1,...,ξ_n k-dimensional unit vectors in t_1,...,t_n directions respectively, such that $\beta^{2n-1}(x,\bar{t_1}+r_1\xi_1,\bar{t_1'},...,\bar{t_n}+r_n\xi_n)$ is one to one on $(r_1,...,r_n) \in [-\delta r,\delta r]^n$, and $Jac_{\xi_1,...,\xi_n}\beta^{2n-1}(x,\bar{t_1}+r_1\xi_1,\bar{t_1'},...,\bar{t_n}+r_n\xi_n) \geq \delta M_{n,n}(x,r)$ for $(r_1,..,r_n) \in [-\delta r,\delta r]^n$. By eq (3.2$a$), writing $\Xi_{x,r}(r_1,...,r_n) = \beta^{4n-2}(x,\bar{t_1}+r_1\xi_1,\bar{t_1'},...,\bar{t_n}+r_n\xi_n,\bar{t_n},\bar{t_{n-1}'}...,\bar{t_1'},\bar{t_1})$, $\Xi_{x,r}$ is one-to-one in $(r_1,..,r_n)$ on $[-\delta r,\delta r]^n$, with $\Xi(0,...0) = x$. Denote $\Xi_{x,r}^{-1}(y)$ by $\bar{Z}_{x,r}$ when it exists. In the next lemma, we will see that $\bar{Z}_{x,r}$ acts as a coordinate system for $B(x,\delta_1 r)$ for an appropriate δ_1. Thus we denote this coordinate system $\bar{Z}_{x,r}$ by $Z_{x,\delta_1 r}$.

Lemma 3.8: There exists $\delta_1 > 0$ such that if $d(x,x') < 6n^2\delta_1 r$, then

a)$\bar{Z}_{x,r}(x')$ exists and $|\bar{Z}_{x,r}(x') - \bar{Z}_{x,r}(x)| = |\bar{Z}_{x,r}(x')| < Cd(x,x')$.

b)If $(t_1,t_1',...,t_{3n^2},t_{3n^2}')$ and $(\dot{t_1},\dot{t_1'},...,\dot{t_{3n^2}},\dot{t_{3n^2}'})$ satisfy $|t_l - \dot{t_l}|$, $|t_l' - \dot{t_l'}| < \delta_1 r$, then $|\bar{Z}_{x,r}(\beta^{6n^2}(x,t_1,t_1',...,t_{3n^2}')) - \bar{Z}_{x,r}(\beta^{6n^2}(\dot{t_1},\dot{t_1'},...,\dot{t_{3n^2}'}))| < C\sum_l(|t_l - \dot{t_l}| + |t_l' - \dot{t_l'}|)$.

Proof: Since $M_{n,n}(x,r) \sim M_{n,2n}(x,r)$ and $\beta^{2n+2}(x,u_1,...,u_{n+1}')$ has Jacobian in x near I, we may assume that for $(r_1,..,r_n) \in [-\delta r,\delta r]^n$,

$$Jac_{\xi_1,...,\xi_n}\beta^{4n}(x,\bar{t_1}+r_1\xi_1,\bar{t_1'},...,\bar{t_n}+r_n\xi_n,\bar{t_n},...,\bar{t_1'},\bar{t_1},0,0) \geq \delta' M_{n,n}(x,r)$$

By Lemma 3.7, $\gamma(x,t) = \beta^{4n}(x,\bar{t_1},\bar{t_1'},\,...\,,\bar{t_n},\bar{t_n},...,\,\bar{t_1'},\bar{t_1},t,0)$ can be written as $\beta^{4n}(x,\bar{t_1}+r_1\xi_1,\bar{t_1'},...,\bar{t_n}+r_n\xi_n,\bar{t_n},...,\bar{t_1'},\bar{t_1},0,0)$ for $\sum_{l=1}^n |r_l| \leq C_0|t| < \delta r$, assuming that $|t| < \delta_1 r$ for an appropriate δ_1. So we have that $\bar{Z}_{x,r}(\gamma(x,t))$ exists and $|\bar{Z}_{x,r}(\gamma(x,t))| \leq C_0|t|$. Repeating this, for some $\{r_l\}$ and $\{r_l'\}$ with $\sum_{l=1}^n |r_l| + |r_l'| \leq C_0(|t| + |t'|) < \delta r$, we have that

$$\beta^2(x,t,t') = \beta^{4n}(x,\bar{t_1}+r_1\xi_1,\bar{t_1'},...,\bar{t_n}+r_n\xi_n,\bar{t_n},...,\bar{t_1'},\bar{t_1},t',0)$$

$$= \beta^{4n}(x, \bar{t_1} + (r_1 + r_1')\xi_1, \bar{t_1'}, ..., \bar{t_n} + (r_n + r_n')\xi_n, \bar{t_n}, ..., \bar{t_1'}, \bar{t_1}, 0, 0)$$

Thus $|\bar{Z}_{x,r}\beta^2(x,t,t')| \leq C_0(|t| + |t'|)$.

We proceed inductively in this fashion, applying Lemma 3.7 with $x' = \beta^2(x, t_1, t_1')$ then $x' = \beta^4(x, t_1, t_1', t_2, t_2')$ and so on. The result is that for any N we have that $|\bar{Z}_{x,r}\beta^{2N}(x, t_1, .., t_N, t_N')| \leq C_0 \sum_{l=1}^N |t_l|$ assuming $\sum_{l=1}^N |t_l| \leq \delta_1 r$; the condition that $\sum_{l=1}^N |t_l| \leq \delta_1 r$ ensures that the resulting $\sum_{m=1}^n (\sum_{l=1}^N (r_{l,m} + r_{l,m}'))\xi_m$ is sufficiently small that we may continue to apply Lemma 3.7, i.e. it is smaller than what was denoted by "$\delta a^2 r$" in the statement of that lemma. Thus by the definition of $d(x,x')$, the proof of a) is complete.

The statement b) is a consequence of a) and Lemma 3.7. Namely, if the hypotheses of b) are satisfied, we may let r_1,...,r_n and $\dot{r_1}$,...,$\dot{r_n}$ be such that

$$\beta^{4n-2}(x, \bar{t_1} + r_1\xi_1, \bar{t_1'}, ..., \bar{t_n} + r_n\xi_n, \bar{t_n}, ..., \bar{t_1'}, \bar{t_1}) = \beta^{6n^2}(x, t_1, t_1', ..., t_{3n^2}, t_{3n^2}')$$

$$\beta^{4n-2}(x, \bar{t_1} + \dot{r_1}\xi_1, \bar{t_1'}, ..., \bar{t_n} + \dot{r_n}\xi_n, \bar{t_n}, ..., \bar{t_1'}, \bar{t_1}) = \beta^{6n^2}(x, \dot{t_1}, \dot{t_1'}, ..., \dot{t_{3n^2}}, \dot{t_{3n^2}'})$$

We apply Lemma 3.7, giving us that $\sum_l |r_l - \dot{r_l}| \leq C \sum_l (|t_l - \dot{t_l}| + |t_l' - \dot{t_l'}|)$. Hence b) is proved.

Lemma 3.9: Suppose $d(x,x') \leq r$. Then $M_{n,n}(x,r) \sim M_{n,n}(x',r)$. In addition, the measure of $B(x,r) \sim M_{n,n}(x,r)r^n$. Furthermore, if $(t_1, t_1', ..., t_{3n^2}, t_{3n^2}')$ is such that each $|t_l|$, $|t_l| < a < r$, then if $b > 0$ with $a + b < r$, the measure of the set $A = \{\beta^{6n^2}(x, t_1 + u_1, t_1' + u_1', ..., t_{3n^2} + u_{3n^2}, t_{3n^2}' + u_{3n^2}') : \ |u_l|, |u_l'| < b\}$ is at most $CM_{n,n}(x,r)b^n$.

Proof: By Lemma 3.8, we can write $x = \beta^{4n-2}(x', \bar{t_1}, \bar{t_1'}, ..., \bar{t_{2n-1}}, \bar{t_{2n-1}'})$ for $|\bar{t_l}|, |\bar{t_l'}| \leq \delta_1^{-1}r$, δ_1 as in Lemma 3.8. Thus if each $|t_l|, |t_l'| \leq \delta_1^{-1}r$ and each $\xi_1 = \xi_n = 1$, then

$$Jac_{\xi_1,...,\xi_n}\beta^{2n}(x, t_1, t_1', ..., t_n') = Jac_{\xi_1,...,\xi_n}\beta^{6n-2}(x', \bar{t_1}, \bar{t_1'}, ..., \bar{t_{2n-1}}, \bar{t_{2n-1'}}, t_1, t_1', ..., t_n, t_n')$$

Hence $M_{n,n}(x,r) \leq M_{n,3n-1}(x', \delta_1^{-1}r) \leq CM_{n,n}(x', \delta_1^{-1}r)$, the last inequality following from Lemma 3.6b). By Lemma 3.2a), if $\delta_2 > 0$ is sufficiently small, we have

$$|\{(t_1, ..., t_n') \in [-\delta_1^{-1}r, \delta_1^{-1}r]^{2n} : Jac_{\xi_1,...,\xi_n}\beta^{2n}(x', t_1, t_1', ..., t_n, t_n') \leq \delta_2 M_{n,n}(x', \delta_1^{-1}r)\}|$$

$$< \frac{1}{(2\delta_1^{-1})^{2n}} |[-\delta_1^{-1}r, \delta_1^{-1}r]^{2n}| = |[-r,r]^{2n}|$$

Thus there exists a $(t_1, t_1', ..., t_n, t_n') \in [-r,r]^{2n}$ with $Jac_{\xi_1,...,\xi_n}\beta^{2n}(x', t_1, t_1', ..., t_n, t_n') \geq \delta_2 M_{n,n}(x', \delta_1^{-1}r)$. Thus $M_{n,n}(x', \delta_1^{-1}r) \leq \delta_2^{-1}M_{n,n}(x', r)$ and we conclude $M_{n,n}(x,r) \leq C\delta_2^{-1}M_{n,n}(x',r)$. By symmetry, we have $M_{n,n}(x',r) \leq C\delta_2^{-1}M_{n,n}(x,r)$ as well and we conclude $M_{n,n}(x,r) \sim M_{n,n}(x',r)$.

Moving to the second statement of this lemma, by Lemma 3.6 we can find a $(t_1, t_1', .., t_n) \in [-r, r]^{2n-1}$, directions $\xi_1,..,\xi_n$ with $|\xi_l| = 1$ for each l such that $(r_1, ..., r_n) \to \beta^{2n-1}(x, (t_1, t_1', .., t_n) + \sum_{l=1}^n r_l\xi_l)$ is one-to-one on $[-\delta r\, ,\, \delta r]^n$ and such that $Jac_{\xi_1,..,\xi_n}\beta^{2n-1}$ $(x, (t_1, t_1', .., t_n) + \sum_{l=1}^n r_l\xi_l) \geq \delta M_{n,n}(x, r)$ for $(r_1, ..., r_n) \in [-\delta r\, ,\, \delta r]^n$. Hence

$$|B(x,r)| > |\{\beta^{2n-1}(x, (t_1, t_1', ..., t_n) + \sum_{l=1}^n r_l\xi_l) : (r_1, ..., r_n) \in [-\delta r, \delta r]^n\}|$$

$$\geq \delta' M_{n,n}(x, r)r^n$$

Next, note by Lemma 3.8 that $B(x, r) \subset$ Image of $Z_{x,r}$. Recall that $\{$Image of $Z_{x,r}\} \subset \{\beta^{4n-2}(x, (\bar{t_1}, \bar{t_1'}, ..., \bar{t_n}) + \sum_{l=1}^n r_l\xi_l, \bar{t_n}, ..., \bar{t_1'}, \bar{t_1}) : (r_1, ..., r_n) \in [-\frac{r}{\delta_1}, \frac{r}{\delta_1}]^n\}$. Thus

$$|\text{Image of} Z_{x,r}| \leq M_{n,2n-1}(x, \frac{r}{\delta_1})(\frac{r}{\delta_1})^n \leq CM_{n,n}(x, \frac{r}{\delta_1})r^n \leq C'M_{n,n}(x, r)r^n$$

(See above for the last inequality.) Hence $\delta' M_{n,n}(x, r)r^n \leq |B(x, r)| \leq C'M_{n,n}(x, r)r^n$. This completes the proof of the second statement of this lemma.

As for the final statement, observe by Lemma 3.8 that if $x' \in A$, then $|Z_{x,r}(x') - Z_{x,r}(x)| < b$. Thus, using the notation of that lemma, if $x = \beta^{4n-2}(x, \bar{t_1} + r_1\xi_1, \bar{t_1'}, ..., \bar{t_n} + r_n\xi_n, \bar{t_n}, ..., \bar{t_1'}, \bar{t_1})$, then $x = \beta^{4n-2}(x, \bar{t_1} + s_1\xi_1, \bar{t_1'}, ..., \bar{t_n} + s_n\xi_n, \bar{t_n}, ..., \bar{t_1'}, \bar{t_1})$ with $|s_l - r_l| < Cb$. Hence the measure of $A \leq CM_{n,2n-1}(x, r)b^n \leq C'M_{n,n}(x, r)b^n$ and we are done.

The following lemma gives us the doubling condition $(CZ1)$ for the metric of this paper.

Lemma 3.10: $|B(x, 2r)| < C|B(x, r)|$.

Proof: By Lemma 3.9, it suffices to show $M_{n,n}(x, 2r) < CM_{n,n}(x, r)$. Suppose ξ_1,...,ξ_n and $(t_1, t_1', ..., t_n, t_n') \in [-2r, 2r]^{(2n-1)k}$ are such that $Jac_{\xi_1,..,\xi_n}\beta^{2n}(x, t_1, t_1', .., t_n, t_n') = M_{n,n}(x, 2r)$. Applying Lemma 3.2 a), for some $\delta > 0$ we have $Jac_{\xi_1,...,\xi_n}\beta^{2n}(x, t_1, t_1', ..., t_n')$ $> \delta M_{n,n}(x, 2r)$ for some $(t_1, t_1', ..., t_n') \in [-r, r]^{2nk}$. Hence $M_{n,n}(x, r) > \delta M_{n,n}(x, 2r)$, and we are done.

Lemma 3.11: The metric defined by the balls $B_{2n}(x, r)$ (as defined in (2.2)) is equivalent to that induced by $B(x, r)$, and satisfies $(CZ1) - (CZ4)$. Thus one may apply the method of the Calderon-Zygmund theorem to proving L^p boundedness of singular integrals with respect to either metric.

Proof: The inclusion $B_{2n}(x, r) \subset B(x, r)$ follows directly from the definitions, and the inclusion $B(x, r) \subset B(x, Cr)$ for some large C follows from the first statement in Lemma 3.8a). The equivalence of the two metrics thus holds.

$(CZ3)$ holds for the metric induced by $B(x, r)$ by definition, and $(CZ1)$ holds by Lemma 3.10. Thus by the equivalence of the two metrics, $(CZ1)$ and $(CZ3)$ hold for the metric induced by $B_{2n}(x,r)$ as well. By the fact that the balls $B_{2n}(x,r)$ are open (Cor. to Lemma 3.6) and the compactness of D_1, the first statement of $(CZ2)$ holds. The second statement of $(CZ2)$ holds since by definition, $B_{2n}(x,r)$ is a subset of the Euclidean ball centered at x of radius Cr for some $C > 0$.

It thus remains to verify $(CZ4)$. Thus let U be an open set, $r > 0$, and $x_p \rightarrow x$ (in the Euclidean topology). Because the sets $B_{2n}(x,r)$ are open in the Euclidean topology, for each $\delta > 0$ there is a P_δ such that $B_{2n}(x,r-\delta) \subset B_{2n}(x_p,r) \subset B_{2n}(x,r+\delta)$ for $p \geq P_\delta$. Hence for each $\delta > 0$

$$\limsup_{p\rightarrow\infty} |B_{2n}(x_p,r)\cap U| \leq |B_{2n}(x,r+\delta)\cap U|,\ \liminf_{p\rightarrow\infty} |B_{2n}(x_p,r)\cap U| \geq |B_{2n}(x,r-\delta)\cap U|$$

Taking limits as $\delta \rightarrow 0$, we see that

$$\liminf_{p\rightarrow\infty} |B_{2n}(x_p,r)\cap U| \geq |B_{2n}(x,r)\cap U|,\ \limsup_{p\rightarrow\infty} |B_{2n}(x_p,r)\cap U| \leq |\bar{B}_{2n}(x,r)\cap U|$$

Here $\bar{B}_{2n}(x,r)$ denotes the closed ball centered at x of radius r. Thus we will have shown $(CZ4)$ if we can show that $\bar{B}_{2n}(x,r) - B_{2n}(x,r)$ has Lebesgue measure zero. Denote $\bar{B}_{2n}(x,r) - B_{2n}(x,r)$ by $C(x,r)$.

Now let ξ_1,...,ξ_n be directions, each ξ_i being one of the k t_i directions, and ∂_z be a directional derivative, such that for some α, $\partial_z^\alpha Jac_{\xi_1,...,\xi_n}\beta^{4n}(x,t_1,t_1',t_2,t_2',...,t_{2n},t_{2n}') \neq 0$ for all $(t_1,t_1',...,t_{2n},t_{2n}') \in D_2$; we may do this by the curvature condition, shrinking D_1 and D_2 if necessary. Then in the $(t_1,t_1',...,t_{2n},t_{2n'})$ variables the zero set of $Jac_{\xi_1,...,\xi_n}\beta^{4n}(x,t_1,t_1',t_2,t_2',...,t_{2n},t_{2n}')$ has measure zero since its intersection with any line segment in the z-direction is a set of cardinality at most α. If we denote the image of this zero set under the map $(t_1,t_1',...,t_{2n},t_{2n}') \rightarrow \beta^{4n}(x,t_1,t_1',t_2,t_2',...,t_{2n},t_{2n}')$ by $E(x,r)$, then since β^{4n} has uniformly bounded first derivatives we must also have that $|E(x,r)| = 0$. In particular $|E(x,r)\cap C(x,r)| = 0$.

It remains to show $|E(x,r)^c\cap C(x,r)| = 0$. Let y be any point in $E(x,r)^c\cap C(x,r)$. Write $y = \beta^{4n}(x,u_1,u_1',...,u_{2n},u_{2n}')$, where $Jac_{\xi_1,...,\xi_n}\beta^{4n}(x,u_1,u_1',u_2,u_2',...,u_{2n},u_{2n}') \neq 0$ and $\sum_i |u_i| + |u_i'| = r$ (by compactness equality is achieved). Then by Lemma 3.6d), there is a $\delta > 0$ such that on $|s_i| < \delta$, $(s_1,...s_n) \rightarrow \beta^{4n}(x,(u_1,u_1',...,u_{2n},u_{2n}') + \sum_i s_i\xi_i)$ smoothly traces out, in a one-to-one fashion, a diffeomorph to an open parallelopiped. However, when each $s_i < 0$, the point $\beta^{4n}(x,(u_1,u_1',...,u_{2n},u_{2n}') + \sum_i s_i\xi_i)$ is of the form $\beta^{4n}(x,v_1,v_1',...,v_{2n},v_{2n}')$ for $\sum_i |v_i|+|v_i'| < r$. Thus the Lebesgue density of the set $C(x,r)$ is less than 1 at the point y. Since this holds for each y in $E(x,r)^c \cap C(x,r)$, we must have that $|E(x,r)^c \cap C(x,r)| = 0$. We conclude that $|C(x,r)| = 0$ and $(CZ4)$ follows, completing the proof of the lemma.

4: L^2 estimates on individual $T_{i,j}$'s.

The main result of this section is the following theorem.

Theorem 4.1: $||T_{i,j}||_{2,2} \leq C2^{-\delta i}$

Proof: Write $T_{i,j}f(x) =$

$$\int [f(\beta^{2n-1}(x, t_1, 2^{-i-j}t_1', ..., 2^{-i-j}t_{n-1}', 2^{-i-j}t_n))-$$
$$f(\beta^{2n-1}(x, t_1, 2^{-i-j+1}t_1', ..., 2^{-i-j+1}t_n))]K_j(x, t_1) \prod_{m=2}^{n} \phi(t_m) \prod_{m=1}^{n-1} \phi(t_m') \prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m' \tag{4.1}$$

For fixed $u = (u_1', u_2, u_2', ..., u_n)$ with $|u_l|$, $|u_l'| < 1$ for each l, we consider the operator $T_{i,j}^u$ where $T_{i,j}^u f(x) =$

$$\int [f(\beta^{2n-1}(x, t_1, 2^{-i-j}u_1', ..., 2^{-i-j}u_{n-1}', 2^{-i-j}u_n))-$$
$$f(\beta^{2n-1}(x, t_1, 2^{-i-j+1}u_1', ..., 2^{-i-j+1}u_{n-1}', 2^{-i-j+1}u_n))]K_j(x, t_1)dt_1 \tag{4.2}$$

We will show the operator $T_{i,j}^u$ satisfies $||T_{i,j}^u||_{2,2} \leq C2^{-\delta i}$ for each u, thereby proving Theorem 4.1. In doing so we will make frequent use of the following case of Schur's test, whose simple proof is included for convenience.

Lemma 4.2: Suppose $T(f)(x) = \int k(x, y)f(y)dy$ and $\int |k(x, y)|dx < C_1$, $\int |k(x, y)|dy < C_2$. Then $||T||_{2,2} < (C_1C_2)^{1/2}$.

Proof: For any functions f and g, $|\langle Tf, g\rangle| =$

$$|\int k(x, y)f(y)g(x)dy\, dx| = |\int k(x, y)(C_2/C_1)^{1/4}f(y)(C_1/C_2)^{1/4}g(x)dy\, dx|$$
$$\leq \frac{1}{2}(C_2/C_1)^{1/2} \int |f(y)|^2|k(x, y)|dydx + \frac{1}{2}(C_1/C_2)^{1/2} \int |g(x)|^2|k(x, y)|dydx$$
$$\leq \frac{1}{2}(C_1C_2)^{1/2}(\int |f(y)|^2dy + \int |g(x)|^2dx)$$

Taking the supremum over all f and g with L^2 norm 1 gives us the lemma.

Our first application of Lemma 4.2 is the following fact, which incidentally proves Theorem 4.1 in the case $i = 0$.

Lemma 4.3: For each u, $||T_{i,j}^u|| < C$.

Proof: Define $\delta_N(y) = cN^n\phi(Ny)$, where $c = (\int \phi(y)dy)^{-1}$. Hence the δ_N are approximations to the Dirac measure at the origin. If $i > 0$, let $U_{i,j}^N$ be the operator with kernel $L(x, y) =$

$$\int [\delta_N(y - \beta^{2n-1}(x, t_1, 2^{-i-j}u_1', ..., 2^{-i-j}u_{n-1}', 2^{-i-j}u_n))-$$

$$\delta_N(y - \beta^{2n-1}(x, t_1, 2^{-i-j+1}u_1', ..., 2^{-i-j+1}u_{n-1}', 2^{-i-j+1}u_n))]K_j(x, t_1)dt_1 \tag{4.3a}$$

If $i = 0$, let $L(x, y) =$

$$\int \delta_N(y - \beta^{2n-1}(x, t_1, 2^{-j}u_1', ..., 2^{-j}u_{n-1}', 2^{-j}u_n))K_j(x, t_1)dt_1 \tag{4.3b}$$

Then $\int |L(x, y)|dy \leq 2\int |K_j(x, t_1)|dt_1 < C$, and since as functions of x, $\beta^{2n-1}(x, t_1, 2^{-i-j+1}u_1', ..., 2^{-i-j+1}u_{n-1}', 2^{-i-j+1}u_n)$, $\beta^{2n-1}(x, t_1, 2^{-i-j}u_1', ..., 2^{-i-j}u_{n-1}', 2^{-i-j}u_n)$ have Jacobian determinants near 1, we similarly have $\int |L(x, y)|dx < 3\int |K_j(x, t_1)|dt_1 < C''$. Thus by Lemma 4.2, $||U_{i,j}^N||_{2,2} < C$ for all i, j. As $N \to \infty$, $U_{i,j}^N \to T_{i,j}^u$, and therefore we are done with the proof.

To prove Theorem 4.1, we will show that for each u, each $i > 0$, $||(T_{i,j}^u(T_{i,j}^u)^*)^n||_{2,2} \leq C2^{-\delta i}$. Write $L(x, y) = L_1(x, y) - L_0(x, y)$, where for $l = 0$ or 1 we define

$$L_l(x, y) = \delta_N(y - \beta^{2n-1}(x, t_1, 2^{-i-j-l+1}u_1', ..., 2^{-i-j-l+1}u_{n-1}', 2^{-i-j-l+1}u_n))K_j(x, t_1)$$

Thus if we let $\gamma_l(x, t_1) = \beta^{2n-1}(x, t_1, 2^{-i-j-l+1}u_1', ..., 2^{-i-j-l+1}u_{n-1}', 2^{-i-j-l+1}u_n)$, the kernel of the operator $(T_{i,j}^u(T_{i,j}^u)^*)^n$ can be expressed as $M^n(x_1, y_{p+1}) =$

$$-\sum_{l_m, l_m'=0,1}(-1)^{\sum_{m=1}^n l_m + \sum_{m=1}^n l_m'}\int\prod_{m=1}^n[\delta(y_m - \gamma_{l_m}(x_m, t_m))K_j(x_m, t_m)$$
$$\delta(y_m - \gamma_{l_m'}(x_{m+1}, t_m'))K_j(x_{m+1}, t_m')]$$
$$\prod_{m=1}^n dt_m \prod_{m=1}^n dt_m' \prod_{m=2}^n dx_m \prod_{m=1}^n dy_m$$

By Lemma 4.2, in order to prove Theorem 4.1, it suffices to prove the following lemma:

Lemma 4.4: For a fixed l_1,...,l_n, l_1',...,l_{n-1}' define $M(x_1, x_{n+1}) =$

$$\int\prod_{m=1}^n \delta(y_m - \gamma_{l_m}(x_m, t_m))K_j(x_m, t_m)\prod_{m=1}^{n-1}\delta(y_m - \gamma_{l_m'}(x_{m+1}, t_m'))K_j(x_{m+1}, t_m')$$
$$[\delta(y_n - \gamma_0(x_{n+1}, t_n')) - \delta(y_n - \gamma_1(x_{n+1}, t_n'))]K_j(x_{n+1}, t_n')$$
$$\prod_{m=1}^n dt_m \prod_{m=1}^n dt_m' \prod_{m=2}^n dx_m \prod_{m=1}^n dy_m \tag{4.4}$$

Then $\int |M(x_1, x_{n+1})|dx_1 < C$, $\int |M(x_1, x_{n+1})|dx_{n+1} < C2^{-\delta i}$ for some $\delta > 0$.

(Note: the reader might be uncomfortable with integrating functions defined via delta functions as we've done here. However, all the arguments in the proof of this lemma [as well as the corresponding arguments in the proof of Lemma 5.2] can be made precise in a straightforward manner by using approximations to the delta function as in the proof of Lemma 4.3; at any rate $M(x_1, x_{n+1})$ is in fact integrable in x_1 and x_{n+1}.)

Proof: We assume x_1 is fixed throughout this proof. Also note that by Lemma 4.3 we only need to prove this lemma for $i > i_0$, where i_0 depends on the fixed parameters of this paper.

By Lemma 3.6c), there is a $\delta > 0$ and directions ξ_1,...,ξ_n in the t_1,...,t_n variables respectively such that $Jac_{\xi_1,...,\xi_n}\ \beta^{2n}(x_1, \hat{t_1}, \hat{t_1'}, ..., \hat{t_n}, \hat{t_n'}) > \delta M_{n,n}(x_1, 2^{-j})$ for some $|\hat{t_1}|$, $|\hat{t_1'}| < 2^{-j}$. Notice that by Lemma 3.2b) and Lemma 3.6b) we may find a $\delta_1 > 0$ such that if $|t_m - \bar{t_m}|$, $|t_m' - \bar{t_m}'| \leq 2\delta_1 2^{\frac{-i}{8n}-j}$ for each m, then for any $1 \leq l \leq n$, $|Jac_{\xi_1,...,\xi_l}\beta^{2n}(x_1, t_1, t_1', ..., t_n, t_n') - Jac_{\xi_1,...,\xi_l}\beta^{2n}(x_1, \bar{t_1}, \bar{t_1'}, ..., \bar{t_n}, \bar{t_n'})| \leq \frac{1}{2}2^{\frac{-i}{8n}}M_{l,l}(x_1, 2^{-j})$. Thus for a fixed x we may find a partition of unity $\{\phi_\alpha\}$ of the $2nk$-dimensional t-t' space such that each ϕ_α is supported on a ball of radius $\delta_1 2^{\frac{-i}{8}-j}$, at most C of the ϕ_α are nonzero at each point, $|\partial_{t_m}\phi|$,$|\partial_{t_m'}\phi| < C2^{\frac{i}{8}+j}$ for each m, and on the support of each ϕ_α either $Jac_{\xi_1,...,\xi_l}\beta^{2n}(x_1, t_1, t_1', ..., t_n, t_n') \leq 2 \times 2^{\frac{-i}{8n}}M_{l,l}(x_1, 2^{-j})$ for some $1 \leq l \leq n$, or $Jac_{\xi_1,...,\xi_l}\beta^{2n}(x_1, t_1, t_1', ..., t_n, t_n') \geq 2^{\frac{-i}{8n}}M_{l,l}(x_1, 2^{-j})$ for all $1 \leq l \leq n$. Say ϕ_α is of type I or II depending on whether 1) or 2) respectively holds. If both hold, for the purpose of our arguments the ϕ_α can be assigned to either type.

We now iteratively define $\bar{\beta}^1$, ..., $\bar{\beta}^{2n}$ by

$$\bar{\beta}^1(x, t) = \gamma_{l_1}(x, t)$$
$$\bar{\beta}^2(x, t_1, t_1') = \text{ the solution y to } \gamma_{l_1}(x, t_1) = \gamma_{l_1'}(y, t_1')$$
$$\bar{\beta}^{2m+1}(x, t_1, t_1', ..., t_m, t_m', t_{m+1}) = \gamma_{l_{m+1}}(\bar{\beta}^{2m}(t_1, t_1', ..., t_m, t_m'), t_{m+1}) \text{ etc}$$

with $\bar{\beta}^{2n}$ defined by letting the final l_n' be a 0. Define $\hat{\beta}^m$ the same way as $\bar{\beta}^m$, except let the final l_n' be 1. By Lemma 3.6b), we have that $M_{n,n}(x_1, 2^{-j}) \sim M_{n,4n^2-2n}(x_1, 2^{-j})$. So by Lemma 3.2b, considering the definitions of γ_0 and γ_1, we have that $Jac_{\xi_1,...,\xi_l}\bar{\beta}^{2n}(x_1, t_1, t_1', ..., t_n, t_n')$, $Jac_{\xi_1,...,\xi_l}\hat{\beta}^{2n}(x_1, t_1, t_1', ..., t_n, t_n') \geq \frac{1}{2}2^{\frac{-i}{8n}}M_{n,n}(x_1,\ 2^{-j})$ on the support of a ϕ_α of type II if i is large enough.

Notice by Lemma 3.2a) that the total measure of the support of the ϕ_α of type I is at most $C2^{-\delta i-2njk}$ for some $\delta > 0$. As a result if we write $1 = \sum_{\alpha\, type\, I}\phi_\alpha + \sum_{\alpha\, type\, II}\phi_\alpha$ and correspondingly write $M(x_1, x_{n+1}) = M_1(x_1, x_{n+1}) + M_2(x_1, x_{n+1})$, then by taking absolute values inside the integral and integrating with respect to x_{n+1}, y_n, x_n,...,y_1, we get $\int |M_1(x_1, x_{n+1})|dx_{n+1} < C2^{2njk}\times$ measure of the total support of ϕ_α's of type I $< 2^{-\delta i}$. Therefore to prove Lemma 4.4 we need to show $\int |M_2(x_1, x_{n+1})|dx_{n+1} < C2^{-\delta i}$. To do this, if M_α denotes a term of M_2 corresponding to α, we will show there is some $\delta > 0$ such that

$$\int |M_\alpha(x_1, x_{n+1})|dx_{n+1} < C2^{\frac{-nki}{4}-\delta i} \tag{4.5}$$

Our first step in proving (4.5) is the observation that the integrand in (4.4) is nonzero iff $\gamma_{l_1}(x_1,t_1)=\gamma_{l_1'}(x_2,t_1')$ i.e. if $\bar{\beta}^2(x_1,t_1,t_1')=x_2$. Analogously, $x_{m+1}=\bar{\beta}^{2m}(x_1,t_1,t_1',..,t_m,t_m')$ $(=\hat{\beta}^{2m}(x_1,t_1,t_1',..,t_m,t_m'))$ for $m<n$, and $x_{n+1}=\bar{\beta}^{2n}(x_1,t_1,t_1',...,t_n,t_n')$ or $\hat{\beta}^{2n}(x_1,t_1,t_1',...,t_n,t_n')$.

We now pick some $(\bar{t_1},\bar{t_1'},...,\bar{t_n},\bar{t_n'})$ in the support of ϕ_α and in each $K_j(x_m,t_m)$ and $K_j(x_{m+1},t_m')$ appearing in the integral defining M_α we "freeze" x_m at $\bar{x_m}=\bar{\beta}^{2m}(x_1,\bar{t_1},\bar{t_1'},...,\bar{t_n},\bar{t_n'})$, t_m at $\bar{t_m}$, and t_m' at $\bar{t_m'}$. Precisely, let $M_\alpha'(x_1,x_{n+1})$ denote the kernel obtained by replacing each $K_j(x_m,t_m)$ by $K_j(\bar{x_m},\bar{t_m})$ and each $K_j(x_{m+1},t_m')$ by $K_j(\bar{x_{m+1}},\bar{t_m'})$. Due to the facts that $|t_m-\bar{t_m}|,|t_m'-\bar{t_m'}|<\delta_1 2^{-\frac{i}{8}-j}$ and $|\bar{Z}_{x_1,2^{-j}}(x_m)-\bar{Z}_{x_1,2^{-j}}(\bar{x}_m)|\leq\delta_1 2^{\frac{-i}{8}-j}$, the first derivative estimates $(K1)$ and $(K2)$ or $(K2')$ tell us that

$$|K(x_m,t_m)-K(\bar{x}_m,\bar{t}_m)|,\ |K(x_{m+1},t_m')-K(\bar{x}_{m+1},\bar{t}_m')|<C2^{\frac{-i}{8}-j}$$

As a result, $\int|M_\alpha(x_1,x_{n+1})-M_\alpha'(x_1,x_{n+1})|dx_{n+1}<$

$$C2^{-\frac{i}{8}+2kjn}\int\phi_\alpha(t_1,t_1',...,t_n,t_n')\prod_{m=1}^{n}\delta(y_m-\gamma_{l_m}(x_m,t_m))\prod_{m=1}^{n-1}\delta(y_m-\gamma_{l_m'}(x_{m+1},t_m'))$$
$$[\delta(y_n-\gamma_0(x_n,t_n))+\delta(y_n-\gamma_1(x_{n+1},t_n'))]\prod_{m=1}^{n}dt_m\prod_{m=1}^{n}dt_m'\prod_{m=2}^{n}dx_m\prod_{m=1}^{n}dy_m$$

This is seen to be at most $C2^{\frac{-i}{8}-\frac{nki}{4}-2nkj}$ after an integration in x_{n+1}, y_n, x_n,...,y_1, then the t and t' variables.

So to prove Lemma 4.4 we need to prove $\int|M_\alpha(x_1,x_{n+1})|dx_{n+1}<C2^{\frac{-nki}{4}-\delta i}$. However since the $K(\bar{x}_m,\bar{t}_m)$ and $K(\bar{x_{m+1}},\bar{t}_m')$ showing up are now constant, we need only to show that $\int|N_\alpha(x_1,x_{n+1})|dx_{n+1}<2^{-2nkj-\frac{nki}{4}-\delta i}$, where $N_\alpha(x_1,x_{n+1})=$

$$\int\phi_\alpha(t_1,t_1',...,t_n,t_n')\prod_{m=1}^{n}\delta(y_m-\gamma_{l_m}(x_m,t_m))\prod_{m=1}^{n-1}\delta(y_m-\gamma_{l_m'}(x_{m+1},t_m'))$$
$$[\delta(y_n-\gamma_0(x_{n+1},t_n'))-\delta(y_n-\gamma_1(x_{n+1},t_n'))]\prod_{m=1}^{n}dt_m\prod_{m=1}^{n}dt_m'\prod_{m=2}^{n}dx_m\prod_{m=1}^{n}dy_m$$

Let ξ_{n+1},...,ξ_{2nk} be unit vectors such that ξ_1,...,ξ_{2nk} are an orthonormal basis for $\mathbf{R}^{2nk}$. Viewing a point $(t_1,t_1',...,t_n,t_n')$ as a point in $\mathbf{R}^{2nk}$, we use the notation $(t_1,t_1',...,t_n,t_n')=(\zeta_1,...,\zeta_{2nk})$, where ζ_m is the projection of $(t_1,t_1',...,t_n,t_n')$ in the ξ_m direction. Recalling that ξ_m is a t_m direction for $1\leq m\leq n$, we may fix the variables other than ζ_1,...,ζ_n and write $\gamma_{l_m}(x_m,t_m)=\gamma_{l_m}(x_m,\zeta_m)$, $\gamma_{l_m'}(x_{m+1},t_m')=\gamma_{l_m'}(x_{m+1},\cdot_m)$, etc. Furthermore, we use the notation $\bar{\phi}_\alpha(\zeta_1,...,\zeta_n)$ to denote $\phi_\alpha(t_1,t_1',...,t_n,t_n')$.

To prove Lemma 4.4, and thus Theorem 4.1, it suffices to prove that if $O_\alpha(x_1,x_{n+1})$ denotes

$$\int\bar{\phi}_\alpha(\zeta_1,...,\zeta_n)\prod_{m=1}^{n}\delta(y_m-\gamma_{l_m}(x_m,\zeta_m))\prod_{m=1}^{n-1}\delta(y_m-\gamma_{l_m'}(x_{m+1},\cdot_m))$$

$$[\delta(y_n-\gamma_0(x_{n+1},\cdot_n)-\delta(y_n-\gamma_1(x_{n+1},\cdot_n))]\prod_{m=1}^n d\zeta_m\prod_{m=2}^n dx_m\prod_{m=1}^n dy_m$$

Then we have $\int|O_\alpha(x_1,x_{n+1})|dx_{n+1}<C2^{-jn-\frac{in}{8}-\delta i}$ for any fixed ζ_{n+1},...,ζ_{2nk}.

Note that the $[\delta(y_n-\gamma_0(x_{n+1},\cdot_n)-\delta(y_n-\gamma_1(x_{n+1},\cdot_n))]$ factor splits $O_\alpha(x_1,x_{n+1}) = O_\alpha^1(x_1,x_{n+1})-O_\alpha^2(x_1,x_{n+1})$. We describe O_α^1 and O_α^2 as functions of x_{n+1} as follows. For any function $f(x_{n+1})$, by doing the x_{n+1}, y_n, x_n,...,y_1 integrals, it is the case that $\int O_\alpha^1(x_1,x_{n+1})f(x_{n+1})dx_{n+1}=$

$$\int f(\bar{B}^n(x_1,\zeta_1,...,\zeta_n))\psi(\zeta_1,...,\zeta_n,2^{-i-j}u)\bar{\phi}_\alpha(\zeta_1,...,\zeta_n)\prod_{m=1}^n d\zeta_m$$

Here $\bar{B}^n(x_1,\zeta_1,...,\zeta_n)$ denotes $\bar{\beta}^{2n}(x_1,\zeta_1,\cdot_1,\zeta_2,...,\cdot_n)$, ψ is a smooth function, and u is the $(u_1',u_2,...,u_n)$ from (4.2). In a similar vein, we may write

$$\int O_\alpha^2(x_1,x_{n+1})f(x_{n+1})dx_{n+1}=$$
$$\int f(\hat{B}^n(x_1,\zeta_1,...,\zeta_n))\psi(\zeta_1,...,\zeta_n,2^{-i-j-1}u)\bar{\phi}_\alpha(\zeta_1,...,\zeta_n)\prod_{m=1}^n d\zeta_m$$

Recalling the definitions of γ_0 and γ_1, since ϕ_α is of type II, Lemma 3.6d) tells us that $\bar{B}^n(x_1,\zeta_1,...,\zeta_n)$ and $\hat{B}^n(x_1,\zeta_1,...,\zeta_n)$ are one-to-one on the support of $\bar{\phi}$. Thus we may write a solution $(\zeta_1,...,\zeta_n)$ to $\bar{B}^n(x_1,\zeta_1,...,\zeta_n)=y$ as $C_0(x_1,y)$, and a solution $(\zeta_1,...,\zeta_n)$ to $\hat{B}^n(x_1,\zeta_1,...,\zeta_n)=y$ as $C_1(x_1,y)$.

Thus $\int(O_\alpha^1(x_1,x_{n+1})-O_\alpha^2(x_1,x_{n+1}))f(x_{n+1})dx_{n+1}=$

$$\int\frac{\psi(C_0(x_1,x_{n+1}),2^{-i-j-1}u)\bar{\phi_\alpha}(C_0(x_1,x_{n+1}))}{Jac_{\xi_1,...,\xi_n}\bar{B^n}(x_1,C_0(x_1,x_{n+1}))}-\frac{\psi(C_1(x_1,x_{n+1}),2^{-i-j}u)\bar{\phi_\alpha}(C_1(x_1,x_{n+1})}{Jac_{\xi_1,...,\xi_n}\bar{B^n}(x_1,C_1(x_1,x_{n+1}))}$$
$$\times f(x_{n+1})dx_{n+1}\tag{4.6}$$

We conclude that $O_\alpha(x_1,x_{n+1})$ is expression of the top line of (4.6).

Unraveling definitions, we see that $C_l(x_1,x_{n+1})=$ the components in the ξ_1,...,ξ_n directions of a vector $v_l=(\tau_1,\tau_1',\ \ldots\ ,\tau_{2n^2-2n}',\tau_{2n^2-2n+1},2^{-i-j+1-l}u_n,2^{-i-j+1-l}u_{n-1}',\ \ldots\ ,2^{-i-j+1-l}u_1',t_n)$ satisfying $\beta^{4n^2-2n}(x_1,v_l)=x_{n+1}$. For $0\leq s\leq 1$, $(t_1,t_1',...,t_n,t_n')\in$ $supp(\phi_\alpha)$ we define $C_s(x_1,x_{n+1})=$ the components in the ξ_1,...,ξ_n directions of the $v_s=(\tau_1\ ,\tau_1',...,\tau_{2n^2-2n}',\tau_{2n^2-2n+1},2^{-i-j+1-s}u_n,2^{-i-j+1-s}u_{n-1}',...2^{-i-j+1-s}u_1',t_n)$ satisfying $\beta^{4n^2-2n}(x_1,v_s)=x_{n+1}$ (assuming it exists; if it does exist it is unique by considerations like those of the previous paragraph.) By applying Lemma 3.7 to the case where $x=x'=x_1$ and $(\bar{t_1},\bar{t_1'},...\bar{t_n},\bar{t_n'})=(\dot{t_1},\dot{t_1'},...,\dot{t_n},\dot{t_n'})=v_s$, we have $|\partial_sC_s(x_1,x_{n+1})|<C2^{-i-j+\frac{i}{8n}}$ whenever the corresponding $(t_1,t_1',...,t_n,t_n')\in supp(\phi_\alpha)$. Consequently, by Lemma 3.2b) we have

$$|\partial_sJac_{\xi_1,...,\xi_n}\bar{B^n}(x_1,C_s(x_1,x_{n+1}))|<C2^{-i+\frac{i}{8n}}M_{n,4n^2-2n}(x_1,2^{-j})$$

$$< C'2^{-i+\frac{i}{8n}} M_{n,n}(x_1, 2^{-j})$$

We conclude that $|O_\alpha(x_1, x_{n+1})| < C2^{\frac{3i}{8n}-i} M_{n,n}(x_1, 2^{-j})^{-1}$.

But $O_\alpha(x_1, x_{n+1})$ is supported when $\bar{C}(x_1, x_{n+1})$ or $\hat{C}(x_1, x_{n+1})$ is in the support of $\bar{\phi_\alpha}$, in other words, when $x_{n+1} = \bar{B}^n(x_1, \zeta_1, ..., \zeta_n)$ or $\hat{B}^n(x_1, \zeta_1, ..., \zeta_n)$ for some $(\zeta_1, ..., \zeta_n) \in supp(\bar{\phi_\alpha})$. By Lemma 3.9, the set of all such x_{n+1} has measure $< CM_{n,n}(x_1, 2^{-j})2^{-\frac{in}{8}-jn}$. Thus $\int |O_\alpha(x_1, x_{n+1})| dx_{n+1} < C2^{\frac{i}{4n}-i-jn}$, and we are done with the proof of Lemma 4.4, and therefore of Theorem 4.1 as well.

5. Estimates on $T_{i,j}T_{i,j}^*$

The main result of this section is the following theorem, which in conjunction with the results of section 4 implies the case $p = 2$ of Theorems 2.1 and 2.2.

Theorem 5.1: There is $\delta > 0$ such that if we are under the hypotheses of Theorem 2.1 and $\delta|j_1 - j_2| > i \geq 0$, or we are under the hypotheses of Theorem 2.2 and $\delta|j_1 - j_2| > i > 0$, then $||T_{i,j_1}T_{i,j_2}^*||_{2,2}$, $||T_{i,j_1}^*T_{i,j_2}||_{2,2} < C2^{-\delta|j_1-j_2|}$.

Proof: Define $U_{i,j}f(x) =$

$$\int f(\beta^{2n-1}(x_1, t_1, 2^{-i-j}t_1', ..., 2^{-i-j}t_n))K_j(x, t_1) \prod_{m=2}^{n} \phi(t_m) \prod_{m=1}^{n-1} \phi(t_m') \prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m'$$

Then if $i > 0$, $T_{i,j}f(x) = U_{i,j}f(x) - U_{i-1,j}f(x)$, and $T_{0,j}f(x) = U_{0,j}f(x)$. Note that $\beta^{2n-1}(x_1, t_1, 2^{-i-j}t_1', ..., 2^{-i-j}t_n) = y$ iff $\beta^{-(2n-1)}(y, 2^{-i-j}t_n, ..., 2^{-i-j}t_1', t_1) = x_1$. Here $\beta^{-(2n-1)}$ denotes the function analogous to β^{2n-1} when γ is replaced by γ^{-1}. As a result, $\int U_{i,j}f(x)g(x)dx =$

$$\int f(x)g(\beta^{-(2n-1)}(x, 2^{-i-j}t_n, ..., 2^{-i-j}t_1', t_1))K_j'(x, 2^{-i-j}t_n, ..., 2^{-i-j}t_1', t_1)$$
$$\prod_{m=2}^{n} \phi(t_m) \prod_{m=1}^{n-1} \phi(t_m') \prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m'$$

Here we define $K_j'(x, 2^{-i-j}t_n, ..., 2^{-i-j}t_1', t_1) = |\det d_x\beta^{-(2n-1)}(x, 2^{-i-j}t_n, ..., 2^{-i-j}t_1', t_1)|$ $K_j(\beta^{-(2n-1)}(x, 2^{-i-j}t_n, ..., 2^{-i-j}t_1', t_1), t_1)$.

Thus $U_{i,j}$ has kernel $L_{i,j}(x, y) =$

$$2^{(i+j)(2n-2)k} \int \delta(y - \beta^{2n-1}(x, t_1, t_1', ..., t_n))K_j(x, t_1) \prod_{m=2}^{n} \phi(2^{i+j}t_m) \prod_{m=1}^{n-1} \phi(2^{i+j}t_m')$$
$$\prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m'$$

And $U_{i,j}^*$ has kernel $L'_{i,j}(x,y) =$

$$2^{(i+j)(2n-2)k}\int \delta(y-\beta^{-(2n-1)}(x,t_n,...,t'_1,t_1))K'_j(x,t_n,...,t'_1,t_1)\prod_{m=2}^{n}\phi(2^{i+j}t_m)$$
$$\prod_{m=1}^{n-1}\phi(2^{i+j}t'_m)\prod_{m=1}^{n}dt_m\prod_{m=1}^{n-1}dt'_m$$

The following lemma will be crucial to proving both Theorem 5.1 and for proving the L^p estimates of the next section.

Lemma 5.2: There is a constant $C_1 > 0$ such that if $2^{-j_1-1} < d(x_1,x_2) < 2^{-j_1}$ and $j_2 \leq j_1 - C_1 i - j$, then

$\int |L_{i,j_2}(x_1,y) - L_{i,j_2}(x_2,y)|dy < C2^{-\delta j}$, $\int |L'_{i,j_2}(x_1,y) - L'_{i,j_2}(x_2,y)|dy < C2^{-\delta j}$

Proof: We prove only the first statement here; the proof to the second is virtually identical. By lemma 3.6, there is a $\delta > 0$ such that we may let ξ_1,...,ξ_n be directions satisfying $Jac_{\xi_1,...,\xi_l}\beta^{2n-1}(x,t_1,t'_1,...,t_n) > \delta M_{l,l}(x_1,2^{-j_2})$ for all $l \leq n$, some $(t_1,t'_1,...,t_n)$ with $|t_m|$, $|t'_m| < 2^{-j_2}$.

Similar to in Section 4, we use a partition of unity to write the integral defining $L_{i,j_2}(x_1,y) - L_{i,j_2}(x_2,y)$ into pieces supported on cubes of radius $\delta_1 2^{\frac{-C_1 i - j}{8} - j_2}$ for an appropriate δ_1 (We will specify how large C_1 must be later). Hence we write $L_{i,j_2}(x_1,y) - L_{i,j_2}(x_2,y) =$

$$\sum_\alpha \int [\delta(y-\beta^{2n-1}(x_1,t_1,t'_1,...,t_n))K_{j_2}(x_1,t_1) - \delta(y-\beta^{2n-1}(x_2,t_1,t'_1,...,t_n))K_{j_2}(x_2,t_1)]$$
$$2^{(i+j_2)(2n-2)k}\phi_\alpha(t_1,t_1,...,t_n)\prod_{m=2}^{n}\phi(2^{i+j_2}t_m)\prod_{m=1}^{n}\phi(2^{i+j_2}t'_m)\prod_{m=1}^{n}dt_m\prod_{m=1}^{n-1}dt'_m \tag{5.1}$$

Like before, if δ_1 were chosen appropriately small, we may divide the ϕ_α's into 2 types. On the first type we have $|Jac_{\xi_1,...,\xi_l}\beta^{2n-1}(x_1,\, t_1,t'_1,...,t_n)| \leq 2^{-\frac{C_1 i - j}{8n}+1}M_{l,l}(x_1,2^{-j_2})$ for some $1 \leq l \leq n$, and on the second type we have $|Jac_{\xi_1,...,\xi_l}\beta^{2n-1}(x_1,t_1,t'_1,...,t_n)| \geq 2^{\frac{-C_1 i - j}{8n}}M_{l,l}(x_1,2^{-j_2})$ for all $1 \leq l \leq n$.

The integral with respect to y of the the absolute value of the sum of the terms in (5.1) corresponding to a ϕ_α of type I is at most the following, where $A = \{(t_1,t'_1,...,t_n) : \exists l Jac_{\xi_1,...,\xi_l}\beta^{2n-1}(x_1,t_1,t'_1,...,t_n)| \leq 2^{\frac{-C_1 i - j}{8n}+1}M_{l,l}(x_1,2^{-j_2})\}$:

$$\sum_\alpha \int_A [\delta(y-\beta^{2n-1}(x_1,t_1,t'_1,...,t_n)|K_{j_2}(x_1,t_1)| + \delta(y-\beta^{2n-1}(x_2,t_1,t'_1,...,t_n)|K_{j_2}(x_2,t_1)|]$$
$$2^{(i+j_2)(2n-2)k}\prod_{m=2}^{n}\phi(2^{i+j_2}t_m)\prod_{m=1}^{n-1}\phi(2^{i+j_2}t'_m)\prod_{m=1}^{n}dt_m\prod_{m=1}^{n-1}dt'_m dy \tag{5.2}$$

Integrating this expression with respect to y first, then the t and t' variables, gives (5.2) is at most a constant times

$$2^{(i+j_2)(2n-2)k}\, 2^{j_2 k} |\{(t_1, t_1', .., t_n) \in [-2^{-j_2}, 2^{-j_2}]^{(2n-1)k} : \exists l\, Jac_{\xi_1,..,\xi_l}\, \beta^{2n-1}\,(x_1, t_1, ..., t_n)$$
$$\leq 2^{\frac{-C_1 i - j}{8n}} M_{l,l}(x_1, 2^{-j_2})\}|$$
$$< C2^{-\delta_1(i+j)}$$

By Lemma 3.2a), the measure of this set is bounded by $2^{-\delta\frac{C_1 i+j}{8n} - j_2(2n-1)k}$ for some $\delta > 0$. Thus if C_1 is large enough, we get that (5.2) $< 2^{\frac{-\delta C_1 i}{16n} - \frac{\delta j}{8n}}$, better than the desired estimate.

We now consider a term of the sum (5.1) corresponding to a ϕ_α of type II. To prove Lemma 5.2, we must show that this term, call it $M_\alpha(x_1, x_2, y)$, satisfies $\int |M_\alpha(x_1, x_2, y)|dy$ $< C2^{\frac{-C_1 i - j}{8}(2n-1)k + (2n-2)ik - \delta j}$ for some $\delta > 0$. During the course of the following arguments, we will at times assume $j > j_0$ for some j_0 depending on the fixed parameters of our setup; for $j \leq j_0$ the desired estimates follow from simple bounds.

Notice that since $d(x_1, x_2) \leq 2^{-j_2 - C_1 i - j}$, by Lemma 3.8 we can write $x_2 = \beta^{4n-2}(x_1, \bar{t_1}, \bar{t_1'}, ..., \bar{t}_{2n-1}, \bar{t}_{2n-1}')$ for $|\bar{t_l}|, |\bar{t_l'}| < C2^{-j_2 - C_1 i - j}$. As a result, by $(K2)$ or $(K2')$ we have $|K_{j_2}(x_2, t) - K_{j_2}(x_1, t)| < C2^{-C_1 i - j - kj_2}$. Furthermore, for a fixed $(\bar{t_1}, \bar{t_1'}, ..., \bar{t_{n-1}'}$, $\bar{t_n}) \in supp(\phi_\alpha)$, by $(K1)$ we have $|K_{j_2}(x_1, t_1) - K_{j_2}(x_1, \bar{t_1})| < C2^{\frac{-C_1 i - j}{8} + kj_2}$ for any t_1 such that $(t_1, t_1', ..., t_{n-1}', t_n) \in supp(\phi_\alpha)$. Thus in $M_\alpha(x_1, x_2, y)$ we may freeze $K_{j_2}(x_1, t_1)$ and $K_{j_2}(x_2, t_1)$ at $x_2 = x_1$ and $t_1 = \bar{t_1}$; the difference is bounded in absolute value by $M_\alpha'(x_1, x_2, y) =$

$$2^{(2n-2)k(i+j_2) - \frac{C_1 i+j}{8} + kj_2} \int_{supp(\phi_\alpha)} \delta(y - \beta^{2n-1}(x_1, t_1, t_1', ..., t_{n-1}', t_n)) +$$
$$\delta(y - \beta^{2n-1}(x_2, t_1, t_1', ..., t_{n-1}', t_n)) \prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m'$$

Therefore $\int |M_\alpha'(x_1, x_2, y)|dy < C2^{(2n-2)ik - \frac{C_1 i+j}{8}(2n-1)k}$, better than the estimate we need if C_1 is appropriately large. Note that $M_\alpha(x_1, x_2, y) - M_\alpha'(x_1, x_2, y) =$

$$2^{(2n-2)(i+j_2)k} K_{j_2}(x_1, \bar{t_1}) \int [\delta(y - \beta^{2n-1}(x_1, t_1, t_1', ..., t_{n-1}', t_n)) -$$
$$\delta(y - \beta^{2n-1}(x_2, t_1, t_1', ..., t_{n-1}', t_n))]\phi_\alpha(t_1, t_1', ..., t_n, t_n') \prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m'$$

So our task is to prove that $\int |N_\alpha(x_1, x_2, y)|dy < C2^{(2n-1)k(-j_2 - \frac{C_1 i+j}{8}) - \delta j}$ for some $\delta > 0$, where $N_\alpha(x_1, x_2, y)$ is the integral in the above expression.

To this end, like before we arrange the $(t_1, t_1', ..., t_{n-1}', t_n)$ directions into orthogonal directions $\xi_1, ..., \xi_n, \xi_{n+1}, ..., \xi_{(2n-1)k}$, and write $(t_1, t_1', ..., t_{n-1}', t_n)$ as $(\zeta_1, ..., \zeta_{(2n-1)k})$, where ζ_m denotes the component of $(t_1, t_1', ..., t_{n-1}', t_n)$ in the ξ_m direction. To prove our needed estimate, we will show that for fixed $\zeta_{n+1}, ..., \zeta_{(2n-1)k}$, $\int |O_\alpha(x_1, x_2, y)| dy < C 2^{-n(\frac{C_1 i + j}{8} + j_2) - \delta j}$ for some $\delta > 0$, where $O_\alpha(x_1, x_2, y) =$

$$\int [\delta(y - B^n(x_1, \zeta_1, ..., \zeta_n)) - \delta(y - B^n(x_2, \zeta_1, ..., \zeta_n))] \bar{\phi}_\alpha(\zeta_1, ..., \zeta_n) \prod_{m=1}^{n} d\zeta_m$$

Here $B^n(x, \zeta_1, ..., \zeta_n) = \beta^{2n-1}(x, t_1, t_1', ..., t_{n-1}', t_n)$, $\bar{\phi}_\alpha(\zeta_1, ..., \zeta_n) = \phi_\alpha(t_1, ..., t_n)$, viewing the other ζ_m's as fixed.

For any $f(y)$, $\int O_\alpha(x_1, x_2, y) f(y) dy =$

$$\int [f(B^n(x_1, \zeta_1, ..., \zeta_n)) - f(B^n(x_2, \zeta_1, ..., \zeta_n))] \bar{\phi}_\alpha(\zeta_1, ..., \zeta_n) \prod_{m=1}^{n} d\zeta_m$$

Observe that for some $\delta_2 > 0$, any $(t_1, t_1', ..., t_{n-1}'\ , t_n) \in supp(\phi_\alpha)$ we have

$$\begin{aligned} Jac_{\xi_1,..,\xi_l} \beta^{6n-3}(x_1, 0, ..., 0, t_1, t_1', ..., t_{n-1}', t_n) &= Jac_{\xi_1,..,\xi_l} \beta^{2n-1}(x_1, t_1, t_1', ..., t_{n-1}', t_n) \\ &\geq 2^{\frac{-C_1 i - j}{8n}} M_{l,l}(x_1, 2^{-j_2}) \\ &\geq \delta_2 2^{\frac{-C_1 i - j}{8n}} M_{l,6n-3}(x_1, 2^{-j_2}) \end{aligned}$$

(The last inequality follows from Lemma 3.6b).) So by Lemma 3.2b), if $1 \leq l \leq n$ and if for each p we have that$|\tau_p|$, $|\tau_p'| < \delta_3 2^{\frac{-C_1 i - j}{8n} - j_2}$ for an appropriate δ_3, we can conclude that

$$Jac_{\xi_1 ... \xi_l} \beta^{6n-3}(\tau_1, \tau_1' ... \tau_{2n-1}, \tau_{2n-1}', t_1, t_1' ... t_n, t_n') \geq \frac{\delta_2}{2} 2^{\frac{-C_1 i - j}{8n}} M_{l,6n-3}(x_1, 2^{-j_2})$$

In particular this holds for $\tau_l = \bar{t}_l$, $\tau_l' = \bar{t}_l'$, where $x_2 = \beta^{4n-2}(x_1, \bar{t_1}, \bar{t_1'}, ..., \bar{t}_{2n-1}, \bar{t}_{2n-1}')$, assuming $j > j_0$ for an appropriate j_0 depending on δ_2 and δ_3.

Therefore, by Lemma 3.6b), if δ_1 is small enough, $B^n(x_1, \zeta_1, ..., \zeta_n)$ and $B^n(x_2, \zeta_1, ..., \zeta_n)$ are one-to-one for ζ_1,...,ζ_n in the support of $\bar{\phi}_\alpha$. In particular for $x = x_1$ or x_2 we may use the notation $C(x, y)$ to denote the solution $(\zeta_1, ... \zeta_n)$ to $B^n(x, \zeta_1, ..., \zeta_n) = y$, and we may rewrite $\int O_\alpha(x_1, x_2, y) f(y) dy =$

$$\int [\frac{\bar{\phi_\alpha}(C(x_1, y))}{Jac_{\xi_1,...,\xi_n} B^n(x_1, C(x_1, y))} - \frac{\bar{\phi_\alpha}(C(x_2, y))}{Jac_{\xi_1,...,\xi_n} B^n(x_1, C(x_2, y))}] f(y) dy$$

Thus $O(x_1, x_2, y)$ is the bracketed factor multiplying $f(y)$ above. Notice that by applying Lemma 3.7 to the situation where $x = x' = x_1$ and $(\bar{t_1}, \bar{t_1'}, ..., \bar{t_m}, \bar{t_m'}) = (\dot{t_1}, \dot{t_1'}, ..., \dot{t_m}, \dot{t_m'}) = (\tau_1, \tau_1', ..., \tau_{2n-1}, \tau_{2n-1}', t_1, t_1', ..., t_{n-1}', t_n, 0)$ for some $|\tau_l|$, $|\tau_l'| < \delta_3 2^{-j_2 - \frac{C_1 i + j}{8n}}$ and some $(t_1, t_1', ...\ , t_n, t_n') \in supp(\phi_\alpha)$, we get that for each l

$$|\nabla_{\tau_l}(C(\beta^{4n-2}(x_1, \tau_1, \tau_1', ..., \tau_{2n-1}, \tau_{2n-1}'), y))| < C 2^{\frac{C_1 i + j}{8n}}$$

$$|\nabla_{\tau_{l'}}(C(\beta^{4n-2}(x_1,\tau_1,\tau_1',...,\tau_{2n-1},\tau_{2n-1}'),y))| < C2^{\frac{C_1 i+j}{8n}}$$

Therefore by Lemma 3.2b), $|\nabla_{\tau_l} Jac_{\xi_1,...,\xi_n} B^n(x_1, C(\beta^{4n-2}(x_1,\tau_1,\tau_1',...,\tau_{2n-1},\tau_{2n-1}'),y))|$ and $|\nabla_{\tau_{l'}}\ Jac_{\xi_1,...,\xi_n} B^n(x_1,\ C(\beta^{4n-2}((x_1,\tau_1,\tau_1',...,\tau_{2n-1},\tau_{2n-1}'),y))|$ are both bounded by $C2^{j_2+\frac{C_1 i+j}{8n}} M_{n,2n-1}(x,2^{-j_2})$, which is in turn at most $C'2^{j_2+\frac{C_1 i+j}{8n}}\ M_{n,n}(x,2^{-j_2})$.

Since $x_2 = \beta^{4n-2}(x_1,\bar{t_1},\bar{t_1'},...,\bar{t}_{2n-1},\bar{t}_{2n-1}')$ with each $|\bar{t_l}|$, $|\bar{t_l}'| < C2^{-j_2-C_1 i-j}$, if $j > j_0$ for an appropriate j_0 we have $|O_\alpha(x_1,x_2,y)| < C2^{\frac{-C_1 i-j}{2}} M_{n,n}(x_1,2^{-j_2})^{-1}$. Because $O_\alpha(x_1,x_2,y) \neq 0$ only if $y = \beta^{2n-1}(x_1,t_1,t_1',...,t_{n-1}',t_n)$ or $\beta^{2n-1}(x_2,t_1,t_1',...,t_{n-1}',t_n)$ for some $(t_1,t_1',...,t_n,t_n') \in supp(\phi_\alpha)$, by Lemma 3.9, $O_\alpha(x_1,x_2,y) \neq 0$ on a set of measure at most $C2^{-j_2 n-\frac{C_1 i+j}{8}n} M_{n,n}(x_1,2^{j_2})$. Hence for some $\delta > 0$,

$$\int |O_\alpha(x_1,x_2,y)|dy < C2^{-j_2 n-\frac{C_1 i+j}{8}n-\frac{C_1 i+j}{2}} < C2^{-j_2 n-\frac{C_1 i+j}{8}n-\delta j}$$

This is the estimate we seek, and we are done with the proof of Lemma 5.2.

Lemma 5.3: Under the hypotheses of Theorem 2.1, $\int |L_{i,j}(x,y)|dy$, $\int |L_{i,j}'(x,y)|dy < C2^{-\delta j}$ for some $\delta > 0$, and under the hypotheses of Theorem 2.2, $\int |L_{i+1,j}(x,y) - L_{i,j}(x,y)|dy$, $\int |L_{i+1,j}'(x,y) - L_{i,j}'(x,y)|dy < 2^{-\delta j}$ for some $\delta > 0$.

Proof: The first statement follows directly from the definition of $L_{i,j}(x,y)$ and $L_{i,j}'(x,y)$ and the cancellation condition $(K3)$. As for the second statement, the first inequality follows from integrating with respect to y in the definition; the result is 0. We thus direct our attention to the second inequality. By Lemma 3.8, if $|t_1'|,..,|t_n| < 2^{-i-j}$, $|K_j'(x,t_n,...,t_1',t_1) - K_j(\gamma^{-1}(x,t_1),t_1)|$ is at most $C2^{j(k-\delta)}$ for some $\delta > 0$. As a result $\int |L_{i+1,j}'(x,y) - L_{i,j}'(x,y)|dy$ is bounded by the sum of 2 terms. The first is

$$2^{(i+j)(2n-2)k} \int |\delta(y - \beta^{-(2n-1)}(x,t_n,...,t_1',t_1))K_j(\gamma^{-1}(x,t_1),t_1)$$

$$[2^{(2n-2)k} \prod_{m=2}^{n} \phi(2^{-i-1-j}t_m) \prod_{m=1}^{n-1} \phi(2^{-i-1-j}t_m') - \prod_{m=2}^{n} \phi(2^{-i-j}t_m) \prod_{m=1}^{n-1} \phi(2^{-i-j}t_m')]dy$$

$$\prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt_m'|dy$$

The second is

$$C2^{(i+j)(2n-2)k+j(k-\delta)} \int \delta(y - \beta^{-(2n-1)}(x,t_n,...,t_1',t_1))\phi(2^{-j}t_1)$$

$$[2^{(2n-2)k} \prod_{m=2}^{n} \phi(2^{-i-j-1}t_m) \prod_{m=1}^{n-1} \phi(2^{-i-j-1}t_m') + \prod_{m=2}^{n} \phi(2^{-i-j}t_m) \prod_{m=1}^{n-1} \phi(2^{-i-j}t_m')]$$

$$\prod_{m=1}^{n} dt_m \prod_{m=1}^{n-1} dt'_m dy$$

The first term is 0; integrate with respect to the t_m and t'_m variables other than t_1, then t_1. Integrating the second term with respect to y, then the t and t' variables, the result is bounded by $C2^{-\delta j}$ and we are done with the proof of Lemma 5.3.

We now can proceed to the proof of Theorem 5.1. First, we assume $0 < i < \frac{|j_1 - j_2|}{2C_1}$, C_1 as in Lemma 5.2. The proofs that $||T_{i,j_1}T^*_{i,j_2}||_{2,2}$, $||T^*_{i,j_1}T_{i,j_2}||_{2,2} < C2^{|j_1-j_2|}$ are entirely symmetric and we write out only the first here. Replacing $T_{i,j_1}T^*_{i,j_2}$ by $(T_{i,j_1}T^*_{i,j_2})^* = T_{i,j_2}T^*_{i,j_1}$ if necessary, assume that $j_1 > j_2$. $T_{i,j_1}T^*_{i,j_2}$ has kernel $\int(L_{i,j_1}(x,z) - L_{i-1,j_1}(x,z))$ $(L'_{i,j_2}(z,y) - L'_{i-1,j_2}(z,y))dz = \int(L_{i,j_1}(x,z) - L_{i-1,j_1}(x,z))L'_{i,j_2}(z,y)dz - \int(L_{i,j_1}(x,z) - L_{i-1,j_1}(x,z))\ L'_{i-1,j_2}(z,y)dz$. These 2 terms are treated identically, so we only consider the first, call it $M(x,y)$. Then $\int |M(x,y)|dy <$

$$\int | \int (L_{i,j_1}(x,z) - L_{i-1,j_1}(x,z))L'_{i,j_2}(x,y)dz|dy +$$

$$\int \int |L_{i,j_1}(x,z) - L_{i-1,j_1}(x,z)||L'_{i,j_2}(x,y) - L'_{i,j_2}(z,y)|dzdy$$

By Lemma 5.3 the first term is bounded by $C2^{-\delta j_1} < C2^{-\delta|j_1-j_2|}$. (The statement that $\int |L'_{i,j_2}(x,\, y)|dy < C$ follows easily from the definition.) By Lemma 5.2, the second term is bounded by $C2^{-\delta|j_1-j_2|}$ since $i < \frac{|j_1-j_2|}{2C_1}$ and thus the j in that lemma is at least $\frac{|j_1-j_2|}{2}$. Hence $\int |M(x,y)|dy < C2^{-\delta|j_1-j_2|}$. $\int |M(x,y)|dx < C$ follows from $\int |L'_{i,j_2}(x,y)|dx < C$, and thus by Lemma 4.2 we have $||T_{i,j_1}T^*_{i,j_2}||_{2,2} < C'2^{-\frac{\delta}{2}|j_1-j_2|}$. We are thus done with Theorem 5.1 for $i > 0$.

If $i = 0$ (and we are under the hypotheses of Theorem 2.1) the argument is much the same. Again considering $T_{0,j_1}T^*_{0,j_2}$ and assuming $j_1 > j_2$, $T_{0,j_1}T^*_{0,j_2}$ has kernel $M(x,y) = \int L_{0,j_1}(x,z)L_{0,j_2}(z,y)dz$, and therefore like before we have $\int |M(x,y)|dy <$

$$\int |L_{0,j_1}(x,z)||L_{0,j_2}(z,y) - L_{0,j_2}(x,y)|dydz + \int | \int L_{0,j_1}(x,z)L_{0,j_2}(x,y)dz|dy < C2^{\delta|j_1-j_2|}$$

like before. Again $\int |M(x,y)|dx < C$ is easy, and via Lemma 4.2 we are done for the case $i = 0$ as well; Theorem 5.1 is proved.

6. L^p estimates for $p \neq 2$

The main result for this section is the following.

Lemma 6.1: For $i > 0$, let $P_i(x,y)$ and $P'_i(x,y)$ be the kernels of $T_i = \sum_j T_{i,j}$ and $T^*_i = \sum_j T^*_{i,j}$ respectively, and let $P_0(x,y)$ and $P'_0(x,y)$ be the kernels of $\tilde{T}$ and $\tilde{T}^*$ respectively. Then for all $i \geq 0$, if $2^{-j_0-1} < d(x_1,x_2) \leq 2^{-j_0}$, then we have the following.

$$\int_{d(y,x_1)\geq(2n-1)2^{-j_0}} |P_i(x_1,y) - P_i(x_2,y)|dy \leq C(i+1)$$

$$\int_{d(y,x_1)\geq (2n-1)2^{-j_0}} |P_i'(x_1,y) - P_i'(x_2,y)|dy \leq C(i+1)$$

Proof: Using the notation of section 5,

$$P_i(x,y) = \sum_j (L_{i-1,j}(x,y) - L_{i,j}(x,y)),\ P_i'(x,y) = \sum_j (L_{i-1,j}'(x,y) - L_{i,j}'(x,y));\ i > 0$$

$$P_0(x,y) = \sum_j L_{0,j}(x,y),\ P_0'(x,y) = \sum_j L_{0,j}'(x,y)$$

Since as functions of y, $L_{i,j}(x,y)$ and $L_{i,j}'(x,y)$ are supported in $\{y : d(y,x) < (2n-1)2^{-j}\}$, to prove Lemma 6.1 it suffices to show that

$$\sum_{j\geq j_0} \int |L_{i,j}(x_1,y) - L_{i,j}(x_2,y)|dy\,,\ \sum_{j\geq j_0} \int |L_{i,j}'(x_1,y) - L_{i,j}'(x_2,y)|dy < C(i+1) \quad (6.1)$$

Notice that each term in these sums is bounded by C as a straightforward consequence of the definitions. Furthermore, by Lemma 5.2, the $j = C_1 i + j_0 + j'$ term in these sums is bounded by $C2^{-\delta j'}$ for some δ. By adding the minimum of these two estimates, (6.1) holds and Lemma 6.1 is proved.

By the generalized Calderon-Zygmund theorem, (see [23] p. 19 for the precise statement) Lemma 6.1 and the estimates of sections 4 and 5 imply that $||T_i||_{p,p} \leq C(i+1)$ for each i. Therefore, by the discussion at the end of section 2, the proofs of Theorems 2.1 and 2.2 are now complete.

7. Future Directions

There are a number of directions in which the methods of this paper might be extended. First, there is the question of maximal analogues to the the theorems of this paper. The definitions of $\tilde{T}$ and T_i we used here were made for compatibility with arguments concerning singular Radon transforms. In particular, although $\tilde{T}$ is well-suited to singular integral methods, its operator kernel can have a rather complicated singular set. It appears likely that by taking a smoother definition of $\tilde{T}$ one can use the usual square function methods (e.g. [3]) to obtain the maximal analogues.

Another issue is finding a natural cancellation condition generalizing $(K3)$. For example it may be that if the hypotheses of the $T(1)$ theorem are satisfied for the metric of this paper, Theorem 2.1 still holds. Some work has been done in this direction; see for example the thesis of Potinton [17].

Next, there is the possibility of relaxing the finite-type condition, the idea being to find a less stringent condition such that for appropriate p the estimates $||T_i||_{p,p} < C2^{-\delta i}$ may be replaced by $||T_i||_{p,p} < n_i$ for some sequence $\{n_i\}$ satisfying $\sum n_i < \infty$. Much work has been done on such "flat" singular Radon transforms, often in a translation-invariant setting with a convexity condition on the submanifolds; see for example [22] [26].

Finally, there is the question of whether the methods of this paper may be used to study boundedness properties of other averaging operators related to those treated here. For example, it would be interesting to see if the relation between the metric of this paper and the Sobolev estimates one may obtain for Radon transforms with a smooth kernel is analogous to the relation between the metric of Fefferman and Phong in [7] and the Sobolev estimates they obtain for solutions to subelliptic problems; the author hopes to consider such issues in later work. Much work has already been done on smooth Radon transforms; we mention [9] [16] [20] [21] to give a small sampling.

References

[1] J. Bourgain, *A remark on the maximal function associated to an analytic vector field*, Analysis at Urbana, Vol. I (Urbana, IL, 1986-1987), 111-132, London Math. Soc. Lecture Note Ser., 137, Cambridge Univ. Press, Cambridge-New York.

[2] M. Christ, *Hilbert transforms along curves. I. Nilpotent groups*, Annals of Mathematics (2) **122** (1985), no.3, 575-596.

[3] M. Christ, A. Nagel, E. Stein, S. Wainger, *Singular and maximal radon transforms*, Annals of Mathematics, (2) **150** (1999), no.2, 489-577.

[4] R. R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogenes*, Lecture Notes in Math. **242**, Springer-Verlag.

[5] S. Cuccagna, *Sobolev estimates for fractional and singular Radon transforms.*, Journal of Functional Analysis **139** (1996), 94-118.

[6] E. Fabes, *Singular integrals and partial differential equations of parabolic type*, Studia Mathematica **28** (1966), 81-131.

[7] C. Fefferman, D.H. Phong, *Subelliptic eigenvalue problems*, Conference on harmonic analysis in honor of Antoni Zygmund, 590-606, Wadsworth Math. Series, Wadsworth, Belmont, California, 1983.

[8] M. Greenblatt, *A method for proving L^p boundedness of singular Radon transforms in codimension one for* $1 < p < \infty$, PhD thesis, Princeton University, 1998.

[9] A. Greenleaf, A. Seeger, S. Wainger, *On X-ray transforms for rigid line complexes and integrals over curves in* $\mathbf{R}^4$, Proceedings of the American Mathematical Society, **127** (1999), no.12, 3533-3545.

[10] A. Greenleaf, G. Uhlmann, *Estimates for singular Radon transforms and pseudodifferential operators with singular symbols*, Journal of Functional Analysis **89** (1990), 202-232.

[11] A. Nagel, N. Riviere, S. Wainger, *On Hilbert transforms along curves*, Bulletin of the

American Mathematical Society **80** (1974), 106-108.

[12] A. Nagel, N. Riviere, S. Wainger, *On Hilbert transforms along curves.II.*, Amer. J. Math, **98** (1976), no. 2, 395-403.

[13] A. Nagel, E. M. Stein, S. Wainger, *Balls and metrics defined by vector fields. I. Basic Properties*, Acta Math, **155** (1985), no 1-2, 103-147.

[14] D. H. Phong, *Singular integrals and Fourier integral operators*, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), 286-320, Princeton Mathematics Series Vol. 42, Princeton University Press, Princeton, NJ, 1995.

[15] D. H. Phong, E. M. Stein, *Hilbert integrals, singular integrals, and Radon transforms I*, Acta Mathematica **157** (1986), 99-157.

[16] D. H. Phong, E. M. Stein, *The Newton polyhedron and oscillatory integral operators*, Acta Mathematica **179** (1997), 105-152.

[17] D. Pottinton, thesis, Princeton University, 1998.

[18] F. Ricci, E. Stein, *Harmonic analysis on nilpotent groups and singular integrals I. Oscillatory integrals*, Journal of Functional Analysis **73** (1987), no. 1, 179-194.

[19] F. Ricci, E. Stein, *Harmonic analysis on nilpotent groups and singular integrals II. Singular integral kernels supported on submanifolds.*, Journal of Functional Analysis **78** (1988), no. 1, 56-84.

[20] A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Mathematical Journal, **71** (1993), no.3, 685-745.

[21] A. Seeger, *Radon transforms and finite type conditions*, Journal of the American Mathematical Society **11** (1998) no.4, 869-897.

[22] A. Seeger, S. Wainger, J. Wright, S. Ziesler, *Classes of singular integral operators along curves and surfaces*, Transactions of the American Mathematical Society **351** (1999), no.9, 3757-3769.

[23] E. Stein, *Harmonic analysis; real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematics Series Vol. 43, Princeton University Press, Princeton, NJ, 1993.

[24] E. Stein, S. Wainger, *Maximal functions associated to smooth curves*, Proceedings of the National Academy of Sciences, USA **73** 1976, no. 12, 4295 - 4296.

[25] E. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc., **84** (1978), 1239-1295.

[26] S. Wainger, J. Wright, S. Ziesler, *Singular integrals associated to hypersurfaces: L^2 theory* Journal of Functional Analysis **167** (1999), no.1, 148-182.